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Antireduction Method for the Exact Solutions of the Porous Media Equation

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Abstract: By the antireduction method we obtain the exact solutions of porous media equation, a nonlinear heat equation that usually arises in mathematical biology, nonlinear problems of mass transfer and combustion theory. This method requires construction of a number of ansatzes that reduce nonlinear heat equations to a system of ordinary differential equations. Solving this system yields the exact solution of the given nonlinear heat equation. Our theoretical results are compared with some other results that appear in the research literature. In addition, numerical results are provided for various values of the porosity constant.

Keywords: Antireduction method, Exact solution, Porous Media equation, Porosity.

1 Introduction

In this paper we obtain the exact solutions of the nonlinear diffusion equation called the porous media equation [9,12,14]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right). \tag{1}$$

Here $u = u(x,t)$ and m is a non-zero real number which is called the porosity constant.

The porous media equation is one of the simplest examples of nonlinear evolution equation of parabolic type. It owes its name to its use in describing the flow of an ideal gas in an homogeneous porous medium [14]. It describes unsteady heat transfer in a quiescent medium with the heat diffusivity being a power-law function of temperature. Eq. (1) has also applications to many physical systems including the fluid dynamics of thin films. Murray [8] describes how this model has been used to represent "population pressure" in biological systems.

Obtaining the exact solutions for nonlinear equations of the form Eq. (1) that have "fundamental importance" plays an important role in understanding the non-linear phenomena.

Recently some methods such as Lie symmetry reduction method [4] and Adomian decomposition method (ADM) [1,3, 6,7,10,11] have been introduced in the research literature to obtain exact solutions of nonlinear partial differential equations (PDE). But, finding an exact solution with these methods is generally tedious or requires additional assumptions.

In the following section the antireduction method [2,5,13] is carried out to get the exact solutions of Eq. (1). This

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method requires construction of a number of ansatzes that reduce Eq. (1) to a system of ordinary differential equations (ODE). By solving this system, one obtains the exact solution to it.

2 The Antireduction Method

The antireduction of Eq. (1) is carried out by means of ansatz

$$f(t,x,u,\phi_1(v),\phi_2(v),...,\phi_n(v)) = 0, \tag{2}$$

where $v = v(t,x,u)$ is a new independent variable and f is an arbitrary function. Ansatz given by Eq. (2) reduces Eq. (1) to a system of ODE for the unknown functions $\phi_i(v), i = 1,2,...,n$.

2.1 Approximation by Two-term Ansatz

In this subsection we look for a non-zero solution of Eq. (1) in the form of

$$u^m = \phi_1(t) + \phi_2(t)x, \tag{3}$$

where $\phi_1(t)$ and $\phi_2(t)$ are to be determined. The partial derivatives of u with respect to t and x are computed as follows

$$u_t = \frac{\phi_1'(t) + \phi_2'(t)x}{mu^{m-1}}, \quad u_x = \frac{\phi_2(t)}{mu^{m-1}}, \tag{4}$$

respectively. Writing these derivatives in Eq. (1) we get

$$\frac{\phi_1' + \phi_2' x}{mu^{m-1}} = \left(u^m \frac{\phi_2}{mu^{m-1}}\right)_x = \frac{1}{m}(u\phi_2)_x = \frac{1}{m^2}\frac{\phi_2^2}{u^{m-1}}.$$

Multiplying both sides by mu^{m-1} and equating coefficients of each power of x on both sides, we obtain the following system of ODE's

$$\phi_1' = \frac{1}{m}\phi_2^2, \quad \phi_2' = 0, \tag{5}$$

whose general solution are

$$\phi_1 = \frac{1}{m}c_2^2 t + c_1, \quad \phi_2 = c_2. \tag{6}$$

Here c_1 and c_2 are some arbitrary constants. Therefore, the desired non-zero exact solution of Eq. (1) is

$$u(x,t) = \left(\frac{1}{m}c_2^2 t + c_1 + c_2 x\right)^{1/m}. \tag{7}$$

2.2 Approximation by Three-term Ansatz

Here we seek a non-zero solution of Eq. (1) in the form of

$$u^m = \phi_1(t) + \phi_2(t)x + \phi_3(t)x^2, \tag{8}$$

where $\phi_1(t), \phi_2(t)$ and $\phi_3(t)$ are unknown functions. If we compute the partial derivatives of u with respect to t and x, respectively, we obtain

$$u_t = \frac{\phi_1'(t)+\phi_2'(t)x+\phi_3'(t)x^2}{mu^{m-1}}, \quad u_x = \frac{\phi_2(t)+2\phi_3(t)x}{mu^{m-1}}. \tag{9}$$

