New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2021.446

# The SOR iterative method for new preconditioned linear algebraic systems

*Abdulrahman Ndanusa and Kuluwa Adamu Al-Mustapha*

Department of Mathematics, Federal University of Technology, Minna, Nigeria,
Department of Mathematical Sciences, Baze University, Abuja, Nigeria,
Department of Mathematics, Airforce Institute of Technology, Kaduna, Nigeria

Received: 13 June 2021, Accepted: 16 June 2021
Published online: 12 July 2021.

**Abstract:** Over the years, a good number of preconditioners have been introduced to improve the convergence of iterative methods for solving linear systems. A common feature of most of these preconditioners is that the preconditioning effect is restricted to only certain entries of the coefficient matrix. In an effort to address this drawback, a new preconditioner is proposed; the effect of its application is observed on every entry of the coefficient matrix; in particular, the preconditioner eliminates the last entry on the leftmost column and scales down every other entry. Convergence and comparison theorems of the resulting preconditioned iteration technique are advanced and established. Simulated solutions of sample numerical examples via Maple 2019 Computer Algebra System are presented. It reveals that the proposed method converges faster than the SOR as well as other preconditioned iterations in literature.

**Keywords:** Successive overrelaxation (SOR), Convergence, Preconditioning, Spectral radius, Iterative Matrix.

## 1 Introduction

Typically, linear systems of equations can be expressed in matrix form as

$$Ax = b \tag{1}$$

Here, the coefficient matrix $A$, being an $n \times n$ square matrix, is assumed to be large and sparse, and usually has certain particular structures and properties, $x$ and $b$ are $n-$dimensional vectors. Suppose $A = M - N$ is a regular splitting of the matrix $A$ in (1). Then, the general basic iteration method for solving (1) is takes the form

$$x^{(k+1)} = Lx^{(k)} + c\ , \qquad k = 0, 1, 2, \tag{2}$$

where $L = M^{-1}N$ and $c = M^{-1}b$. The necessary and sufficient condition for convergence of the iterative method (2) entails that the spectral radius of the method be less than 1, and the smaller it is, the faster its convergence. Following (2), the SOR and AOR iterative matrices are respectively defined as

$$L_\omega = [(D - \omega E)]^{-1} [(1-\omega) D + \omega F] \tag{3}$$

and

$$L_{r,\ \omega} = [(D - rE)]^{-1} [(1-\omega) D + (\omega - r) E + \omega F] \tag{4}$$

* Corresponding author e-mail: as.ndanusa@futminna.edu.ng

© 2021 BISKA Bilisim Technology

In order to improve the convergence of the iterative method (2), the following preconditioned linear system is considered:

$$PAx = Pb \tag{5}$$

where $P \in R^{n\times n}$is a nonsingular matrix called a preconditioner. Without loss of generality, it is assumed that $A$ has the usual splitting $A = I - E - F$, where $I$ is the $n \times n$ identity matrix, and $-E$ and $--F$ are strictly lower and strictly upper triangular parts of $A$, respectively. Following this, alternative forms of (3) and (4) are $L_\omega = [(I-\omega E)]^{-1}[(1-\omega)I+\omega F]$ and $L_{r,\ \omega} = [(I-rE)]^{-1}[(1-\omega)I+(\omega-r)E+\omega F]$ respectively.

In 2001, Evans *et al.* [1] proposed the modified AOR method $P = I + S'$ where

$$S' = (s_{ij}) = \begin{cases} -a_{1n} & \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Consequently, the preconditioned matrix $A' = (I+S')A$ can be written as:

$$A' = \left(I+S'\right)A = \left(I+S'-F-S'E\right)-E = D'-E'-F'$$

, with $D' = \operatorname{diag}(d_1', d_2', \cdots d_n')$ where $d_i' = 1,\ \ i = 2,\cdots,n$ and $d_1' = 1 - a_{1n}a_{n1}$. If $0 < a_{i,i+1}a_{i+1,i}\ ,\ i = 1,\cdots,n-1$ and $0 < a_{n1}a_{1n} < 1$ and $0 \le r \le w \le 1\ \ (w \ne 0)\ \ (r \ne 1)$ then there exists $[(I+S')-rS'E)]^{-1}$.

