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# A note for the Ulam-Hyers-Rassias stability of differential equations on bounded intervals

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**Abstract:** In this paper, we considered the stability problem of non-linear differential equations in the sense of Ulam-Hyers and Ulam-Hyers-Rassias on bounded intervals. We show that some widely used restrictions commonly assumed in similar problems are not necessary. Therefore we extend and improve some well-known results by dropping some of their assumptions.

**Keywords:** Differential equations; Stability theory; Ulam-Hyers-Rassias Stability; Fixed point theory; Generalized metric spaces.

## 1 Introduction

In 1940, S.M. Ulam [1] posed the following problem in a talk given at Wisconsin University: "Under what conditions does there exists a homomorphism near an approximately homomorphism of a complete metric group?, i.e. Given a metric group (G,d), a number $\varepsilon > 0$ and a mapping $f : G \to G$ satisfying the inequality

$$d(f(tu), f(t)f(u)) < \varepsilon$$

for all $t,u \in G$, does there exist a homomorphism $g$ of $G$ and a constant $K$, depending only on $G$, such that

$$d(f(t), g(t)) < K\varepsilon$$

for all $t,u \in G$?" And if there is such homomorphism, then the equation $g(tu) = g(t)g(u)$ of the homomorphism is called stable.

In 1941, Hyers [2] answered this problem: Let $B_1$ and $B_2$ be real Banach spaces and $\varepsilon > 0$. Then for each $f : B_1 \to B_2$ satisfying the inequality

$$\|f(t+u) - f(t) - f(u)\| \leq \varepsilon$$

for all $t,u \in B_1$, there exists a unique $g : B_1 \to B_2$ such that

$$\|f(t) - g(t)\| \leq \varepsilon$$

holds for all $t \in B_1$.

In 1978, Rassias [3] introduced a remarkable generalization of this problem considering the constant $\varepsilon$ as a variable function, this type problems known as Ulam-Hyers-Rassias stability problems in the literature.

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First paper on the Ulam-Hyers type stability of the differential equations is given by Obloza [4,?]. After that Alsina and Ger [6] proved that every differentiable mapping $y : I \to \mathbb{R}$, where $I$ is an open interval of real numbers, satisfying $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$ and a given $\varepsilon > 0$, there exists a solution $t_0$ of the differential equation $y'(t) = y(t)$ such that $|y(t) - y_0(t)| \leq 3\varepsilon$ for all $t \in I$. This result was later extended by Takahasi et. al. to the equation $y(t) = \lambda y(t)$ in Banach spaces, and to higher-order linear differential equations with constant coefficients, see [7,?,?]. Following these pioneering works, many authors studied this subject, see [10,11,12,13,14] and references therein.

In 2010, S.M. Jung [10] proved Ulam-Hyers and Ulam-Hyers-Rassias stability of the equation

$$y'(t) = f(t, y),$$

using the fixed point technique. This result is important since it extends the previous results to the non-linear case. Bojor [15] modified this technique to investigate the stability of the linear equation

$$y'(t) + f(t)y(t) = g(t).$$

The fixed point technique proposed by Jung is used widely used to analyze the Ulam-Hyers and Ulam-Hyers-Rassias stability of the differential equations in different types.

Başcı et.al. [11] investigated the Ulam-Hyers and Ulam-Hyers-Rassias stability of the equation

$$y'(t) = f(t, y), \tag{1}$$

using Jung's method with a modification on the metric. Contrary to Jung, they used the generalized metric defined as

$$d(f,g) := \inf\left\{C \in [0,\infty] : |f(t) - g(t)|\, e^{-M(t-t_0)} \leq C\Phi(t),\ t \in I\right\}.$$

Here, we will show the condition

$$\left| \int_{t_0}^{t} \varphi(s)ds \right| \leq K\varphi(t),$$

used by Başcı et.al. to prove the Ulam-Hyers-Rassias stability of the (1) is unnecessary, and then Ulam-Hyers-Rassias stability implies Ulam-Hyers stability the on the bounded intervals.

This paper is organized as follows: In Section 2, we gave the definitions of Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized metric. And, we stated the fixed point theorem that will be used in the sequel. In Section 3, we stated and proved our main result and noted some remarks about it. We devoted last section to the evaluation of the article.

## 2 Preliminaries

Let $I$ be an open interval. For every $\varepsilon \geq 0$ and $y \in \mathscr{C}^1(I)$ satisfying

$$|y(t) - f(t, y(t))| \leq \varepsilon,$$

if there exists a solution $y_0$ of the Eq. (1) such that

$$|y(t)-y_0(r)| \leq K\varepsilon,$$

where $K$ is a constant which does not depend on $\varepsilon$ and $y$, then the differential equation (1) is said to be stable in the sense of Ulam-Hyers. If the above statement remains true after replacing the constants $\varepsilon$ and $K$ with the functions $\phi, \Phi : I \to [0,\infty)$ respectively, where these functions do not depend on $y$ and $y_0$, then the differential equation (1) is said to be stable in the sense of Ulam-Hyers-Rassias. This definition may be applied to different classes of differential equations, we refer to Jung [10] and references cited therein for more detailed definitions of Ulam-Hyers stability and Ulam-Hyers-Rassias stability.

