JACM **6**, No. 2, 16-29 (2021)

Journal of Abstract and Computational Mathematics

http://www.ntmsci.com/jacm

A Class of Nonlocal Elliptic Equations in Orlicz-Sobolev Spaces

Berat SÜER [1], *Mustafa AVCI* [2], *Veyis TURUT* [3*]

[1] Institute of Graduate Studies in Science and Engineering, Batman University, Batman Turkey
https://orcid.org/0000-0003-3264-0443
[2] Edwards School of Business, University of Saskatchewan, Saskatoon Canada
https://orcid.org/0000-0002-6001-627X
[3] Department of Mathematics, Batman University, Batman Turkey
https://orcid.org/0000-0002-8148-7935

Received: 02 May 2021, Accepted: 04 Jun 2021
Published online: 27 Sep 2021

Abstract: In this article, we are concerned with some classes of nonlocal elliptic equations. We apply the homogenous Dirichlet boundary conditions. Our problem is settled in Orlicz-Sobolev spaces. To obtain the nontrivial solutions, we apply variational approach. The variational approach is a very helpful tool especially to obtain solutions of nonlinear differential equations. The main idea in the variational approach is to designate the corresponding energy functional, which is also known as Euler-Lagrange functional, and then using some auxiliary theorems from functional analysis and nonlinear analysis, such as Hölder inequality, Lebesgue convergence theorem, continuous and compact embeddings theorems, and Frechet derivative, to find local or global minimizers of the corresponding energy functional, that is, the solutions of the corresponding differential equations.

Keywords: Nonlocal elliptic equations, Ginzburg-Landau energy, variational approach, Mountain-Pass theorem, Orlicz-Sobolev spaces

1 Introduction

We are concerned with the following nonlocal elliptic equation

$$-A\left(\int_\Omega \Phi(|\nabla u(x)|)dx\right)\operatorname{div}(a(|\nabla u(x)|)\nabla u(x))+\alpha(x)\left(\frac{|u(x)|^q}{q}-\beta(x)\right)\|\,(x)|^{q-2}u \quad = f(x,u) \text{ on } \Omega, \qquad u=0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $R^N (N\geq 3)$ with smooth boundary $\partial\Omega$; $u:\Omega\to R$ denotes primal field; $q\geq 2$; $\alpha,\beta\in L^\infty(\Omega)$ with $\inf_{xa\bar{a}}\alpha(x)\beta(x)>0; A-(0,\infty)\to(0,\infty)$ is a continuous function; $f:\bar{\Omega}xR\to R$ is a Caratheodory function, and the function $\varphi(t)=\int_0^t a(|t|)t$ is an increasing homeomorphism from $\bar{\Omega}xR$ onto R such that $\Phi(t)=\int_0^t\varphi(s)ds$. We want to mention that equations like (1) have been intensively studied by many authors over the past twenty years due to its significant role in many fields of mathematics, such as calculus of variations, non-linear potential theory, non-Newtonian fluids, image processing (see, e.g., [2,4,5,6,9,10,12,13,18,21,23,27,34,40,42]). Therefore, equations of type (1) may represent a variety of mathematical models corresponding to certain phenomenons:

For $\varphi(t)=p|t|^{p-2}t$, (1) turns into the well-known p-Kirchhoff-type equation

$$-A\left(\int_\Omega \Phi\frac{|\nabla u(x)^p|}{p}dx\right)div\left(|\nabla u(x)|^{p-2}\nabla u(x)\right)+\alpha(x)\left(\frac{|u(x)|^q}{q}-\beta(x)\right)|u(x)|^{q-2}u=f(x,u) \text{ on}\Omega, \qquad u=0 \text{ on } \partial\Omega. \tag{2}$$

* Corresponding author e-mail: veyisturut@gmail.com

Problem (2) is a stationary counterpart of (3).

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \left[p_0 + p_1 \int_0^1 \left(\frac{\partial u(x)}{\partial x} \right)^2 dx \right] \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad x \in (0,1), \ t > 0, \tag{3}$$

when we put $A(t) = p_0 + p_1 t$ and $p(x) = 2$, where p_0 is connected with the initial tension, p_1 is dependent on the characteristic of the material of the string and $u(x,t)$ denotes the vertical displacement of the point x of the string at a time t (3) was proposed by Kirchhoff in [29] as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings. Such problems are often called nonlocal since the equation is no longer a pointwise identity because it contains an integral over Ω .

2 Preliminaries

We start with some basic concepts of Orlicz spaces. For more details we refer the readers to the monographs [1,30,31, 37,39], and the papers [17,21,25,26,34]. The function $\alpha(t) : R \to R$ is a function such that the mapping $\varphi(t) : R \to R$, defined by

$$\varphi(t) = \begin{cases} \alpha(|\,t)t, & for\, t \neq 0 \\ 0, & for\, t = 0 \end{cases}, \tag{4}$$

is an odd, increasing homeomorphism. For the function φ above, if we define

$$\Phi(t) = \int_0^t \varphi(s) ds, \tag{5}$$

then the function $\Phi : [0,+\infty) \to [0,+\infty)$ is called a $N-$function if it satisfies the following conditions (see e.g., [1,37, 39]):

$(\Phi_0)\Phi(t)$ is a convex, nondecreasing and continuous function such that, $\Phi(0) = 0,\ \Phi(t) > 0$ for all $t > 0$, and

$$lim_{t\to 0} \frac{\Phi(t)}{t} = 0,\ \ lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty.$$

The set of all generalized $N-$functions is denoted by $N(\Omega)$ .The function $\bar{\Omega}$ defined by

$$\bar{\Phi}(t) = \int_0^t \varphi^{-1}(s) ds, t \geq 0, \tag{6}$$

is called the complementary (or conjugate) function to Φ , where $\bar{\Phi}$ satisfies the following

$$\bar{\Phi}(t) = \sup_{\infty 0}\{st - \Phi(s) : s \in R\}, t \geq 0.$$

It is well known that $\bar{\Phi} \in N(\Omega)$,and then the following Young inequality holds

$$st \leq \Phi(t) + \bar{\Phi}(s),\ t \in R. \tag{7}$$

The function Φ allows us to define the Orlicz space, by

$$L^{\Phi}(\Omega) = \left\{ u : \Omega \to R \text{ measurable } \exists \lambda > 0 \text{ such that } \int_\Omega \Phi(\frac{|u(x)|}{\lambda}) dx < \infty \right\}.$$