Similarly, in 2012, Ndanusa and Adeboye [2] proposed the preconditioner $P = I + \overline{S}$ for the SOR method where,

$$\overline{S} = (s_{ij}) = \begin{cases} -a_{i1}, & i = 2,\cdots,n \\ -a_{i,i+1}, & i = 1,\cdots,n-1 \\ 0, & \text{otherwise } 00 \end{cases} \tag{7}$$

Then, the preconditioned matrix $\overline{A} = \left(I+\overline{S}\right)A$ can be written as follows:

$$\overline{A} = \left(I+\overline{S}\right)A = I - E - F + (S) - (SE + SF)$$

$$= (I+D_1) - (E+E_S+E_1) - (F+F_S+F_1) = \overline{D}\ -\overline{E}\ -\overline{F}$$

Several researchers have proposed similar preconditioners [3-13]. However, the preconditioned effect is almost always restricted to only a segment of the coefficient matrix A. To enable the preconditioned effect to be felt on every entry of the matrix, Abdullahi and Ndanusa [14] proposed a preconditioner for the AOR iterative matrix

$$P = I + \overline{S} + S'$$

Then the preconditioned linear system for $\hat{A}$ can be written as

$$\hat{A}x = \hat{b} \tag{8}$$

where, $\hat{A} = \left(I+\hat{S}\right)A = (I+D_1) - \left(E+E_{\hat{S}}+E_1\right) - \left(F+F_{\hat{S}}+F_1\right) = \hat{D}-\hat{E}-\hat{F}$ and $\hat{b} = Pb$. Then the preconditioned AOR iterative matrix for $\hat{A}$ is defined by

$$\widehat{\mathscr{L}_{r,\ \omega}} = (\hat{D}-r\hat{E})^{-1}\left[(1-\omega)\hat{D}+(\omega-r)\hat{E}+\omega\hat{F}\right] \tag{9}$$

© 2021 BISKA Bilisim Technology

This present work is an attempt to investigate the application of the preconditioner $P = I + \hat{S}$ to the SOR method in order to accelerate its convergence.

## Materials and methods

Following [15], we propose the preconditioner $P = I + \hat{S} = I + (-E_{\hat{S}} - F_{\hat{S}})$, from whence the preconditioned coefficient matrix $\hat{A}$ is written as, $\hat{A} = \left(I + \hat{S}\right) A = \left(I + (-E_{\hat{S}} - F_{\hat{S}})\right) A = \hat{D} - \hat{E} - \hat{F}$. Consequently, the preconditioned SOR iteration matrix for $\hat{A}$ is obtained as

$$\widehat{\mathscr{L}_\omega} = (\hat{D} - \omega\hat{E})^{-1}\left[(1-\omega)\hat{D} + \omega\hat{F}\right] \tag{10}$$

In the succeeding sections, a discussion of convergence theorems for SOR method with $\left(I + \hat{S}\right)$ is advanced. In order to prove the theorems, some and lemmas that will be used are briefly explained.

**Lemma 1.** ([17]). Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

(1) $A$ has a positive real eigenvalue equal to its spectral radius.
(2) To $\rho(A)$ there corresponds an eigenvector $x > 0$.
(3) $\rho(A)$ increases when any entry of $A$ increases.
(4) $\rho(A)$ is a simple eigenvalue of $A$.

**Lemma 2.** ([4]). Let $A$ be a nonnegative matrix. Then

(1) If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$.
(2) If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector $x$, then $\alpha \leq \rho(A) \leq \beta$ and $x$ is a positive vector.

**Lemma 3.** ([17]). Suppose $A = M - N$ is an $M-$splitting of A. Then the splitting is convergent iff $A$ is a nonsingular $M-$matrix.

**Lemma 4.** ([17]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of $A$, where $A^{-1} \geq O$. If $N_2 \geq N_1 \geq O$, then $1 > \rho\left({M_2}^{-1}N_2\right) \geq \rho\left({M_1}^{-1}N_1\right) \geq O$.

If moreover, $A^{-1} > O$ and if $N_2 \geq N_1 \geq O$, equality excluded (meaning that neither $N_1$ nor $N_2 - N_1$ is the null matrix), then $1 > \rho\left({M_2}^{-1}N_2\right) > \rho\left({M_1}^{-1}N_1\right) > O$.

