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Solving Convolution Type Linear Volterra Integral Equations with Kashuri Fundo Transform

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Abstract: In this paper, Kashuri Fundo Transform is used for solving convolution type linear Volterra integral equations of the first kind and also convolution type linear Volterra integral equations of the second kind. Some applications are given to explain the procedure of solution of linear Volterra integral equations using Kashuri Fundo transform.

Keywords: Integral equations, Integral transforms, Kashuri Fundo transform, Volterra integral equations.

1 Introduction

Integral equations have been used for the solution of several problems in engineering, applied mathematics and mathematical physics. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations which have a wide range of applications and so their importance has increased in recent years. One can find relevant terminology related to integral equations in [13,17,18,19,20,21,22,23].

The linear Volterra integral equation of the second kind has the form

$$y(x) = f(x) + \lambda \int_0^x k(x,t)\, y(t)\, dt, \tag{1}$$

where $k(x,t)$ is the bivariate kernel of the equation and λ is a constant parameter. Here $f(x)$ and $k(x,t)$ are specified functions but $y(x)$ is an unknown function that needs to be determined. The linear Volterra integral equation of the first kind is given by

$$f(x) = \int_0^x k(x,t)\, y(t)\, dt. \tag{2}$$

There are numerous integral transforms to solve integral equations, the most widely used of these transforms is Laplace transform. Asiru [7] demonstrated solving convolution type integral equations by using Sumudu transform. Song and Kim [16] checked the Volterra integral equations of the second kind with an integral of the form of a convolution by using the Elzaki transform. Haarsa [9] solved Volterra integral equations of the first kind using the Elzaki transform. Abdallah and Shama [1] solved the applications of differential transform method to integral equations. Aggarwal et al. [2,3,4,5,6] applied Aboodh, Kamal, Mahgoub and Shehu transforms for solving linear Volterra integral equations. Senthil Kumar et al. [14,15] obtained the exact results of Mohand transforms for solving linear Volterra integro-differential equations and linear Volterra integral equations of the first kind. Gnanavel et al. [8] applied Tarig transform for solving linear Volterra integral equations of the first kind.

Kashuri and Fundo [10] defined a new integral transform which is called Kashuri Fundo transform. They presented some properties of this transform and so they are provided a new mathematical tool to solve linear ordinary differential equation of variable and constant coefficients with initial conditions. Kashuri, Fundo and Kreku [11] mixtured Kashuri Fundo integral transform and homotopy perturbation method to solve nonlinear partial differential equations. Kashuri,

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Fundo and Liko [12] applied Kashuri Fundo integral transform for solving some families of fractional differential equations. The aim of this paper is to give exact solutions for the convoluiton type linear Volterra integral equation of the first kind and also the second kind by using Kashuri Fundo integral transform without large computational work.

Now, we will give some necessary information about Kashuri Fundo integral transform.

We consider functions in the set F defined by:

$$F=\left\{f(t)\mid \exists M,k_1,k_2>0,\text{ such that}|f(t)|\leq Me^{\frac{|t|}{k_i^2}},\text{ if } t\in(-1)^i\times[0,\infty)\right\}. \tag{3}$$

For a given function in the set F, the constant M must be finite number k_1, k_2 may be finite or infinite. A new integral transform denoted by the operator $\mathscr{K}(.)$ is defined as [10],

$$\mathscr{K}[f(t)](v)=A(v)=\frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}}f(t)dt,\quad t\geq 0,-k_1<v<k_2. \tag{4}$$

Kashuri Fundo transforms of some functions are given the follows:

$f(x)$	$\mathscr{K}[f(x)]=A(v)$
1	v
x	v^3
x^n	$n!v^{2n+1}$
e^{ax}	$\frac{v}{1-av^2}$
$\sin(ax)$	$\frac{av^3}{1+a^2v^4}$
$\cos(ax)$	$\frac{v}{1+a^2v^4}$
$\sinh(ax)$	$\frac{av^3}{1-a^2v^4}$
$\cosh(ax)$	$\frac{v}{1-a^2v^4}$

Kashuri Fundo Transforms of Some Standart Result

Theorem 1(*Linearity property [10]*). *Let $f(x)$ and $g(x)$ be functions whose Kashuri Fundo integral transforms exit and c be a constant, then*

(i) $\mathscr{K}[(f+g)(x)]=\mathscr{K}[(f)(x)]+\mathscr{K}[(g)(x)]$
(ii) $\mathscr{K}[(cf)(x)]=c\,\mathscr{K}[f(x)]$.