Moreover, by Δ_2 condition (see below), $L^{\bar{\Phi}}(\Omega)$ is the dual space of $L^{\Phi}(\Omega)$, i.e., $\left(L^{\Phi}(\Omega)\right)^* = L^{\bar{\Phi}}(\Omega)$.
In the sequel, we also use the following assumptions for Φ :

$$1 < \varphi_0 = \inf_{D0} \frac{t\varphi(t)}{\Phi(t)} \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 = \sup_{\infty 0} \frac{t\varphi(t)}{\Phi(t)} < \infty,\ \forall t > 0; \tag{8}$$

$$\inf_{t>0}\Phi(t)>0, \tag{9}$$

$$\text{The function } t\to\Phi(\sqrt{t}) \text{ is convex, } \forall t\geq 0. \tag{10}$$

By the help of assumption (8), the Orlicz spaces $L^{\Phi}(\Omega)$ coincides with the equivalence classes of measurable functions $u:\Omega\to R$ such that

$$\int_{n}\Phi(|u(x)|)dx<\infty, \tag{11}$$

and is equipped with the Luxemburg norm

$$|u|_{\Phi}=\inf\left\{\mu>0:\int_{\Omega}\Phi\left(\frac{|u(x)|}{\mu}\right)dx\leq 1\right\}. \tag{12}$$

For the Musielak-Orlicz spaces, Hölder inequality reads as follows (see [1,39])
$\int_{\Omega}uvdx\leq 2\|u\|_{L^{\Phi}(\Omega)}\|v\|_{L^{\bar{\Phi}}(\Omega)}$ for all $u\in L^{\Phi}(\Omega)$ and $v\in L^{\bar{\Phi}}(\Omega)$.

$$W^{1,\Phi}(\Omega):=\left\{u\in L^{\Phi}(\Omega):\frac{\partial u(x)}{\partial x_i}\in L^{\Phi}(\Omega)\ i=1,2,...N\right\},$$

under the norm

$$\|u\|_{1,\Phi}:=|u|_{\Phi}+|\nabla u|_{\Phi}. \tag{13}$$

Now we introduce Orlicz-Sobolev spaces with zero boundary traces $W_0^{1,\Phi}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ under the norm $\|u\|_{1,\Phi}$. Moreover, by the help of the well-known Poincar'e inequality, we can define an equivalent norm $\|.\|_{\Phi}$ on $W_0^{1,\Phi}(\Omega)$ by

$$\|u\|_{1,\Phi}:=|\nabla u|_{\Phi}. \tag{14}$$

Proposition 2.1 ([1,21]). If (8)-(10) hold then the spaces $L^{\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.2 ([17,34]). Let define the modular $\rho(u)=\int_{\Omega}\Phi(|\nabla_n|)\,dx=W_0^{1,\Phi}(\Omega)\to R$ Then for every $u_n,u\in W_0^{1,\Phi}(\Omega)$ we have

1. $\|u\|_{\Phi}^{\varphi^0}\leq\rho(u)\leq\|u\|_{\Phi}^{\varphi_0}$ if $\|u\|_{\Phi}<1$,
2. $\|u\|_{\Phi}^{\varphi_0}\leq\rho(u)\leq\|u\|_{\Phi}^{\varphi^0}$ if $\|u\|_{\Phi}>1$,
3. $\|u\|_{\Phi}\leq\rho(u)+1$,
4. $\|u_{\mathrm{n}}-u\|_{\Phi}\to 0\Leftrightarrow\rho(u_{\mathrm{a}}-u)\to 0$,
5. $\|u_{\mathrm{n}}-u\|_{\Phi}\to\infty\Leftrightarrow\rho(u_{\mathrm{n}}-u)\to\infty$.

Remark 2.3 The functional ρ is from $C^1\left(W_0^{1,\Phi}(\Omega),R\right)$ with the derivative

$$\langle\rho'(u),v\rangle=\int_{\Omega}a(|\nabla_u|)\nabla u\nabla v dx,$$

where $\langle...\rangle$ is the dual pairing between $W_0^{1,\Phi}(\Omega)$ and its dual $\left(W_0^{1,\Phi}(\Omega)\right)^*$. Moreover, the operator ρ' is of type (S+), that is, $u_n\to u$ in $W_0^{1,\Phi}(\Omega)$ and $\limsup\langle\rho'(u_n),u_n-u\rangle\leq 0$ imply $u_n\to u$ in $W_0^{1,\Phi}(\Omega)$ (see [34]).

It is said that Φ satisfies the Δ_2-condition if there is a positive constant M such that

$$\Phi(2t)\leq M\Phi(t),\forall t\geq 0. \tag{15}$$

Proposition 2.4 ([20]). Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous and compact provided $p > 1$, $1 < r < p^*$ where

$$p^* := \frac{Np}{N-p},\ p < N \text{ and } p^* := +\infty,\ p \geq N.$$

Remark 2.5 The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is continuously embedded in the Sobolev space $W^{1,\varphi^0}(\Omega)$ On the other hand, $W^{1,\varphi^0}(\Omega)$ is compactly embedded in the Lebesgue space $L^r(\Omega)$ for all $1 \leq r < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$ As a result, $W^{1,\Phi}(\Omega)$ is continuously and compactly embedded in the Lebesgue space $L^r(\Omega)$.

3 Main Results

First, we shall give the variational framework of the problem (1). The energy functional corresponding to problem (1) is defined as $\varepsilon : W_0^{1,\Phi}(\Omega) \to R$

$$\varepsilon(u) = \hat{A}\left(\int_\Omega \Phi(|\nabla u|)\,dx\right) + \int_\Omega \frac{\alpha(x)}{2}\left(\frac{|u|^q}{q} - \beta(x)\right)^2 dx - \int_\Omega F(x,u)dx, \tag{16}$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $\hat{A}(t) = \int_0^t A(s)ds$. Then the problem will be to find some $u_0 \in W_0^{1,\Phi}(\Omega)$, which satisfy the equation (1), such that

$$\varepsilon(u_0) = \min_{u\in W_0^{1,\Phi}(\Omega)} \{\varepsilon(u)\}. \tag{17}$$

We say that $W_0^{1,\Phi}(\Omega)$ is a weak solution of problem (1) if

$$A\left(\rho(u)\left\langle\rho'(u),v\right\rangle\right) + \int_\Omega \alpha(x)\left(\frac{|u|^q}{q} - \beta(x)\right)|u|^{q-2}uvdx - \int_\Omega f(x,u)vdx = 0 \tag{18}$$

for all $v \in W_0^{1,\Phi}(\Omega)$.

We will study problem (1) under the following assumptions: Throughout the paper we always assume that

$$2 \leq q < p < \infty.$$

(A0) $A : (0,\infty) \to (0,\infty)$ is a continuous function and satisfies the condition

$$m_1 t^{\sigma-1} \leq A(t) \leq m_2 t^{\sigma-1}.$$

where $t > 0, \sigma > 1$ and $m_2 \geq m_1 > 1$.