**Theorem 1.** Let $\mathscr{L}_\omega$ and $\widehat{\mathscr{L}}_\omega$ be the SOR iterative matrices corresponding to the linear systems (1) and (8) respectively. Suppose $0 < \omega < 1$, $A$ is an irreducible $L-$matrix with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$ $(i = 2(1)n-1)$ and $0 < a_{12}a_{21} + a_{1n}a_{n1} < 1$. Then $\mathscr{L}_{,\omega}$ and $\widehat{\mathscr{L}}_\omega$ are nonnegative and irreducible matrices.

**Proof.** Since $A$ is an $L-$matrix, $E \geq 0$ and $F \geq 0$. Thus $(I - \omega E)^{-1} = I + \omega E + \omega^2E^2 + \cdots + \omega^{n-1}E^{n-1} \geq 0$. We know that $\mathscr{L}_\omega = (I - \omega E)^{-1}[(1-\omega)I + \omega F]$. Thus,

$$\begin{aligned}\mathscr{L}_\omega =& [I + \omega E + \omega^2E^2 + \cdots + \omega^{n-1}E^{n-1}]\,[(1-\omega)I + \omega F] \\ =& (1-\omega)I + \omega(1-\omega)E + \omega F + \omega^2EF + \omega^2(1-\omega)E^2 + \omega^3E^2F + \cdots \\ =& (1-\omega)I + \omega(1-\omega)E + \omega F + \text{nonnegative terms}\end{aligned}$$

© 2021 BISKA Bilisim Technology

It is clear that $(1-\omega)I+\omega(1-\omega)E+\omega F\geq 0$. Consequently, $\mathscr{L}_\omega=(1-\omega)I+\omega(1-\omega)E+\omega F+\cdots\geq 0$. Hence, $\mathscr{L}_\omega$ is a nonnegative matrix. Since $A=I-E-F$ is irreducible, so also is $(1-\omega)I+\omega(1-\omega)E+\omega F$ since the coefficients of $I$, $E$, and $F$ are different from zero and less than 1 in absolute value. Hence, $\mathscr{L}_\omega$ is an irreducible matrix. Now, consider the preconditioned SOR iterative matrix $\widehat{\mathscr{L}}_\omega=(\hat{D}-\omega\hat{E})^{-1}\left[(1-\omega)\hat{D}+\omega\hat{F}\right]$. Equation (5) ensures that the $L-$matrix structure of $A$ is preserved in $\hat{A}$. Since $\hat{A}$ is an $L-$matrix, it is evident that $\hat{E}\geq 0$ and $\hat{F}\geq 0$. Also, by the conditions of Theorem 1, it is easy to get that $\hat{D}\geq 0$. Thus,

$$\begin{aligned}\widehat{\mathscr{L}_{r,\,\omega}}&=\left[\hat{D}(I-\omega\hat{D}^{-1}\hat{E})\right]^{-1}\left[(1-\omega)\hat{D}+\omega\hat{F}\right]=(I-\omega\hat{D}^{-1}\hat{E})^{-1}\left[(1-\omega)I+\omega\hat{D}^{-1}\hat{F}\right]\\&=\left[I+\omega\hat{D}^{-1}\hat{E}+\omega^2\left(\hat{D}^{-1}\hat{E}\right)^2+\cdots+\omega^{n-1}\left(\hat{D}^{-1}\hat{E}\right)^{n-1}\right]\times\left[(1-\omega)I+\omega\hat{D}^{-1}\hat{F}\right]\\&=(1-\omega)I+\omega(1-\omega)\hat{D}^{-1}\hat{E}+\omega\hat{D}^{-1}\hat{F}+\text{nonnegative terms}\end{aligned}$$

Using similar arguments, it is conclusive that $\widehat{\mathscr{L}}_\omega$ is also nonnegative and irreducible. The following equalities are essential to prove Theorem 2.

$$\hat{F}=F+F_{\hat{S}}+F_1,\hat{D}=\overline{D}-S'\overline{S}=I+D_1,\hat{E}=\overline{E}=E+E_{\hat{S}}+E_1,\hat{D}-\hat{E}=\overline{D}-S'\overline{S}-\overline{E}$$