Theorem 2(*Convolution Theorem [10]*). *Let $f(x)$ and $g(x)$ defined in F having Kashuri Fundo integral transforms $M(v)$ and $N(v)$, respectively. Then Kashuri Fundo integral transform of f and g is given by*

$$\mathscr{K}[f(x)*g(x)]=vM(v)N(v). \tag{5}$$

Here, the convolution of two functions $f(x)$ and $g(x)$ is defined by

$$f(x)*g(x)=\int_0^x f(x-t)g(t)dt. \tag{6}$$

Theorem 3(*Kashuri Fundo transform of* $x^nf(x)$, $n\geq 1$ [10]). *If $\mathscr{K}[f(x)]=A(v)$, then*

$$\begin{aligned}\mathscr{K}[xf(x)]&=\frac{v^3}{2}A'(v)+\frac{v^2}{2}A(v),\\ \mathscr{K}\left[x^2f(x)\right]&=\frac{v^4}{4}\left[v^2A''(v)+5vA(v)+3A(v)\right].\end{aligned} \tag{7}$$

2 Main Results

In this work, we deal with the kernel $k(x,t)$ as a convolution type kernel that can be expressed by the difference $(x-t)$. The convolution type linear Volterra integral equation of the second kind (LVIESK) with the form of

$$y(x) = f(x) + \lambda \int_0^x k(x-t)\, y(t)\, dt, \tag{8}$$

and the convolution type linear Volterra integral equation of the first kind (LVIEFK) with the form of

$$f(x) = \int_0^x k(x-t)\, y(t)\, dt. \tag{9}$$

Theorem 4. *The solution of Volterra integral equation of the first kind*

$$f(x) = \int_0^x k(x-t)\, y(t)\, dt, \tag{10}$$

is expressed by

$$y(x) = \mathscr{K}^{-1}[A(v)] = \mathscr{K}^{-1}\left[\frac{1}{v}\frac{\mathscr{K}[f(x)]}{\mathscr{K}[k(x)]}\right], \tag{11}$$

where k is the kernel and $\mathscr{K}[y(x)] = A(v)$.

Proof. Let taking the Kashuri Fundo transform to both sides of VIEFK (9), we have

$$\begin{aligned} \mathscr{K}[f(x)] &= \mathscr{K}\left[\int_0^x k(x-t)\, y(t)\, dt\right], \\ \mathscr{K}[f(x)] &= \mathscr{K}[k(x) * y(x)]. \end{aligned} \tag{12}$$

Using convolution theorem of Kashuri Fundo transform, we find

$$\begin{aligned} \mathscr{K}[f(x)] &= v\mathscr{K}[k(x)]\,\mathscr{K}[y(x)], \\ \mathscr{K}[y(x)] &= \frac{1}{v}\frac{\mathscr{K}[f(x)]}{\mathscr{K}[k(x)]}. \end{aligned} \tag{13}$$

Operating inverse Kashuri Fundo transform on both sides of (13), we get

$$y(x) = \mathscr{K}^{-1}\left[\frac{1}{v}\frac{\mathscr{K}[f(x)]}{\mathscr{K}[k(x)]}\right], \tag{14}$$

which is the required solution of (9).