Plugging these derivatives in Eq. (1) one obtains

$$\frac{\phi_1'+\phi_2'x+\phi_3'x^2}{mu^{m-1}} = \left(u^m\frac{\phi_2+2\phi_3 x}{mu^{m-1}}\right)_x = \frac{1}{m}(u(\phi_2+2\phi_3 x))_x = \frac{u_x(\phi_2+2\phi_3 x)+2\phi_3 u}{m}.$$

If we multiply both sides by mu^{m-1} we get

$$\phi_1'+\phi_2'x+\phi_3'x^2 = u^{m-1}u_x(\phi_2+2\phi_3 x)+2u^m\phi_3 = \frac{1}{m}(\phi_2+2\phi_3 x)^2+2(\phi_1+\phi_2 x+\phi_3 x^2)\phi_3.$$

Equating coefficients of each power of x on both sides, one obtains the following non-linear system of ODE's

$$\begin{aligned}\phi_1' &= \frac{1}{m}\phi_2^2+2\phi_1\phi_3,\\ \phi_2' &= (2+\frac{4}{m})\phi_2\phi_3,\\ \phi_3' &= (2+\frac{4}{m})\phi_3^2.\end{aligned} \tag{10}$$

If $m \neq -2$, the general solution of the last differential equation of the system (10) is easy to obtain

$$\phi_3 = -\frac{1}{c_3+(2+\frac{4}{m})t}.$$

Writing this ϕ_3 in the second differential equation of the system (10) and solving it yields

$$\phi_2 = \frac{c_2}{c_3+(2+\frac{4}{m})t}.$$

Finally, putting these ϕ_2 and ϕ_3 in the first differential equation of the system (10) and solving it we obtain

$$\phi_1 = \frac{c_1}{\left(c_3+(2+\frac{4}{m})t\right)^{\frac{2m}{2m+4}}} - \frac{c_2^2}{4\left(c_3+(2+\frac{4}{m})t\right)}.$$

Here we have assumed that c_1, c_2 and c_3 are some arbitrary constants. Hence, the non-zero exact solution of Eq. (1) that we look for is

$$u(x,t) = \left[\frac{c_1}{\left(c_3+(2+\frac{4}{m})t\right)^{\frac{2m}{2m+4}}} + \frac{4c_2x-4x^2-c_2^2}{4\left(c_3+(2+\frac{4}{m})t\right)}\right]^{1/m}. \tag{11}$$

Also, if $m=-2$, solving the system (10) gives the exact solution of Eq. (1) as follows

$$u(x,t) = \left[\frac{c_2^2}{4c_3}+c_1e^{2c_3t}+c_2x+c_3x^2\right]^{-1/2}, \tag{12}$$

which is obtained in [12] as a functional seperable solution.

3 Applications

Example 1. We solve the following initial value problem(IVP)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right), \ \ u(x,0) = u_0(x) = x. \tag{13}$$

By letting $c_1 = 0$ and $c_2 = 1$ in Eq. (7) we readily get the two-term exact solution of this (IVP)

$$u(x,t) = t + x,$$

which is exactly the same as what we obtained in [9] by ADM.

Example 2. We now find the three-term exact solution of the following initial value problem(IVP)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right), \ \ u(x,0) = u_0(x) = x - x^2/8. \tag{14}$$

To do this, we let $c_1 = 4, c_2 = c_3 = 8$ in Eq. (11) and get the solution

$$u(x,t) = \frac{4}{(8+6t)^{1/3}} - \frac{16}{8+6t} + \frac{8}{8+6t}x - \frac{1}{8+6t}x^2.$$

In Fig. 1 we compare these two solutions for $0 \le x \le 0.3$ and $0 \le t \le 0.05$. For these values of x and t, the two-term and the three-term solutions seem close to each other.