(f1) $f : \Omega \times R \to R$ is a Carathéodory function and there exists $c_1 > 0$ such that

$$|f(x,t)| \leq c_1 |t|^{s-1},$$

where $s \in C(\bar{\Omega})$ such that $s < \alpha\varphi_0$.

(f2) There exist constants $M, \Theta > 0$ with $2q \leq \sigma\varphi^0 < \theta < \varphi_0^*$ such that

$$0 < \theta F(x,t) \leq f(x,t)t, |t| \geq M, \forall x \in \Omega.$$

(f3) $f(x,t) = o\left(|t|^{q-1}\right)$ as $t \to 0$ uniformly in $x \in \Omega$.

(f4) $f(x,-t) = -f(x,t)$.

Remark 3.1 The function $f(x,t) = o\left(|t|^{\sigma-2}\right)t$, where $\sigma > q$, satisfies assumptions (f1)-(f4).

Remark 3.2 By assumption (f2) there exists a constant $c > 0$ such that $F(x,t) \geq c|t|^{\theta}$ for all $x \in \Omega$ and $|t| \geq M$.
The main results of the present paper are the following.

Theorem 3.3 Assume that (f1)-(f3) and (A0) hold. Then problem (1) has a nontrivial solution in $W_0^{1,\Phi}(\Omega)$.
First, we need to show that functional ε satisfies the main smoothness properties which are the essential part of the main proofs of the paper.

Lemma 3.4 The functional ε is well-defined on $W_0^{1,\Phi}(\Omega)$ and Fréchet differentiable, i.e., $\varepsilon \in C^1(W_0^{1,\Phi}(\Omega), R)$ whose derivative is

$$\left\langle \varepsilon'(u), v \right\rangle = A\left(\rho(u)\left\langle \rho'(u), v \right\rangle\right) + \int_{\Omega} \alpha(x)\left(\frac{|u|^q}{q} - \beta(x)\right)|u|^{q-2}uvdx - \int_{\Omega} f(x,u)vdx. \tag{19}$$

Proof From the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{2q}(\Omega) \hookrightarrow L^q(\Omega)$,for any $u \in W_0^{1,\Phi}(\Omega)$ it is easy to see that

$$\left(\frac{|u|^q}{q} - \beta(x)\right)^2 \in L^1(\Omega). \tag{20}$$

By conditions (f1), (A0), the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^s(\Omega)$ and (20), it follows

$$|\varepsilon(u)| \leq \hat{A}(\rho(u)) + \int_{\Omega} \frac{\alpha(x)}{2}\left(\frac{|u|^q}{q} - \beta(x)\right)^2 dx - \int_{\Omega} F(x,u)dx < \infty,$$

which means that ε is well-defined on $W_0^{1,\Phi}(\Omega)$.

Denote $K := W_0^{1,\Omega}(\Omega) \to R$ by $K(u) := \hat{A}(\rho(u))$ .Considering the fact that the functional ρ is of class $C^1(W_0^{1,\Phi}(\Omega))$, and A is a continuous function satisfying the growth condition (A0), it is easy to see that the composition functional $K(u) := \hat{A}(\rho(u))$ is well-defined on $W_0^{1,\Phi}(\Omega)$ and of class $C^1(W_0^{1,\Phi}(\Omega))$ with the derivative $K' := W_0^{1,\Phi}(\Omega) \to \left(W_0^{1,\Phi}(\Omega)\right)^*$

$$\left\langle K'(u), v \right\rangle = A(\rho(u))\langle \rho'(u), v\rangle,$$

for all $u, v \in W_0^{1,\Omega}(\Omega)$,Therefore, to obtain that $\varepsilon \in C^1(W_0^{1,\Omega}(\Omega), R)$, it is enough to show that the operator $\Lambda : W_0^{1,\Omega}(\Omega) \to R$ given by

$$\Lambda(u) = \int_{\Omega} \frac{\alpha(x)}{2}\left(\frac{|u|^q}{q} - \beta(x)\right)^2 dx - \int_{\Omega} F(x,u)dx,$$

is of class $C^1(W_0^{1,\Omega}(\Omega), R)$ .To this end, first, it must be shown that for all $v \in W_0^{1,\Omega}(\Omega)$

$$\langle \Lambda'(u), v\rangle = \lim_{t \to 0} \frac{\Lambda(u+tv) - \Lambda(u)}{t} = \int_{\omega} \alpha(x)\left(\frac{|u|^q}{q} - \beta(x)\right)|u|^{q-2}uvdx - \int_{\Omega} f(x,u)vdx,$$

and then it must be obtained that $\Lambda' := W_0^{1,\Omega}(\Omega) \to \left(W_0^{1,\Omega}(\Omega)\right)^*$ is continuous.
The continuity properties of $|.|$ and f along with the definition of f , allow us to apply the mean value theorem, that is,

$$\begin{aligned}\langle \Lambda'(u), v\rangle &= \lim_{t \to 0} \int_{\Omega} \frac{\alpha(x)}{2t}\left(\left(\frac{|u+tv|^q}{q} - \beta(x)\right)^2 - \left(\frac{u|^q}{q} - \beta(x)\right)^2\right) dx - \lim_{t \to 0} \int_{\Omega} \frac{F(x,u+tv) - F(x,u)}{t} dx, \\ &= \lim_{t \to 0} \int_{\Omega} \alpha(x)\left(\frac{|u+t\varepsilon v|}{q}\right)|u+t\varepsilon v|^{q-2}(u+t\varepsilon v)vdx - \lim_{t \to 0} \int_{\Omega} f(u+t\varepsilon v)vdx,\end{aligned}$$

where $u,v \in W_0^{1,\Omega}(\Omega)$ and $0<\varepsilon<1$. Now, if we apply the Young's inequality along with the inequality $|a+b|^m \leq 2^{m-1}(|a|^m+|b|^m)$, for all $a,b \in R^N$ and $m \geq 1$, consecutively to all integrands on the right-hand side of the above expression, and use condition (f1), it reads

$$\begin{aligned}
&\left|a(x)\left(\frac{|u+t\varepsilon v|^q}{q}-\beta(x)\right)|u+t\varepsilon v|^{q-2}(u+t\theta v)\right|,\\
&\leq \alpha(x)\left(\frac{|u+t\varepsilon v|^q}{q}+\beta(x)\right)|u+t\varepsilon v|^{q-1}|v|,\\
&\leq \alpha(x)\left(\frac{|u+t\varepsilon v|^{2q-1}}{q}|v|+\beta(x)|u+t\varepsilon v|^{q-1}|v|\right).
\end{aligned}$$