Now we introduce the generalized metric which will be used in the sequel. For a non-empty set $X$, a function $d : X \times X \to [0,\infty]$ is called a generalized metric on $X$ if and only if satisfies,

M(1) $d(t,u) = 0$ if and only if $t = u$,
M(2) $d(t,u) = d(u,t)$ for all $t,u \in X$,
M(3) $d(t,w) \leq d(t,u) + d(u,w)$ for all $t,u,w \in X$.

We will use the following fixed point theorem in the proof of our main theorem.

**Theorem 1.** *[16] Let (X, d) is a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there is a non-negative integer m such that $d\left(T^{m+1}x, T^m x\right) < \infty$ for some $x \in X$, then the following are true:*

(a) *The sequence $\{T^m x\}$ converges to a fixed point $\bar{x}$ of $T$,*
(b) *$\bar{x}$ is the unique fixed point of $T$ in*

$$\overline{X} = \{y \in X : d(T^m x, y) < \infty\},$$

(c) *If $y \in \overline{X}$, then*

$$d(y,\bar{x}) \leq \frac{1}{1-L} d(Ty,y).$$

## 3 Main results

In this section, we define the interval $I$ as $I := [t_0, t_0 + r], t_0, r \in \mathbb{R}$ and $r > 0$. And we define the set of all continuous functions defined on $I$ by

$$X := \{f : I \to \mathbb{R};\ f \text{ is continuous}\} = \mathscr{C}(I,\mathbb{R}). \tag{2}$$

**Lemma 1.** *$(X,d)$ is a generalized complete metric space with the metric*

$$d(f,g) := \inf\left\{c \in [0,\infty];\ |f(t)-g(t)|\, e^{M(t-t_0)} \leq c\Phi(t),\ t \in I\right\}, \tag{3}$$

*where $M > 0$ is a constant and $\Phi : I \to (0,\infty)$ is a continuous.*

For the proof of this lemma, we refer to [11].

Our next result is about the Ulam-Hyers-Rassias stability of the Eq. (1).

**Theorem 2.** *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*

$$|f(t,g) - f(t,h)| \leq L|g-h| \tag{4}$$

*for all $t \in I$ and all $g,h \in \mathbb{R}$. For a non-decreasing continuous function $\varphi : I \to (0,\infty)$ if a continuously differentiable function $y : I \to \mathbb{R}$ satisfies*

$$\left|y'(t) - f(t,y(t))\right| \leq \varphi(t) \tag{5}$$

*for all $t \in I$, then there exists a unique solution $y_0$ of the Eq. (1) satisfying*

$$|y(t) - y_0(t)| \leq r(1+L)\varphi(t) \tag{6}$$

*for all $t \in I$.*

*Proof.*Let the set $X$ be defined by (2). Define the $d : X \times X \to [0,\infty]$ by

$$d(f,g) := \inf\left\{c \in [0,\infty]\ ;\ |f(t) - g(t)|\, e^{-(\alpha+1)(t-t_0)} \leq c\varphi(t),\ t \in I\right\}.$$

Given Lemma 1, one can conclude that $(X,d)$ is a generalized complete metric space. Define the operator $\mathscr{R} : X \to X$ by

$$\mathscr{R}y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y(\tau))d\tau,$$

$t \in I$ and $y \in X$. It is clear that fixed points of $\mathscr{R}$ are solutions of Eq. (1) . Since functions $f$ and $y$ are continuous functions, we can say that $\mathscr{R}y$ is continuous and it belongs to the set $X$. Hence, it is easy to see $d(\mathscr{R}h_0, h_0) < \infty$ for all $h_0 \in X$ and $\{h \in X\ ;\ d(h_0,h) < \infty\} = X$.