Example 3. In this example we find the two-term solution of the IVP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial u}{\partial x}\right), \ \ u(x,0) = u_0(x) = \frac{1}{x}, \tag{15}$$

by plugging $c_1 = 0$ and $c_2 = 1$ in Eq. (7). And, the solution becomes

$$u(x,t) = \frac{1}{x-t},$$

which is exactly the same solution obtained in [9] by ADM.

Example 4. Here we obtain the three-term solution of the IVP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial u}{\partial x}\right), \ \ u(x,0) = u_0(x) = \frac{1}{0.001 + x + 0.002x^2}. \tag{16}$$

We do this by letting $c_1 = \frac{124999}{500000}$ and $c_2 = c_3 = -500$ in Eq. (11) and get the solution

$$u(x,t) = \left[\frac{-124999(500+2t)}{500000} + \frac{62500+500x+x^2}{500+2t}\right]^{-1}.$$

For numerical purposes, these two solutions are compared for $0.15 \le x \le 0.3$ and $0 \le t \le 0.14$ in Fig. 2. It seems that the two-term and the three-term solutions get close to each other for these choices of x and t.

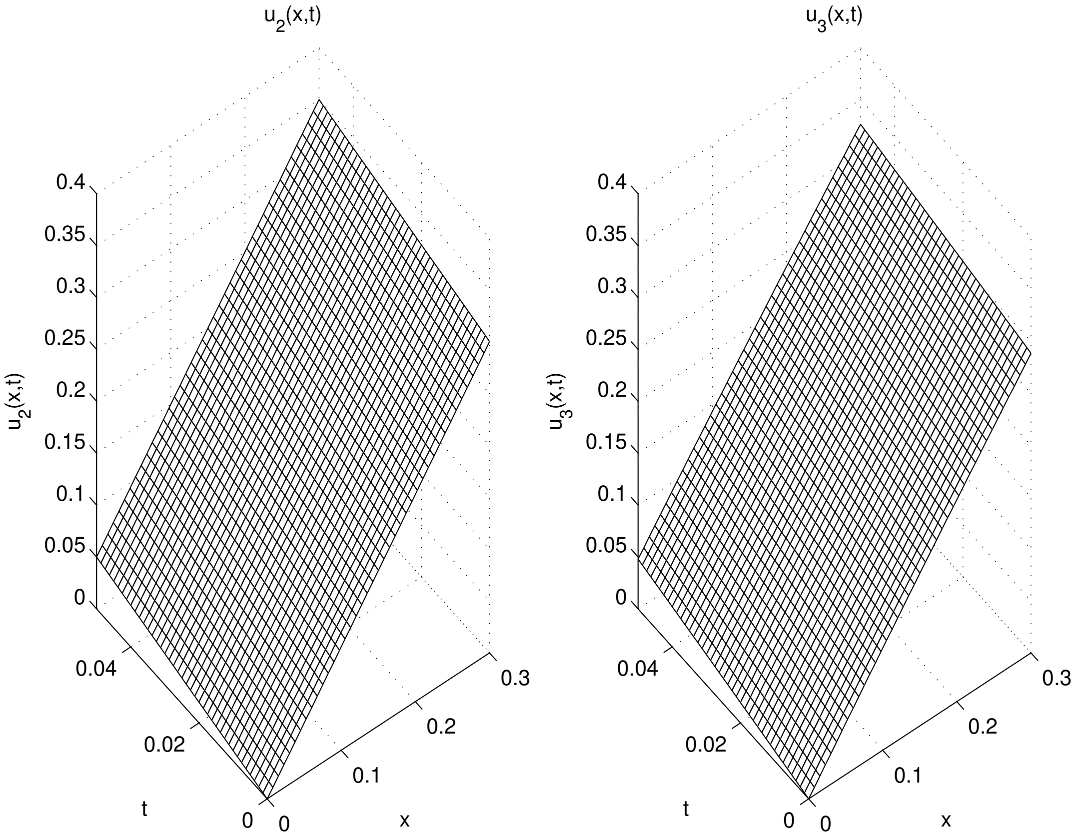

Fig. 1: Comparison of the two-term and the three-term solutions to Eq. (1) with $m = 1$.