**Theorem 2.** Let $\mathscr{L}_\omega=(I-\omega E)^{-1}[(1-\omega)I+\omega F]$ and $\widehat{\mathscr{L}_\omega}=(\hat{D}-\omega\hat{E})^{-1}[(1-\omega)\hat{D}+\omega\hat{F}]$ be the SOR and preconditioned SOR iterative matrices respectively. Suppose $0<\omega<1$, $A$ is an irreducible $L-$matrix with $0<a_{1n}a_{n1}<1$, $0<a_{1i}a_{i1}+a_{i,i+1}a_{i+1,i}<1$ $(i=2(1)n-1)$ and $0<a_{12}a_{21}+a_{in}a_{n1}<1$. Then

(1) $\rho(\widehat{\mathscr{L}_\omega})<\rho(\mathscr{L}_\omega)$, if $\rho(\mathscr{L}_\omega)<1$;
(2) $\rho(\widehat{\mathscr{L}_\omega})=\rho(\mathscr{L}_\omega)$, if $\rho(\mathscr{L}_\omega)=1$;
(3) $\rho(\widehat{\mathscr{L}_\omega})>\rho(\mathscr{L}_\omega)$, if $\rho(\mathscr{L}_\omega)>1$.

**Proof.** Theorem 1 established $\mathscr{L}_\omega$ and $\widehat{\mathscr{L}}_\omega$ to be nonnegative and irreducible matrices. Therefore, suppose $\eta=\rho(\mathscr{L}_\omega)$, then by Lemma 1 there exists a positive vector $y$, such that

$$\mathscr{L}_\omega y=\eta y$$

Equivalently,

$$(I-\omega E)^{-1}[(1-\omega)I+\omega F]y=\eta y$$

$$[(1-\omega)I+\omega F]=\eta(I-\omega E)$$

$$(1-\omega)I+\omega F=\eta(I-\omega E)$$

Therefore, for this $y>0$,

$$\begin{aligned}\widehat{\mathscr{L}}_\omega y-\eta y&=\left(\hat{D}-\omega\hat{E}\right)^{-1}\left[(1-\omega)\hat{D}+\omega\hat{F}\right]y-\eta y\\&=\left(\hat{D}-\omega\hat{E}\right)^{-1}\left[(1-\omega)\hat{D}+\omega\hat{F}\right]y-\left(\hat{D}-\omega\hat{E}\right)^{-1}\left(\hat{D}-\omega\hat{L}\right)\eta y\\&=\left(\hat{D}-\omega\hat{E}\right)^{-1}[(1-\omega)\hat{D}+\omega\hat{F}-\eta\left(\hat{D}-\omega\hat{E}\right)]y\\&=\left(\hat{D}-\omega\hat{E}\right)^{-1}[(1-\omega-\eta)\hat{D}+\eta\omega\hat{E}+\omega\hat{F}]y\\&=\left(\hat{D}-\omega\hat{E}\right)^{-1}\{(\eta-1)\left(-D_1+\omega E_1+\omega E_{\hat{S}}\right)+\eta\hat{S}(I-\omega E)-\hat{S}(I-\omega E)\}y\\&=\left(\hat{D}-\omega\hat{E}\right)^{-1}\{(\eta-1)\left(-D_1+\omega E_1+\omega E_{\hat{S}}\right)+(\eta-1)\hat{S}(I-\omega E)\}y\end{aligned}$$

© 2021 BISKA Bilisim Technology

By employing the relation $(I-\omega E)=[(1-\omega)I+\omega F]/\eta$ we have,

$$\begin{aligned}\widehat{\mathscr{L}}_{\omega}y-\eta y=&(\eta-1)\left(\hat{D}-\omega\hat{E}\right)^{-1}\{-D_1+\omega E_1+\omega E_{\hat{S}}+(1/\eta)((1-\omega)\hat{S}+\omega\hat{S}F)\}y\\ =&[(\eta-1)/\eta]\left(\hat{D}-r\hat{E}\right)^{-1}[-\eta D_1+\eta\omega E_{\hat{S}}+\eta\omega E_1+(1-\omega)\hat{S}+\omega\hat{S}F]y\end{aligned}$$