However, by the Young's inequality it reads

$$\frac{|u+t\varepsilon v|^{2q-1}}{q}|v| \leq \frac{(2q-1)2^{2q-1}}{2q^2}\left[|u|^{2q}+|v|^{2q}\right]+\frac{1}{2q}|v|^{2q}, \tag{21}$$

and

$$|u+t\varepsilon v|^{q-1}|v| \leq \frac{(q-1)2^{q-1}}{qq^2}(|u|^q+|v|^q)+\frac{1}{q}|v|^q. \tag{22}$$

Similarly,

$$|f(x,u+t\varepsilon v)v| \leq c\left(\frac{2^{s-1}(s-1)}{s}|u|^s+\left(\frac{2^{s-1}(s-1)}{s}|u|^5+\left(\frac{2^{s-1}(s-1)+1}{s}\right)|v|^5\right)\right). \tag{23}$$

The right hand sides of the inequalities (21)-(23) belong to $L^1(\Omega)$. Therefore, by the Lebesgue dominated convergence theorem, which make it possible to change the order of lim and integral signs, along with the continuity properties of f and $|.|$, it reads that

$$\begin{aligned}
\langle \Delta', v\rangle &= \int_\Omega \alpha(x) lim_{t\to 0}\left(\frac{|u+t\varepsilon v|^q}{q}-\beta(x)\right)|u+t\varepsilon v|^{q-2}(u+t\varepsilon v)vdx - lim_{t\to 0}\int_\Omega f(x,u+t\varepsilon v)vdx,\\
&= \int_\Omega \alpha(x)\left(\frac{|u|^q}{q}-\beta(x)\right)|u|^{q-2}uvdx-\int_\Omega f(x,u)vdx.
\end{aligned}$$

Since the right-hand side of the above expression, as a function of v , is a continuous linear functional on $W_0^{1,\Omega}(\Omega)$, it is the Gateaux differential of Λ.

Next, we proceed to the continuity of Λ. To this end, we assume, for a sequence $(u_n) \subset W_0^{1,\Omega}(\Omega)$, that $(u_n) \to u \in W_0^{1,\Omega}(\Omega)$. Then

$$\left|\langle \Lambda'(u_n)-\Lambda'(u),v\rangle\right| \leq \left|\int_\Omega \alpha(x) I_n v dx\right| + \left|\int_\Omega (f(x,u)-f(x,u_n))vdx\right|,$$

where

$$I_n := \Theta(u_n)-\Theta(u) = \left[\left(\frac{|u_n|^q}{q}-\beta(x)\right)|u_n|^{q-2}u_n - \left(\frac{|u|^q}{q}-\beta(x)\right)|u|^{q-2}u\right],$$

and

$$\theta(.) = \left(\frac{|.|^q}{q}-\beta(x)\right)|.|^{q-2}. \tag{24}$$

By the Hölder inequality it reads

$$\left|\int_{\Omega}\alpha(x)I_n v dx\right| \leq c|I_n|_{\frac{q}{q-1}}|v|_q. \tag{25}$$

Note that because of the embeddings $W_0^{1,\Phi}(\Omega)\hookrightarrow L^q(\Omega)\hookrightarrow L^{\frac{q}{q-1}}(\Omega)$ we can apply $(u_n)\to u\in W_0^{1,\Omega}(\Omega)$ to (25).

On the other hand, we can write

$$\begin{aligned}|I_n| = |\Theta(u_n)-\Theta(u)| &\leq \beta(x)\left(|u_n|^{q-1}+|u|^{q-1}\right)+\frac{1}{q}\left(|u_n|^{2q-1}+|u|^{2q-1}\right),\\ &\leq C\left(|u_n|^{q-1}+|u|^{q-1}+|u_n|^{2q-1}+|u|^{2q-1}\right),\end{aligned} \tag{26}$$

where $C := \max\left(\frac{1}{p},\sup_{x\in\Omega}\beta(x)\right)$, since $(u_n)\to u\in W_0^{1,\Omega}(\Omega)$, by the compact embeddings $W_0^{1,\Phi}(\Omega)\hookrightarrow L^{\frac{(q-1)q}{q-1}}(\Omega)$, $W_0^{1,\Phi}(\Omega)\hookrightarrow L^{\frac{(2q-1)q}{q-1}}(\Omega)$ and $W_0^{1,\Phi}(\Omega)\hookrightarrow L^s(\Omega)$ up to a subsequence still denoted by u_n , we have

$$\begin{aligned} u_n &\to u \text{ in } L^{\frac{(q-1)q}{q-1}}(\Omega),\\ u_n &\to u \text{ in } L^{\frac{(2q-1)q}{q-1}}(\Omega),\\ u_n &\to u \text{ in } L^s(\Omega),\\ u_n &\to u \text{ a.e. } x\in\Omega,\end{aligned}$$

and there exist $w_1\in L^{\frac{(q-1)q}{q-1}}(\Omega)$, $w_2\in L^{\frac{(2q-1)q}{q-1}}(\Omega)$ and $w_3\in L^s(\Omega)$ such that $|u_n(x)|\leq w_1(x)$, $|u_n(x)|\leq w_2(x)$ and $|u_n(x)|\leq w_3(x)$ a.e. $x\in\Omega$, respectively, for all $n\in N$ Therefore, using this information in (26), we obtain

$$|I_n|_{\frac{q}{q-1}} = |\Theta(u_n)-\Theta(u)|_{\frac{q}{q-1}} = \left(\int_{\Omega}|\Theta(u_n)-\Theta(u)|^{\frac{q}{q-1}}dx\right)^{\frac{q}{q-1}},$$

and

$$\begin{aligned}|\Theta(u_n)-\Theta(u)|_{\frac{q}{q-1}} &\leq c\left(1+|u_n|^{q-1}+|u|^{q-1}+|u_n|^{2q-1}+|u|^{2q-1}\right)^{\frac{q}{q-1}},\\ &\leq c\left(1+|w_1|^{\frac{(q-1)q}{q-1}}+|u|^{\frac{(q-1)q}{q-1}}+|w_2|^{\frac{(2q-1)q}{q-1}}+|u|^{\frac{(2q-1)q}{q-1}}\right)\in L^1(\Omega).\end{aligned}$$

Now, we show that $|\Theta(u_n(x))-\Theta(u(x))|\to 0$ as $n\to\infty$. Indeed

$$\begin{aligned}|\Theta(u_n(x))-\Theta(u(x))| &= \left|\left(\frac{|u_n|^q}{q}-\beta(x)\right)|u_n|^{q-2}u_n-\left(\frac{|u|^q}{q}-\beta(x)\right)|u|^{q-2}u\right|,\\ &\leq \frac{1}{q}\left||u_n|^{2q-2}u_n-|u|^{2q-2}u\right|+\beta(x)\left||u_n|^{q-2}u_n-|u|^{q-2}u\right|.\end{aligned}$$