Theorem 5. *The solution of Volterra integral equation of the second kind*

$$y(x) = f(x) + \lambda \int_0^x k(x-t)\, y(t)\, dt, \tag{15}$$

is expressed by

$$y(x) = \mathscr{K}^{-1}[A(v)] = \mathscr{K}^{-1}\left[\frac{\mathscr{K}[f(x)]}{1 - \lambda v \mathscr{K}[k(x)]}\right], \tag{16}$$

where k is the kernel and $\mathscr{K}[y(x)] = A(v)$.

*Proof.*Applying the Kashuri Fundo transform to both sides of VIESK (15), we have

$$\begin{aligned} \mathscr{K}[y(x)] &= \mathscr{K}\left[f(x)+\lambda\int_0^x k(x-t)\,y(t)\,dt\right], \\ \mathscr{K}[y(x)] &= \mathscr{K}[f(x)]+\lambda\mathscr{K}\left[\int_0^x k(x-t)\,y(t)\,dt\right], \\ \mathscr{K}[y(x)] &= \mathscr{K}[f(x)]+\lambda\mathscr{K}[k(x)*y(x)]. \end{aligned} \tag{17}$$

Using convolution theorem of Kashuri Fundo transform, we find

$$\begin{aligned} \mathscr{K}[y(x)] &= \mathscr{K}[f(x)]+\lambda v\mathscr{K}[k(x)]\,\mathscr{K}[y(x)], \\ \mathscr{K}[y(x)] &= \frac{\mathscr{K}[f(x)]}{1-\lambda v\mathscr{K}[k(x)]}. \end{aligned} \tag{18}$$

Operating inverse Kashuri Fundo transform on both sides of (18), we get

$$y(x) = \mathscr{K}^{-1}\left[\frac{\mathscr{K}[f(x)]}{1-\lambda v\mathscr{K}[k(x)]}\right], \tag{19}$$

which is the required solution of (15).

2.1 APPLICATIONS

In this section, some applications are given to explain the procedure of the solution of convolution type linear Volterra integral equations using Kashuri Fundo transform.

Example 1. Solve the following convolution type Volterra integral equation of the first kind

$$\sinh x = \int_0^x e^{x-t}\,y(t)\,dt, \tag{20}$$

by using Kashuri Fundo transform method.

Let us write $\mathscr{K}[y(x)] = A(v)$. Apply the Kashuri Fundo transform

$$\mathscr{K}[\sinh x] = \mathscr{K}\left[\int_0^x e^{x-t}\,y(t)\,dt\right], \tag{21}$$

$$\frac{v^3}{1-v^4} = \mathscr{K}[e^x * y(x)]. \tag{22}$$

Using convolution theorem of Kashuri Fundo transform on (22), we write

$$\begin{aligned} \frac{v^3}{1-v^4} &= v\mathscr{K}[e^x]\,\mathscr{K}[y(x)], \\ \frac{v^3}{1-v^4} &= v\frac{v}{1-v^2}A(v), \\ A(v) &= \frac{v}{1+v^2}. \end{aligned} \tag{23}$$

Hence, we find

$$\mathscr{K}[y(x)] = A(v) = \frac{v}{1+v^2}. \tag{24}$$

Operating inverse Kashuri Fundo transform, we get

$$y(x) = \mathcal{K}^{-1}\left[\frac{v}{1+v^2}\right] = e^{-x}. \tag{25}$$

Therefore, we have the answer

$$y(x) = e^{-x}. \tag{26}$$

Example 2. Solve the following convolution type Volterra integral equation of the second kind

$$y(x) = \cos x - \int_0^x (x-t)\cos(x-t)\, y(t)\, dt, \tag{27}$$

by using Kashuri Fundo transform method.