It is obvious that $-\eta D_1+\eta\omega E_{\hat{S}}+\eta\omega E_1\geq 0$, provided $a_{q,q+1}a_{q+1,1}+a_{q1}\geq 0$ $(q=2,\cdots,n-1)$ and $\eta r a_{n1}-(1-\omega)a_{n1}\geq 0$, $(1-\omega)\hat{S}\geq 0$, $(\omega-r)\hat{S}E\geq 0$ and $\omega\hat{S}F\geq 0$. Suppose $\hat{D}-\omega\hat{E}$ is a splitting of some matrix $X$. From observation, $\hat{D}$ is an $M-$matrix and $\omega\hat{E}\geq 0$. Consequently, $\hat{D}-\omega\hat{E}$ is an $M-$splitting of $X$. Also, $\omega\hat{D}^{-1}\hat{E}$, being a strictly lower triangular matrix, has its eigenvalues lying on its main diagonal, and they are all zeros. Therefore, $\rho(\omega\hat{D}^{-1}\hat{E})=0<1$. And by Lemma 3, $X$ is a nonsingular $M-$matrix. consequently, $X^{-1}=\left(\hat{D}-\omega\hat{E}\right)^{-1}\geq 0$. We are now ready to deduce $(i)-(iii)$, by employing Lemma 2 thus: (1)If $\eta<1$, then $\widehat{\mathscr{L}_{\omega}}y-\eta y\leq 0$ but not equal to 0. Therefore, $\widehat{\mathscr{L}}_{\omega}y\leq\eta y$. By Lemma 2, we obtain $\rho\left(\widehat{\mathscr{L}_{\omega}}\right)<\eta=\rho\left(\mathscr{L}_{\omega}\right)$. (2) If $\eta=1$, then $\widehat{\mathscr{L}_{\omega}}y-\eta y=0$. Therefore, $\widehat{\mathscr{L}_{\omega}}y=\eta y$. By Lemma 2, we obtain $\rho\left(\widehat{\mathscr{L}_{\omega}}\right)=\eta=\rho\left(\mathscr{L}_{\omega}\right)$, and (3) If $\eta>1$, then $\widehat{\mathscr{L}_{\omega}}y-\eta y\geq 0$ but not equal to 0. Therefore, $\widehat{\mathscr{L}}_{\omega}y\geq\eta y$. By Lemma 2, we obtain $\rho\left(\widehat{\mathscr{L}_{\omega}}\right)>\eta=\rho\left(\mathscr{L}_{\omega}\right)$.

Following Kohno *et al.* [5], a more general case of the preconditioner introduced in this work is advanced by introducing the preconditioner $P=I+\hat{s}(\alpha)$ for the preconditioned linear system $\hat{A}(\alpha)x=\hat{b}(\alpha)$. Thus, the iteration matrix of the preconditioned SOR method takes the form $\widehat{\mathscr{L}_{\omega}}(\alpha)=(\hat{D}(\alpha)-\omega\hat{E}(\alpha))^{-1}\left[(1-\omega)\hat{D}(\alpha)+\omega\hat{F}(\alpha)\right]$.

**Theorem 3.** Let $\mathscr{L}_{\omega}$ and $\widehat{\mathscr{L}_{r,\,\omega}}(\alpha)$ be the SOR iterative matrices corresponding to the linear systems $Ax=b$ and $\hat{A}(\alpha)x=\hat{b}(\alpha)$ respectively. Suppose $0<\omega<1$, $A$ is an irreducible $L-$matrix with $0<\alpha_n a_{1n}a_{n1}<1$, $0<a_{1i}a_{i1}+\alpha_i a_{i,i+1}a_{i+1,i}<1$ $(i=2(1)n-1)$ and $0<\alpha_1 a_{12}a_{21}+a_{in}a_{n1}<1$. Then $\mathscr{L}_{\omega}$ and $\widehat{\mathscr{L}_{\omega}}$ are nonnegative and irreducible matrices.