Now, we mention the following inequality given in [11]: for $1<k<\infty$ there exist constants $c_k>0$ such that

$$\left||\xi|^{k-2}\xi-|\zeta|^{k-2}\zeta\right| \leq C_k|\xi-\zeta|\left(|\xi|+|\zeta|\right)^{k-2},\forall\xi,\zeta\in R^N.$$

Therefore, since $(u_n)\to u\in W_0^{1,\Omega}(\Omega)$, we obtain that

$$\lim_{n\to\infty}|\Theta(u_n(x))-\Theta(u(x))| = 0.$$

As for the term $\left|\int_\Omega (f(x,u) - f(x,u_n))\,vdx\right|$, using (f1), the Hölder inequality and the continuous embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^s(\Omega) \hookrightarrow L^{s-1}(\Omega)$,we have

$$\left|\int_\Omega (f(x,u) - f(x,u_n))\,vdx\right| \le c_2 \int_\Omega \left(|w_3|^{s-1} + |u|^{s-1}\right)|v|dx,$$
$$\le c_3 \int_\Omega \left(||w_3|s-1|_{\frac{s}{s-1}} + ||u|s-1|_{\frac{s}{s-1}}\right)|v| \in L^1(\Omega).$$

Moreover, considering that $u_n(x) \to u(x)$ a.e. $x \in \Omega$ and f is continuous, we obtain that

$$\lim_{n\to\infty}\left|\int_\Omega (f(x,(u_n)) - f(x,u(x)))\right| = 0.$$

If we take into account all information obtained above and apply the Lebesgue dominated convergence theorem, it reads

$$\lim_{n\to\infty}\left|\int_\Omega (f(x,u_n) - f(x,u))\right| = 0,$$

and

$$\lim_{n\to\infty}\int_\Omega |\Theta(u_n) - \Theta(u)|^{\frac{q}{q-1}}\,dx = 0,$$

where these two results together mean, as a conclusion, that

$$\lim_{n\to\infty}\sup\|\Lambda'(u_n) - \Lambda'(u)\|_{\left(W_0^{1,\Phi}(\Omega)\right)^*} = \lim_{n\to\infty}\sup_{\|x\|_\Phi \le 1}\left|\langle\Lambda'(u_n) - \Lambda'(u), v\rangle\right|.$$

Therefore, $\Lambda' := W_0^{1,\Phi}(\Omega) \to \left(W_0^{1,\Phi}(\Omega)\right)^*$ is continuous. □

Remark 3.5 Due to Lemma 3.4, we have $\varepsilon \in C^1\left(W_0^{1,\Phi}(\Omega), R\right)$ whose derivative is given by (19). If we compare (18) with (19), it is clear to see that the critical points of functional ε are those functions which satisfy both (17) and (18). Therefore, we will seek the critical points of functional ε to obtain the solutions of problem (1) To this end, we will obtain some results in what follows next, which allow us to prove that functional ε has critical points.

Lemma 3.6 Assume that (f1) and (f3) hold. Then,

i.There exist two positive real numbers η and τ such that $\varepsilon(u) \ge \tau > 0$, for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_\Phi \eta < 1$.
ii.There exists $e \in W_0^{1,\Phi}(\Omega)$ such that $\|e\|_\Phi > 1, \varepsilon(e) < 0$.

Proof (i) By assumption (f3), given $\varepsilon \in (0, \delta^{\varphi^0}/2\sigma c_0^q)$ with $\delta \in (0,1)$, we can write

$$|F(x,t)| \le \frac{\varepsilon|t|^q}{q}, \;\; \forall x \in \Omega, \;\; |t| \le \delta.$$

Let $u \in W_0^{1,\Phi}(\Omega)$ be such that

$$\|u\|_\Phi = \eta := \left(\frac{1}{m_1 q}\right)^{1/\sigma\varphi^0 - q} \delta^{\varphi^0/\sigma\varphi^0 - q} < 1.$$

Then, by Proposition 2.3 and the continuous embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$ i.e., $\exists c_0 = c(|\Omega|) > 0$ such that $|u|_q \le c_0\|u\|_\Phi$, $\forall u \in W_0^{1,\Phi}(\Omega)$, it follows

$$\begin{aligned}\varepsilon(u) &\geq \frac{m_1}{\sigma}\left(\int_\Omega \Phi(x,|\nabla|)dx\right)^\sigma - \frac{\varepsilon}{q}\int_\Omega |u|^q dx,\\ &\geq \frac{m_1}{\sigma}\|u\|_\Phi^{\sigma\varphi^0} - \frac{\varepsilon}{q}c_0^q\|u\|_\Phi^q,\\ &\geq \left(\frac{m_1}{\sigma}\|u\|_\Phi^{\sigma\varphi^0-q} - \frac{\varepsilon}{q}\right)\|u\|_\Phi^q = \left(\frac{1}{\sigma q}\delta^{\varphi^0} - \frac{\varepsilon}{q}c_0^q\right)\eta^q = \tau,\end{aligned}$$

i.e., we obtain that $\varepsilon(u) \geq \tau > 0$.

(ii) First, we note that for $t > 1$ and $s > 0$ it holds $\Phi(ts) \leq t^{\varphi^0}\Phi(s)$. Indeed, from the assumption (2.5), we have

$$\frac{z\varphi(z)}{\Phi(z)} \leq, \ \forall z \geq 0,$$

from which we can proceed as follows

$$\begin{aligned}\int_s^{ts}\frac{\varphi(z)}{\Phi(z)} &\leq \int_s^{ts}\frac{\varphi^0}{z}dz,\\ \log\Phi(ts) - \log\Phi(s) &\leq \log t^{\varphi^0},\\ \Phi(ts) &\leq t^{\varphi^0}\Phi(s).\end{aligned}$$

Let $0 \neq \phi \in W^{1,\Phi}(\Omega)$ and $1 < t \in R$. By Remark 3.2, we obtain that

$$\begin{aligned}\varepsilon(t\phi) =& \frac{m_2}{\sigma}t^{\sigma\varphi^0}\left(\int_\Omega \Phi(x,|\nabla u|)dx\right)^\sigma + \frac{t^{2q}}{2q^2}\int_\Omega \alpha(x)|\phi|^{2q(x)}dx + t^q\int_\Omega \alpha(x)\beta(x)|\phi|^q dx\\ &+ \int_\Omega \alpha(x)\beta^2(x)dx - ct^0\int_\Omega |\phi|^\theta dx.\end{aligned} \tag{27}$$

Since $\theta > \sigma\varphi^0 > 2q$, we obtain that $\varepsilon(t\phi) \to -\infty$ as $t \to +\infty$. Then, for $t > 1$ large enough, if we set $t\phi = e$ with $\|e\|_\Phi > \eta$ we obtain that $\varepsilon(e) < 0$. □

Definition 3.7 Let X be a Banach space and $I : X \to R$ be a C^1-functional. We say that a functional $I : X \to R$ satisfies the Palais-Smale condition (shortly, (P S)-condition), if any Palais-Smale sequence, i.e., a sequence $u_n \subset X$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$, contains a convergent subsequence.