Let us write $\mathcal{K}[y(x)] = A(v)$. Let applying the Kashuri Fundo transform

$$\begin{aligned} \mathcal{K}[y(x)] =& \mathcal{K}\left[\cos x - \int_0^x (x-t)\cos(x-t)\, y(t)\, dt\right], \\ =& \mathcal{K}[\cos x] - \mathcal{K}\left[\int_0^x (x-t)\cos(x-t)\, y(t)\, dt\right], \end{aligned} \tag{28}$$

$$= \mathcal{K}[\cos x] - \mathcal{K}[x\cos(x) * y(x)]. \tag{29}$$

Applying convolution theorem of Kashuri Fundo transform in (29) , we have

$$\begin{aligned} A(v) =& \mathcal{K}[\cos x] - v\mathcal{K}[x\cos(x)]\,\mathcal{K}[y(x)], \\ =& \frac{v}{1+v^4} - v\left(\frac{v^3}{2}\frac{1-3v^4}{(1+v^4)^2} + \frac{v^2}{2}\frac{v}{1+v^4}\right)A(v), \\ =& \frac{v}{1+v^4} - v\left(\frac{-2v^7+2v^3}{2(1+v^4)^2}\right)A(v), \\ =& \frac{v}{1+v^4} - \frac{-v^8+v^4}{(1+v^4)^2}A(v). \end{aligned} \tag{30}$$

Hence, we find

$$\mathcal{K}[y(x)] = A(v) = \frac{v^5+v}{3v^4+1}. \tag{31}$$

Operating inverse Kashuri Fundo transform, we get

$$\begin{aligned} y(x) =& \mathcal{K}^{-1}\left[\frac{v^5+v}{3v^4+1}\right], \\ =& \mathcal{K}^{-1}\left[\frac{1}{3}v + \frac{2}{3}\frac{v}{3v^4+1}\right], \\ =& \frac{1}{3}\mathcal{K}^{-1}[v] + \frac{2}{3}\mathcal{K}^{-1}\left[\frac{v}{1+3v^4}\right], \\ =& \frac{1}{3} + \frac{2}{3}\cos\left(\sqrt{3}x\right). \end{aligned} \tag{32}$$

Therefore, we have the answer

$$y(x) = \frac{1}{3} + \frac{2}{3}\cos\left(\sqrt{3}x\right). \tag{33}$$

Example 3. Solve the following convolution type Volterra integral equation of the second kind

$$y(x) = e^x - \cos x - 2\int_0^x e^{x-t} y(t)\,dt, \tag{34}$$

by using Kashuri Fundo transform method.
Let us write $\mathscr{K}[y(x)] = A(v)$. Let applying the Kashuri Fundo transform

$$\begin{aligned} \mathscr{K}[y(x)] &= \mathscr{K}\left[e^x - \cos x - 2\int_0^x e^{x-t} y(t)\,dt\right], \\ &= \mathscr{K}[e^x - \cos x] - 2\mathscr{K}\left[\int_0^x e^{x-t} y(t)\,d\right], \end{aligned} \tag{35}$$

$$\mathscr{K}[y(x)] = \mathscr{K}[e^x] - \mathscr{K}[\cos x] - 2\mathscr{K}[e^x * y(x)]. \tag{36}$$

Applying convolution theorem of Kashuri Fundo transform in (36), we have

$$\begin{aligned} A(v) &= \mathscr{K}[e^x] - \mathscr{K}[\cos x] - 2v\mathscr{K}[e^x]\mathscr{K}[y(x)], \\ &= \frac{v}{1-v^2} - \frac{v}{1+v^4} - 2v\frac{v}{1-v^2}A(v), \\ &= \frac{v^3+v^5}{(1-v^2)(1+v^4)} - \frac{2v^2}{1-v^2}A(v). \end{aligned} \tag{37}$$

Hence, we find

$$\mathscr{K}[y(x)] = A(v) = \frac{v^3}{1+v^4}. \tag{38}$$

Operating inverse Kashuri Fundo transform, we get

$$y(x) = \mathscr{K}^{-1}\left[\frac{v^3}{1+v^4}\right] = \sin x. \tag{39}$$

Therefore, we have the answer

$$y(x) = \sin x. \tag{40}$$

3 CONCLUSION

In this paper, we apply Kashuri Fundo transform to find the exact solution of the convolution type linear Volterra integral equation of the first and also the second kind. The given applications show the exact solutions of integral equations that have been obtained by less computational work and spending very little time.

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