**Theorem 4.** Let $\mathscr{L}_{\omega}=(I-\omega E)^{-1}[(1-\omega)I+\omega F]$ and $\widehat{\mathscr{L}_{\omega}}(\alpha)=(\hat{D}(\alpha)-\omega\hat{E}(\alpha))^{-1}\left[(1-\omega)\hat{D}(\alpha)+\omega\hat{F}(\alpha)\right]$ be the SOR and preconditioned SOR iterative matrices respectively. Suppose $0<\omega<1$, $A$ is an irreducible $L-$matrix with $0<\alpha_n a_{1n}a_{n1}<1$, $0<a_{1i}a_{i1}+\alpha_i a_{i,i+1}a_{i+1,i}<1$ $(i=2(1)n-1)$ and $0<\alpha_1 a_{12}a_{21}+a_{in}a_{n1}<1$. Then

(1) $\rho\left(\widehat{\mathscr{L}_{\omega}}(\alpha)\right)<\rho(\mathscr{L}_{\omega})$, if $\rho\left(\mathscr{L}_{\omega}\right)<1$;
(2) $\rho\left(\widehat{\mathscr{L}_{\omega}}(\alpha)\right)=\rho(\mathscr{L}_{\omega})$, if $\rho\left(\mathscr{L}_{\omega}\right)=1$;
(3) $\rho\left(\widehat{\mathscr{L}_{\omega}}(\alpha)\right)>\rho(\mathscr{L}_{\omega})$, if $\rho\left(\mathscr{L}_{\omega}\right)>1$

Similar to the proofs of Theorems 1 and 2, Theorems 3 and 4 can be easily established, and the proofs are omitted here.

**Theorem 5.** Let $0<\omega_1<\omega_2\leq 1$ and $A^{-1}\geq 0$. Under the hypothesis of Theorem 2, then $1>\rho\left(\widehat{\mathscr{L}_{\omega_1}}\right)>\rho\left(\widehat{\mathscr{L}_{\omega_2}}\right)>0$, if $0<\eta<1$.

**Proof.** Let $\hat{A}=\hat{M}_{\omega}-\hat{N}_{\omega}$, where $\hat{M}_{\omega}=(1/\omega)(\hat{D}-\omega\hat{E})$ and $\hat{N}_{\omega}=(1/\omega)[(1-\omega)\hat{D}+\omega\hat{F}]$. Suppose also that $\hat{A}=\hat{M}_{\omega_1}-\hat{N}_{\omega_1}$ and $\hat{A}=\hat{M}_{\omega_2}-\hat{N}_{\omega_2}$ are two regular splittings of $\hat{A}$, where $\hat{M}_{\omega_1}=(1/\omega_1)(\hat{D}-\omega_1\hat{E})$, $\hat{N}_{\omega_1}=(1/\omega_1)[(1-\omega_1)\hat{D}+\omega_1\hat{F}]$, $\hat{M}_{\omega_2}=(1/\omega_2)(\hat{D}-\omega_2\hat{E})$ and $\hat{N}_{\omega_2}=(1/\omega_2)[(1-\omega_2)\hat{D}+\omega_2\hat{F}]$. Since$0<\omega_1<\omega_2\leq 1$, then $\hat{N}_{\omega_1}\geq\hat{N}_{\omega_2}\geq 0$, equality excluded, then in the light of Theorem 2, we have that $1>\rho\left(\widehat{\mathscr{L}_{\omega_1}}\right)>\rho\left(\widehat{\mathscr{L}_{\omega_2}}\right)>0$. Consequently, Theorem 6 applies.

**Theorem 6.** Let $0<\omega_1<\omega_2\leq 1$ and $A^{-1}\geq 0$. Under the hypothesis of Theorem 4, then $1>\rho\left(\widehat{\mathscr{L}_{\omega_1}}(\alpha)\right)>\rho\left(\widehat{\mathscr{L}_{\omega_2}}(\alpha)\right)>0$, if $0<\eta<1$.