Lemma 3.8 Assume that (f1) and (f2) hold. Then, ε satisfies the (P S)-condition.

Proof From the proof of Lemma 3.7, ε satisfies the Mountain-Pass geometry which assures the existence of a Palais-Smale sequence $u_n \in W_0^{1,\Phi}(\Omega)$ such that

$$\varepsilon(u_n) \to \hat{c} \ \text{and} \ \|\varepsilon'(u_n)\|_{\left(W^{1,\Phi}(\Omega)\right)^*} \to 0, \tag{28}$$

where $\hat{c}$ is a critical value of ε and characterized in Theorem 3.10(ii).

First, let us show that (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Assume the contrary. Then, along a subsequence, $\|u_n\|_\Phi \to \infty$ and, in addition, we may assume that $\|u_n\|_\Phi > 1$. Then, from (28) there exists a real number $C > 0$ such that

$$\begin{aligned} C+\|u_n\| \geq \varepsilon(u_n) =& \hat{A}(\rho(u_n)) + \int_\Omega \frac{\alpha(x)}{2}\left(\frac{|u_n|^q}{q} - \beta(x)\right)^2 dx - \int_\Omega F(x,u_n)\,dx, \\ &\geq \frac{m_1}{\sigma}\|u_n\|_\Phi^{\sigma\varphi_0} - \frac{c_1}{s}\int_\Omega |u_n|^s\,dx, \\ &\geq \frac{m_1}{\sigma}\|u_n\|_\Phi^{\sigma\varphi_0} - c\|u_n\|_\Phi^s. \end{aligned}$$

Since $\sigma\varphi_0 > s > 1$, if we divide above inequality by $\|u_n\|_\Phi^s$ and take the limit as $n \to +\infty$, we obtain a contradiction. Therefore, (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Since $W_0^{1,\Phi}(\Omega)$ is reflexive, there exists a subsequence, still denoted by (u_n), which converges weakly to a $u \in W_0^{1,\Phi}(\Omega)$. Then, by (28) it reads

$$\begin{aligned} \langle \varepsilon'(u_n), u_n - u\rangle =& A\left(\rho(u_n)\langle \rho'(u_n), u_n - u\rangle\right) + \int_\Omega \alpha(x)\left(\frac{|u_n|^q}{q} - \beta(x)\right)|u_n|^{q-2}u_n(u_n-u)\,dx \\ &- \int_\Omega f(x,u_n)(u_n-u)\,dx \to 0. \end{aligned}$$

By (f1), the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^s(\Omega)$ and Hölder inequality we have

$$\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| \leq \int_\Omega |u_n|^{s-1}|u_n-u|\,dx \leq \left||u_n|^{s-1}\right|_{s-1}|u_n-u|_s \to 0. \tag{29}$$

Similarly, by the compact embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{2q}(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L^q(\Omega)$ and Hölder inequality we have

$$\begin{aligned} &\left|\int_\Omega \alpha(x)\left(\frac{|u_n|^q}{q} - \beta(x)\right)\left|u_n\right|^{q-2}u_n(u_n-u)\,dx\right| \leq \bar{c}\int_\Omega (|u_n|^q+1)\left|u_n\right|^{q-1}|u_n-u|\,dx, \\ &\leq \bar{c}\left(\int_\Omega |u_n|^{2q-1}|u_n-u|\,dx + \int_\Omega |u_n|^{q-1}|u_n-u|\,dx\right), \\ &\leq \bar{c}\left(\left||u_n|^{2q-1}\right|_{\frac{2q}{2q-1}}|u_n-u|_{2q} + \left||u_n|^{q-1}\right|_{\frac{q}{q-1}}|u_n-u|_q\right) \to 0, \end{aligned} \tag{30}$$

where $\bar{c} := 2\max\left(\sup_{x\in\bar{\Omega}}\alpha(x) X \sup_{x\in\bar{\Omega}}\beta(x), \frac{1}{q}\right)$.

By (29) and (30), we must have

$$A\left(\rho(u_n)\langle\rho'(u_n), u_n - u\rangle\right) \to 0.$$

From Remark 2.4 and condition (A0), it follows at once that (u_n) converges strongly to $u \in W_0^{1,\Phi}(\Omega)$. As a conclusion, ε satisfies the (P S)-condition. □

Proof (Proof of Theorem 3.3) By Lemmas 3.7 and 3.9, functional ε satisfies the assumptions of Mountain-Pass theorem ([3]). Hence, there exists a nontrivial critical point of functional ε which is a solution of problem (1). □

4 Examples

In this section, we present a concrete problem that our results can be applied successfully. Suppose that

$$\begin{cases} A(t) = 1, \\ a(t) = |t|^{p-2}, \\ \beta(x) = 1, \\ \alpha(x) = \alpha = \text{ const } > 0, \\ f(x,t) = |t|^{\gamma-2}t, \gamma \in R, \gamma > 1, \\ p = q = 2. \end{cases} \tag{31}$$

Then equation (1) turns into the Ginzburg-Landau-type (GL) equation

$$\begin{cases} -\nabla^2\psi + \alpha\left(\frac{|\psi|^2}{2} - 1\right)\psi = |\psi|^{\gamma-2}\psi, \Omega, \\ \psi = 0, \partial\Omega, \end{cases} \tag{32}$$

where ψ denotes the macrowave function describing a superconducting state and $|\psi|^2$ is the density of superconducting electrons. This equation was originally proposed in [24]. In the field of superconductivity the GL equation has been playing an important role for the understanding of macroscopic superconducting phenomena. We refer the reader to [7,8,14,28,32,33,35,36,41] and the references there in for detailed background regarding the GL equations.

The energy functional, called the GL energy, corresponding to problem (32) is defined as $\varepsilon_* : W_0^{1,2} \to R$

$$\varepsilon_*(\psi) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx + \int_\Omega \left(\frac{|\psi^2}{2} - 1\right)^2 dx - \frac{1}{\gamma}\int_\Omega |\psi|^\gamma dx,$$

and the critical points of functional ε_* lie in

$$W_0^{1,2}(\Omega) = \left\{\psi \in W_0^{1,2}(\Omega) : \psi = 0, \partial\Omega\right\},$$

with an equivalent norm to $\|.\|_{W^{1,2}}$ (due to the Poincar'e inequality)

$$\|\psi\|_{W_0^{1,2}} = \|\psi\|_W = \left(\int_\Omega |\nabla\psi|^2 dx\right)^{1/2},$$

where

$$W^{1,2}(\Omega) = \left\{\psi \in L^2(\Omega) : \frac{\partial\psi}{\partial x_i} \in L^2(\Omega), i = 1,2,\ldots\ldots N\right\}.$$

Theorem 4.1 Assume that $4 < \gamma < 2^* = \frac{2N}{N-2}$ and $0 < \alpha < 1$ hold. Then problem (32) has a nontrivial solution in $W_0^{1,2}(\Omega)$.