Example 5. In this example we study the IVP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right), \;\; u(x,0) = u_0(x) = x^{-1/2}. \tag{17}$$

If we take $c_1 = 0$ and $c_2 = 1$ in Eq. (7), we get the two-term solution

$$u(x,t) = (x - \frac{t}{2})^{-1/2}.$$

Example 6. In this example we study the IVP

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right), \;\; u(x,0) = u_0(x) = \left(\frac{93}{3140} + x + 3.14x^2\right)^{-1/2}. \tag{18}$$

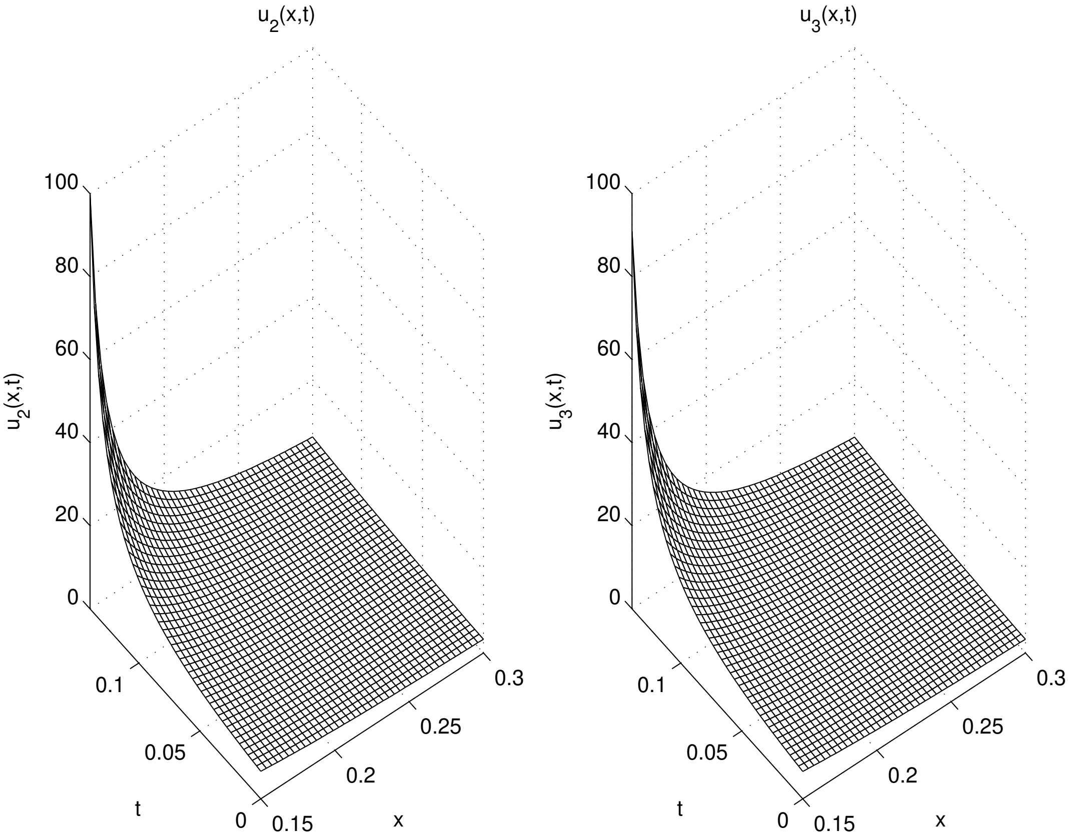

Fig. 2: Comparison of the two-term and the three-term solutions to Eq. (1) with $m = -1$.

If we take $c_1 = -0.05$, $c_2 = 1$ and $c_3 = 3.14$ in Eq. (12), we get the three-term solution

$$u(x,t) = \left(\frac{25}{314} - \frac{1}{20}e^{6.28t} + x + 3.14x^2\right)^{-1/2}.$$

As a numerical comparison of the two-term and the three-term solutions, we provide Fig. 3 for $0.15 \leq x \leq 0.3$ and $0 \leq t \leq 0.28$. It is clear that the two pictures look almost identical.