© 2021 BISKA Bilisim Technology

## Results and Discussion

### Numerical Experiments

In order to illustrate the results of the theorems, the following example is presented. All numerical experiments were carried out using MAPLE 2019 computer algebra system. Consider a $6\times 6$ matrix of the form

$$A=\begin{pmatrix} 1 & -1/6 & -1/7 & -1/8 & -1/6 & -1/7 \\ -1/8 & 1 & -1/6 & -1/7 & -1/8 & -1/6 \\ -1/6 & -1/8 & 1 & -1/6 & -1/7 & -1/8 \\ -1/7 & -1/6 & -1/8 & 1 & -1/6 & -1/7 \\ -1/8 & -1/7 & -1/6 & -1/8 & 1 & -1/6 \\ -1/6 & -1/8 & -1/7 & -1/6 & -1/8 & 1 \end{pmatrix}$$

Source: ([14]). Numerical results for this matrix are provided in Tables 1 – 3. For convenience, we denote the iterative matrix of the classical SOR method by $\mathscr{L}_\omega$, under the conditions of Theorem 2; $\overline{\mathscr{L}}_{r,\,\omega}$ denotes the iterative matrix of [2] under the conditions of theorem 2, and $\widehat{\mathscr{L}_\omega}$ be the iteration matrix (10).

| $\omega$ | $\rho\left(\widehat{\mathscr{L}_\omega}\right)$ | $\rho\left(\overline{\mathscr{L}}_{r,\,\omega}\right)$ | $\rho(\mathscr{L}_\omega)$ |
|---|---|---|---|
| 0.90 | 0.4892856621 | 0.5276840729 | 0.6279802179 |
| 0.80 | 0.5794241055 | 0.6068707293 | 0.6905399204 |
| 0.70 | 0.6552161106 | 0.6661423662 | 0.7447637378 |
| 0.60 | 0.7210485253 | 0.7361121449 | 0.7926676311 |
| 0.50 | 0.7793837568 | 0.7904979556 | 0.8355639464 |
| 0.40 | 0.8317916620 | 0.8397793239 | 0.8743691634 |
| 0.30 | 0.8793561388 | 0.8847984990 | 0.9097557850 |
| 0.20 | 0.9228676010 | 0.9261931550 | 0.9422357830 |
| 0.10 | 0.9629258430 | 0.9637940570 | 0.9722101200 |

**Table 1:** Numerical validation of Theorem 2

| $\alpha$ | $\omega$ | $\rho\left(\widehat{\mathscr{L}_\omega}(\alpha)\right)$ | $\rho\left(\widehat{\mathscr{L}_\omega}\right)$ | $\rho(\mathscr{L}_\omega)$ |
|---|---|---|---|---|
| (2,1,3,1,1,2) | 0.90 | 0.4163523991 | 0.4892856621 | 0.6279802179 |
| (1,1,2,1,1,1) | 0.80 | 0.5625182108 | 0.5794241055 | 0.6905399204 |
| (2,1,2,3,6,8) | 0.70 | 0.6156915826 | 0.6552161106 | 0.7447637378 |

**Table 2:** Numerical validation of Theorem 4

| $\omega_1$ | $\omega_2$ | $\rho\left(\widehat{\mathscr{L}_{\omega_2}}\right)$ | $\rho\left(\widehat{\mathscr{L}_{\omega_1}}\right)$ |
|---|---|---|---|
| 0.10 | 0.15 | 0.9432965130 | 0.9629258430 |
| 0.40 | 0.50 | 0.7793837568 | 0.8317916620 |
| 0.60 | 0.65 | 0.6891951876 | 0.7210485253 |
| 0.80 | 0.90 | 0.4892856621 | 0.5794241055 |

**Table 3:** Numerical validation of Theorem 5

© 2021 BISKA Bilisim Technology

Table 1 is a comparison of spectral radii of three iterative matrices: the proposed preconditioned SOR method of (10), the preconditioned SOR method of Ndanusa and Adeboye [2] and the classical SOR method of Young [15]. The results show that the proposed preconditioned SOR iteration converges faster than the preconditioned iteration of [2], more so, the two preconditioned iterations converge faster than the unpreconditioned (classical) SOR method. In Table 2, the proposed preconditioned SOR iteration is parametrized, and it is shown to converge faster than the unparametrized preconditioned iteration. And lastly, Table 3 is an affirmation of the Theorem that says when there exists two regular splittings of the preconditioned matrix $\hat{A} = \hat{M}_{\omega_1} - \hat{N}_{\omega_1} = \hat{M}_{\omega_2} - \hat{N}_{\omega_2}$, such that $\hat{N}_{\omega_1} \geq \hat{N}_{\omega_2} \geq 0$, then $1 > \rho\left(\widehat{\mathscr{L}_{\omega_1}}\right) > \rho\left(\widehat{\mathscr{L}_{\omega_2}}\right) > 0$.