Proof Since all functions appearing in problem (1) satisfy (31) here, all the necessary conditions (A0) and (f1)-(f4) hold for the problem (32). Therefore, it can be easily shown that Lemma 3.7 and Lemma 3.9 hold for the problem (32) as well. So we skipped the details. Having said that, we would like to give a concise outline of the proof of Lemma 3.9 since it requires a slightly different approach. To this end, from Lemma 3.7 there is a Palais-Smale sequence $(u_n) \subset W_0^{1,2}(\Omega)$ of ε_* such that

$$\varepsilon_*(u_n) \to c \text{ and } \|\varepsilon_*(u_n)\|_{\left(W_0^{1,2}(\Omega)\right)^*} \to 0. \tag{33}$$

Again assume by contradiction that $\|u_n\|_W > 1$. Using the fact that by (33) there exist positive real numbers C and κ with $4 < \kappa < \gamma$ along with the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, it follows

$$
\begin{aligned}
C &\geq \varepsilon_*(u_n) - \frac{1}{\kappa}\left\langle \varepsilon_*'(u_n), u_n \right\rangle, \\
&\geq \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{\alpha}{2}\int_\Omega \left(\frac{|u_n|^2}{2} - 1\right)^2 dx - \frac{1}{\gamma}\int_\Omega |u_n|^\gamma dx, \\
&-\frac{1}{\kappa}\int_\Omega |\nabla u_n|^2 dx - \frac{\alpha}{\kappa}\int_\Omega \left(\frac{|u_n|^2}{2} - 1\right) dx + \frac{1}{\kappa}\int_\Omega |u_n|^\gamma dx, \\
&\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right)\|u_n\|_w^2 + \frac{\alpha}{2}\int_\Omega \left(\frac{|u_n|^4}{4} - |u_n|^2 + 1\right) dx, \\
&-\frac{\alpha}{\kappa}\int_\Omega \left(\frac{|u_n|^4}{2} - |u_n|^2\right) dx + \left(\frac{1}{\theta} - \frac{1}{\gamma}\right)\int_\Omega |u_n|^\gamma dx, \\
&\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right)\|u_n\|_w^2 + \left(\frac{\alpha}{8} - \frac{\alpha}{2\kappa}\right)\int_\Omega |u_n|^4 dx, \\
&+\left(\frac{1}{2} - \frac{1}{\kappa}\right)\int_\Omega |u_n|^2 dx + \left(\frac{1}{\kappa} - \frac{1}{\gamma}\right)\int_\Omega |u_n|^\gamma dx, \\
&\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right)\|u_n\|_w^2 + \left(\frac{\alpha}{\kappa} - \frac{\alpha}{2}\right)\|u_n\|_w^2, \\
&\geq (1-\alpha)\left(\frac{1}{2} - \frac{1}{\kappa}\right)\|u_n\|_w^2.
\end{aligned}
$$

Since $\kappa > 4$ and $0 < \alpha < 1$, (u_n) is bounded in $W_0^{1,2}(\Omega)$. Considering that $W_0^{1,2}(\Omega)$ is reflexive, there exists a subsequence (u_n) (not relabelled) which converges weakly to $u \in W_0^{1,2}(\Omega)$. Then, using (33), it reads

$$
\left\langle \varepsilon_*'(u_n), u_n - u \right\rangle = \int_\Omega \nabla u_n (\nabla u_n - \nabla u)\, dx + \int_\Omega \left(\frac{|u_n|^2}{2} - 1\right) u_n (u_n - u)\, dx - \int_\Omega |u_n|^{\gamma-2} u_n (u_n - u)\, dx \to .
$$

By the Hölder inequality and the compact embeddings $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and $W_0^{1,2}(\Omega) \hookrightarrow L^\gamma(\Omega)$, it is easy to see that

$$
\left|\int_\Omega |u_n|^{\gamma-2} u_n (u_n - u)\, dx\right| \to 0, \tag{34}
$$

and

$$
\left|\int_\Omega \left(\frac{|u_n|^2}{2} - 1\right) u_n (u_n - u)\, dx\right| \to 0, \tag{35}
$$

hold as in (29) and (30), respectively. Hence, from (34) and (35), we have

$$
\int_\Omega \nabla u_n (\nabla u_n - \nabla u)\, dx \to 0. \tag{36}
$$

On the other hand, the operator $L : W_0^{1,2}(\Omega) \hookrightarrow \left(W_0^{1,2}(\Omega)\right)^*$ defined by

$$
\langle L(v) w \rangle = \int_\Omega |\nabla v|^{p-2} \nabla v \nabla w dx, \;\; p > 1, \tag{37}
$$

is of type S+ (see Theorem 3.1, [19]). So, if we put $p = 2, v = u_n$ and $w = u_n - u$ in (37) we obtain $\Omega \int_\Omega \nabla u_n(\nabla u_n - \nabla u) dx$, which means it is of type S+. Therefore, by (36), $u_n \to u$ in $W_0^{1,2}(\Omega)$, and hence, ε_* satisfies the (P S)-condition. Overall, problem (32) has a nontrivial solution in $W_0^{1,2}(\Omega)$. □

5 Conclusion

In this paper, to obtain a nontrivial solution for a nonlocal problem with Dirichlet boundary condition including a nonlocal and nonhomogeneous differential operator. To do so, some well-known results from functional analysis and the theory of Orlicz-Sobolev spaces are applied. Additionally, a concrete example, where it is shown that a Ginzburg-Landau-type equation has a nontrivial solution, is provided that our main results can be applied successfully.