Notice that the initial conditions (IC) in Example 1 and Example 2 were chosen close to each other for the x and t values from which the comparison made between the two solutions. The reason we made this is to see how the two-term and the three-term solutions get close to each other for these IC. Similar choices were made for IC in Examples 3-4 and in Examples 5-6, as well.

4 Conclusion

In this work we have obtained the two-term and the three-term exact solutions of the porous media equation by the antireduction method. Then, we have compared these solutions numerically and seen that they get close to each other for some specific values of x and t in the domain of u.

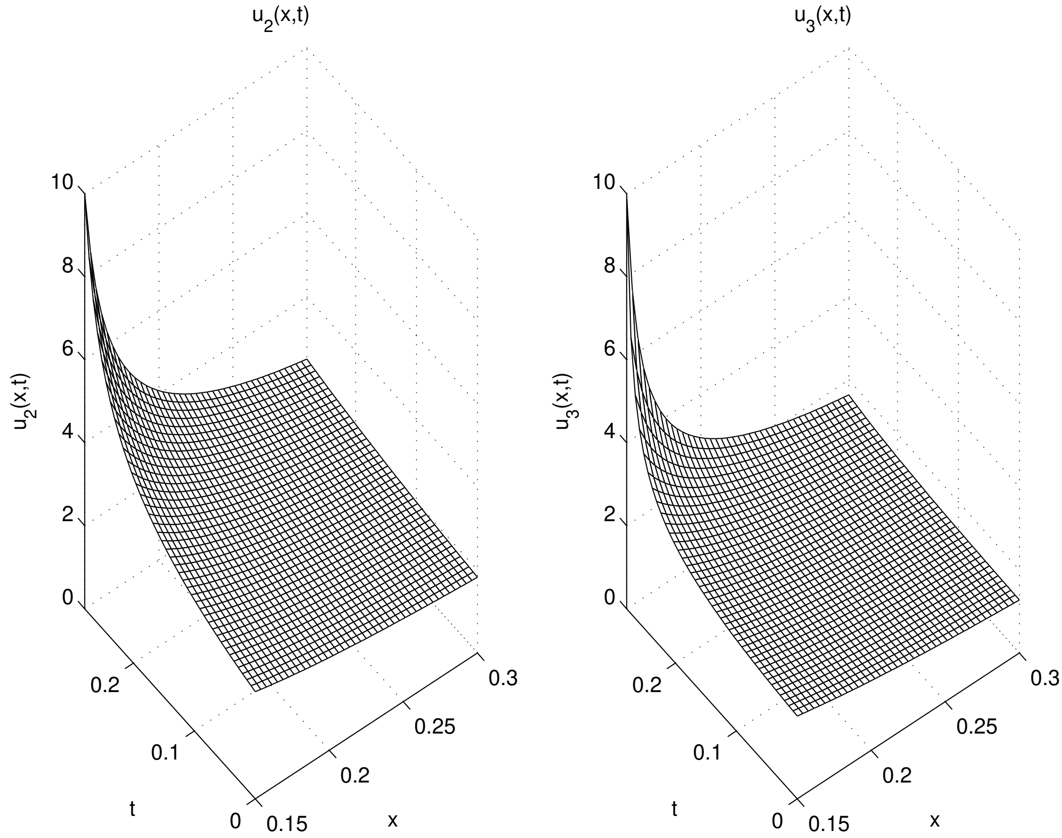

Fig. 3: Comparison of the two-term and the three-term solutions to Eq. (1) with $m = -2$

This method provides an easy computation of exact solutions for nonlinear problems compared to other methods. For example, Lie symmetry reduction method is known as the most effective and universal method used in obtaining the exact solutions of nonlinear PDE. But there exist very important equations that admit poor Lie symmetries [13]. Also, in ADM one has to compute the Adomian Polynomials, which are kind of tedious. This method also avoids linearization of the PDE and physically unrealistic assumptions.

As it is known the porous media equation appears in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t = \Delta u$, its most famous relative. Hence, both the pure and the applied mathematicians will get benefit of our results.

One, of course, may add a nonlinear function of u to the right hand side of Eq. (1) for a “nice” inhomogenous nonlinear problem, which may be solved by this method. Some solutions to such an inhomogenous problem by other methods may be found in [12].

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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