For any $h_1, h_2 \in X$, let $c_{h_1,h_2} \in [0,\infty]$ be a constant such that $d(h_1,h_2) \leq c_{h_1,h_2}$, i.e.

$$|h_1(t) - h_2(t)|\, e^{-(\alpha+1)(t-t_0)} \leq c_{h_1,h_2}\varphi(t),\ t \in I.$$

Now, we can show that operator $\mathscr{R}$ is strictly contractive. For any $h_1, h_2 \in X$ and all $t \in I$, using (3) , (4) , and monotonicity of $\varphi$, we obtain that

$$\begin{aligned}
|\mathscr{R}h_1(t) - \mathscr{R}h_2(t)| &= \left|\int_{t_0}^{t} [f(\tau,h_1(\tau)) - f(\tau,h_2(\tau))]\, d\tau\right| \\
&\leq \int_{t_0}^{t} |f(\tau,h_1(\tau)) - f(\tau,h_2(\tau))|\, d\tau \\
&\leq L\int_{t_0}^{t} |h_1(\tau) - h_2(\tau)| \\
&= L\int_{t_0}^{t} |h_1(\tau) - h_2(\tau)|\, e^{-(\alpha+1)(\tau-t_0)} e^{(\alpha+1)(\tau-t_0)} d\tau \\
&\leq Lc_{h_1,h_2}\int_{t_0}^{t} \varphi(\tau)e^{(\alpha+1)(\tau-t_0)} d\tau \\
&\leq Lc_{h_1,h_2}\varphi(t)\int_{t_0}^{t} e^{(\alpha+1)(\tau-t_0)} d\tau \\
&\leq \frac{L}{L+1} c_{h_1,h_2}\varphi(t)e^{(\alpha+1)(t-t_0)}.
\end{aligned}$$

Therefore, we have

$$|\mathscr{R}h_1(t)-\mathscr{R}h_2(t)|\,e^{-(\alpha+1)(t-t_0)}\leq\frac{L}{L+1}c_{h_1,h_2}\varphi(t),$$

that is

$$d(\mathscr{R}h_1,\mathscr{R}h_2)\leq\frac{L}{L+1}d(h_1,h_2).$$

So, operator $\mathscr{R}$ is strictly contractive on $X$, and all conditions of Theorem 1 are satisfied for $m=1$.

From (5), we have

$$-\varphi(t)\leq y'(t)-f(t,y(t))\leq\varphi(t)$$

for all $t\in I$. Integrating this inequality from $t_0$ to $t$ and using the monotonicity of the function $\varphi$, we obtain

$$\left|y'(t)-\mathscr{R}y(t)\right|\leq\int_{t_0}^{t}\varphi(\tau)\,d\tau\leq\varphi(t)\int_{t_0}^{t}d\tau\leq\varphi(t)r$$

for all $t\in I$. Multiplying this last inequality with $e^{-(\alpha+1)(t-t_0)}$, we get

$$\left|y'(t)-\mathscr{R}y(t)\right|e^{-(\alpha+1)(t-t_0)}\leq\varphi(t)re^{-(\alpha+1)(t-t_0)}$$

for all $t\in I$, which implies

$$d(\mathscr{R}y,y)\leq\varphi(t)re^{-(\alpha+1)(t-t_0)}$$

for all $t\in I$.

According to Theorem 1, there exists a unique solution $y_0:I\to\mathbb{R}$ of Eq. (1) satisfying

$$d(y,y_0)\leq\frac{1}{1-L/(L+1)}d(\mathscr{R}y,y)\leq\frac{\varphi(t)re^{-(\alpha+1)(t-t_0)}}{1-L/(L+1)}=r(1+L)\varphi(t)e^{-(\alpha+1)(t-t_0)}$$

for all $t\in I$. Hence, from 3, we obtain

$$|y(t)-y_0(t)|\,e^{-(\alpha+1)(t-t_0)}\leq r(1+L)\varphi(t)e^{-(\alpha+1)(t-t_0)}$$

for all $t\in I$. This inequality implies (6) and completes the proof.

*Remark.*Notice that we do not need the assumption

$$\left|\int_{t_0}^{t}\varphi(s)ds\right|\leq K\varphi(t)$$

in the proof of Theorem 2, which is required in [11]. This condition is required also in many papers concerning similar problems on bounded intervals, so we improve the literature with this result.

By taking $\varphi(t)=\varepsilon$ in Theorem 2, we obtain the following result on Ulam-Hyers stability of (1).

**Corollary 1.***Let $f:I\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the Lipschitz condition (4) for all $t\in I$ and all $g,h\in\mathbb{R}$. If a continuously differentiable function $y:I\to\mathbb{R}$ satisfies*

$$\left|y'(t)-f(t,y(t))\right|\leq\varepsilon \tag{7}$$

*for all $t \in I$, then there exists a unique solution $y_0$ of the Eq. (1) satisfying*

$$|y(t) - y_0(t)| \leq r(1+L)\varepsilon \tag{8}$$

*for all $t \in I$.*

## 4 Conclusion

In this study, we have considered the stability problem of the non-linear differential equation (1) in the sense of Ulam-Hyers and Ulam-Hyers-Rassias on bounded intervals. With Theorem 2, we proved the Ulam-Hyers-Rassias stability of (1) with fewer assumptions than the analogue result stated in [11]. Also we have obtained Ulam-Hyers stability of the differential equation (1) as a corollary of Theorem 2.

The results we obtained here with fewer conditions can be applied to many different equations that previous methods cannot be used.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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