References

[1] Adams, R.A., Sobolev spaces, Academic Press, New York, (1975).
[2] Alves, C.O., Ferreira, M. C., Existence of solutions for a class of p(x)-Laplacian equations involving a concave- convex nonlinearity with critical growth in RN, Topological Methods in Nonlinear Analysis, 45 (2),399-422, (2015).
[3] Ambrosetti, A., Rabinowitz, P. H., Dual variational methods in critical point theory, J. Funct. Anal., 14,349-381,(1973).
[4] Avci, M., Cekic, B. and Mashiyev, R.A., Existence and multiplicity of the solutions of the p(x)-Kirchhoff type equation via genus theory, Math. Meth. Appl. Sci., 34 (14), 1751-1759,(2011).
[5] Avci, M., Pankov, A., Multivalued Elliptic Operators with Nonstandard Growth, Advances in Nonlinear Analysis,7 (1), 35-48,(2018).
[6] Ayazoglu (Mashiyev), R., Avci, M., N.T. Chung, Existence of Solutions for Nonlocal Problems in Sobolev- Orlicz Spaces via Monotone Method, Electronic Journal of Mathematical Analysis and Appl., 4 (1), 63-73,(2016).
[7] Bethuel, F., Brezis, H. and Hlein, F., Ginzburg-Landau,Vortices, Birkhuser, 1994.
[8] Bethuel, F., Brezis, H. and Hlein, F., Asymptotics for the minimization of a Ginzburg-Landau functional, Calculus of Variations and PDE, 1,123-148,(1993).
[9] Bonanno, G. , Molica Bisci, G.and Radulescu, V. Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces, Nonlinear Anal. TMA,75, 4441-4456,(2012).
[10] Boureanu, M.M., Udrea, D.N., Existence and multiplicity results for elliptic problems with p(.)-Growth conditions, Nonlinear Analysis: Real World Applications, 14 (4), 1829-1844, (2013).
[11] Chipot, M., Elliptic Equations: An Introductory Course, Birkhuser Verlag AG, Basel (2009).
[12] Colasuonno, F., Pucci, P., Multiplicity of solutions for p(x)-polyharmonic Kirchhoff equations, Nonlinear Anal., 74, 5962-5974, (2011).
[13] CorrÂ´ea, F.J.S.A., Figueiredo, G.M., On a p-Kirchhoff equation via Krasnoselskiis genus, Appl. Math. Letters, 22, 819-822, (2009).
[14] Comte, M., Mironescu, P., Minimizing properties of arbitrary solutions to the GinzburgLandau equation, Proceedings of the Royal Society of Edinburgh Section A Mathematics,129 (6), 1157-1169, (1999).
[15] Cruz-Uribe, D.V., Fiorenza, A., Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Springer, Basel, (2013).
[16] Diening, L., Harjulehto, P., Hasto, P. and Ruzicka, M., Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg, (2011).
[17] Fan, X.L., Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method, J. Math. Anal. Appl., 386, 593-604, (2012).
[18] Fan, X.L., On nonlocal p(x)-Laplacian Dirichlet problems, Nonlinear Anal., 72, 3314-3323, (2010).
[19] Fan, X.L., Zhang, Q.H., Existence of solutions for p(x)-Laplacian Dirichlet problem, Nonlinear Anal., 52, 1843-1852, (2003).
[20] Fan, X.L., Zhao, D., On the spaces $Lp(x)(\Omega)andWm,p(x)(\Omega)$, J. Math. Anal. Appl., 263, 424-446, (2001).
[21] Fang, F., Tan, Z., Existence and Multiplicity of solutions for a class of quasilinear elliptic equations: An Orlicz-Sobolev setting, J. Math. Anal. Appl., 389 420-428, (2012).
[22] Fukagai, N., Ito, M. and Narukawa, K., Positive solutions of quasilinear elliptic equations with critical Orlicz- Sobolev nonlinearity on RN, Funkcial. Ekvac. 49, 235-267, (2006).
[23] Galewski, M., A new variational method for the p(x)-Laplacian equation, Bull. Australian Math. Soc., 72, (1), 53-65, (2005).
[24] Ginzburg, V., Landau, L., On the theory of superconductivity, Zh.exper.teor.Fiz. 20,1064-1082, (1950).
[25] Harjulehto, P., Hasto, P. and KlÂ´en, R., Generalized Orlicz spaces and related PDE, Nonlinear Analysis: Theory, Methods and Appl., 143, 155173, (2016).
[26] Hudzik, H., On generalized Orlicz-Sobolev space, Funct. Approx. Comment. Math., 4, 37-51, (1976).
[27] Heidarkhani, S., Caristi, G. and Ferrara, M., Perturbed Kirchhoff-type Neumann problems in Orlicz-Sobolev spaces, Comput. Math. Appl., 71, 2008-2019, (2016).
[28] Jimbo, S., Morito, Y., Notes on the limit equation of vortex motion for the Ginzburg-Landau equation with Neumann condition, Japan J. Indust. Appl. Math., 18, 483-501, (2001).
[29] Kirchhoff, G., Mechanik, Teubner, Leipzig, (1883).
[30] Krasnosels'kii, M., and Rutic'kii, J., Convex Functions and Orlicz Spaces, Noordhoff, Groningen, (1961).
[31] Kufner, A., John, O. and Fucik, S., Function Spaces, Noordhoff, Leyden, (1977).
[32] Lefter, C. and Radulescu, V.D. On the Ginzburg-Landau energy with weight, Annales de l'Institut Henri Poincar C, Analyse Non Linaire, 13, (2), 171-184, (1996).

[33] Lu, K. and Pan, X.B., Ginzburg-Landau equation with DeGennes boundary condition, J. Differential Equations, 129, 136-165, (1996).
[34] Mihailescu, M. and Radulescu, V., Neumann problems associated to non-homogeneous differential operators in Orlicz-Sobolev spaces, Ann. Inst. Fourier, 58, 2087-2111,(2008).
[35] Mironescu, P., On the stability of radial solutions of the Ginzburg-Landau equation, J.Func. Anlaysis, 130, 334-344, (1995).
[36] Morito, Y., Stable solutions to the Ginzburg-Landau equation with magnetic effect in a thin domain, Japan J. Indust. Appl., 21, 129-147, (2001).
[37] Musielak, J., Modular spaces and Orlicz spaces, Lecture Notes in Math, vol.1034, Springer-Verlag, Berlin, (1983).
[38] Rabinowitz, P.H., Minimax Methods in Critical Point Theory with Applications to Differential Equations, Amer. Math. Soc., Providence, R.I., (1986).
[39] Rao, M.M. and Ren, Z.D., Theory of Orlicz Spaces, Marcel Dekker Inc., New York, (1991).
[40] Radulescu, V.D. and Repovs, D.D., Partial Differential Equations with Variable Equations: Variational Methods and Qualitative Analysis, CRC press, New York., (2015)
[41] Struwe, M., Une estimation asymptotique pour le modle de Ginzburg-Landau, C. R. Acad. Sc. Paris., 317, 677-680, (1993).
[42] Yucedag, Z., Existence of Solutions for p(x)-Laplacian Equations Without Ambrosetti-Rabinowitz Type Condition, Bull. Malaysian Math. Sci. Soc., 38, (3), 1023-1033, (2015).