## Conclusion

The application of a new preconditioner of the type $P = I + \hat{S} = I + S' + \overline{S}$ to the SOR method is undertaken. The preconditioned iteration is shown to converge under certain conditions imposed on the coefficient matrix. Results of numerical experiments validate the various theorems advanced. The proposed method is proven to be effective in solving linear systems, as it converges faster than some known methods in literature.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

## References

[1] Evans, D. J., Martins, M. M. and Trigo, M. E. "The AOR iterative method for new preconditioned linear systems", Journal of Computational and Applied Mathematics, 132, pp. 461-466, 2001.

[2] Ndanusa, A. and Adeboye, K. R. "Preconditioned SOR iterative methods for *L*-matrices", American Journal of Computational and Applied Mathematics, 2(6), pp. 300-305, 2012.

[3] Dehghan, M. and Hajarian, M. "Improving preconditioned SOR-type iterative methods for *L*-matrices", International Journal for Numerical Methods in Biomedical Engineering, 27, pp. 774-784.

[4] Gunawardena, A. D., Jain, S. K. and Snyder, L. "Modified iterative methods for consistent linear systems", Linear Algebra and its Applications, 154-156, pp. 123-143, 1991.

[5] Kohno, T., Kotakemori, H., Niki, H. and Usui, M. "Improving modified Gauss–Seidel method for Z-matrices", Linear Algebra and its Applications, 267: 113–123, 1997.

[6] Kotakemori, H., Harada, K., Morimoto, M. and Niki, H. "A comparison theorem for the iterative method with the preconditioner $(I + S_{max})$", Journal of Computational and Applied Mathematics, 145, pp. 373-378, 2002.

[7] Kotakemori, H., Niki, H. and Okamoto, N. "Accelerated iteration method for *Z*-matrices", Journal of Computational and Applied Mathematics, 75, pp. 87–97, 1996.

[8] Li, W. and Sun, W. "Modified Gauss–Seidel type methods and Jacobi type methods for *Z*-matrices", Linear Algebra and its Applications, 317, pp. 227-240, 2000.

[9] Milaszewicz, J. P. "Improving Jacobi and Gauss-Seidel iterations", Linear Algebra and its Applications, 93, pp. 161-170, 1987.

[10] Morimoto, M., Harada, K., Sakakihara, M. and Sawami, H. "The Gauss–Seidel iterative method with the preconditioning matrix $(I + S_{max} + S_m)$", Japan J. Indust. Appl. Math., 21, pp. 25-34, 2004.

© 2021 BISKA Bilisim Technology

[11] Niki, H., Harada, K., Morimoto, M. and Sakakihara, M. "The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method", Journal of Computational and Applied Mathematics, 164-165, pp. 587-600, 2004.
[12] Mayaki, Z. and Ndanusa, A. "Modified successive overrelaxation (SOR) type methods for *M*-matrices", Science world Journal, 14(4), pp. 1-5, 2019.
[13] Faruk, A. I. and Ndanusa, A. "Improvements of successive overrelaxation (SOR) methods for *L*-matrices", Savanna Journal of Basic and Applied Sciences, 1(2), pp. 218-223, 2019.
[14] Abdullahi, I. and Ndanusa, A. "A new modified preconditioned accelerated overrelaxation (AOR) iterative methods for *L*-matrix linear algebraic systems", Science World Journal, 15(2), pp. 45-50, 2020.
[15] Young, D. M. "Iterative solution of large linear systems", Academic Press, 1971.
[16] Saad, Y. "Iterative methods for sparse linear systems", $2^{nd}$ ed., Society for Industrial and Applied Mathematics, 2000.
[17] Varga, R. S. "Matrix iterative analysis", Prentice-Hall, 1962.

© 2021 BISKA Bilisim Technology