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Stability analysis of nonlinear differential equations depending on a parameter with conformable derivative

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Abstract: In this paper, we study the uniformly fractional exponential stability for some class of systems like class of perturbed systems and class of nonlinear fractional-order equations with control using the Lyapunov-like function.

Keywords: Stability analysis, nonlinear differential equations, conformable derivative.

1 Introduction

An other type of non classical derivative is the so called conformable derivative that was introduced by Khalil et al. in [4]. This interesting fractional derivative is based on a limit form as in the classical derivative and has similar properties than the classical one. The new conformable fractional derivative is now knowing a great interest and is the subject of several articles concerning boundary value problems, see [1], [2], [3] . Recently, Khalil et al. gave a new definition of integral and derivative of non-integer order [4]. This new definition is used as a limit form as in the case of the classical derivative. They proved the product rule, the fractional Rolle theorem and the mean value theorem. Later, this theory is developed by Abdeljawad who gave definitions of the left and right conformable derivatives of higher order, integration by part formulas, chain rule, Taylor power series representation, see [1]. The paper is organized as follows. In section 2, preliminaries and useful results with respect to the conformable derivative are introduced. In sections 3 the main results of the work are introduced detailing the stability of differential equations depending on a parameter with conformable derivative, the stability for a class of perturbed systems and the stability of a class for nonlinear differential equations with uncertainties. Finally, some examples and conclusions are given in section 4.

2 Preliminaries

2.1 Conformable derivative

Definition 1. *Let $n<\alpha<n+1$, and set $\beta=\alpha-n$. For a function g defined on $[a,\infty)$, we define the conformable integral by*

$$I_a^\alpha g(t)=\int_a^t (s-a)^{\alpha-1}g(s)ds,\ \ 0<\alpha<1, \tag{1}$$

and

$$I_a^\alpha g(t)=\frac{1}{n!}\int_a^t (t-s)^n g(s)d\beta(s,a)=\frac{1}{n!}\int_a^t (t-s)^n (s-a)^{\beta-1}g(s)ds,\ \ n\geq 1.$$

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Definition 2. *The conformable fractional derivative of order* $\alpha > 0$, $0 < \alpha < 1$, *of a function h defined on* $[a,\infty)$ *is given by*

$$T_a^{\alpha} h(t) = \lim_{\varepsilon \to 0} \frac{h\left(t + \varepsilon (t-a)^{1-\alpha}\right) - h(t)}{\varepsilon}, \tag{2}$$

for all $t > a$. *If* $T_a^{\alpha} h(t)$ *exists* $\forall t \in (a,b)$, $b > a$ *and* $\lim_{t \to a^+} T_a^{\alpha} h(t)$ *exists, then by definition*

$$T_a^{\alpha} h(a) = \lim_{t \to a^+} T_a^{\alpha} h(t). \tag{3}$$

The conformable derivative of order α, $n < \alpha < n+1$ *of a function h, when* $h^{(n)}$ *exists, is defined by*

Definition 3. *The conformable fractional derivative of order* $\alpha > 0$, $0 < \alpha < 1$, *of a function h defined on* $[a,\infty)$ *is given by*

$$T_a^{\alpha} h(t) = \lim_{\varepsilon \to 0} \frac{h\left(t + \varepsilon (t-a)^{1-\alpha}\right) - h(t)}{\varepsilon}, \tag{4}$$

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$$T_a^{\alpha} h(a) = \lim_{t \to a^+} T_a^{\alpha} h(t). \tag{5}$$

The conformable derivative of order α, $n < \alpha < n+1$ *of a function h, when* $h^{(n)}$ *exists, is defined by*
For $0 < \alpha < 1$, *it yields*

$$\lim_{\alpha \to 1} T_a^{\alpha} h(t) = h'(t)$$

and

$$\lim_{\alpha \to 0} T_a^{\alpha} h(t) = (t-a) h'(t),$$

i.e. the zero order derivative of a differentiable function does not return to the function itself.

3 Main results

In this section, we shall give sufficient conditions on the stability of conformable nonlinear systems depending on a parameter.

Consider the following system for differential equations involving conformable derivative of order α:

$$T_{t_0}^{\alpha} x = f(t,x,\varepsilon),\ t > t_0 \tag{6}$$

$$x(t_0) = x_0 \tag{7}$$

where $0 < \alpha < 1$, $t_0 > 0$, $x \in \mathbb{R}^n$, $f(.,.,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function.

Our objective is to establish the stability of system $(6)-(7)$ by using Lyapunov techniques. We will assume that for any $\varepsilon > 0$ and initial data $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$, the system $(6)-(7)$ has a unique solution $x_\varepsilon(t) \in C([t_0,+\infty),\mathbb{R}^n)$.

Definition 4. *The system* $(6)-(7)$ *is said to be* ε^*- *uniformly practically fractional exponentially stable if for all* $0 < \varepsilon < \varepsilon^*$ *there exists positive scalars* $K(\varepsilon)$, $\lambda\,(\varepsilon)$ *and* $\rho\,(\varepsilon)$ *such that*

$$\|x_\varepsilon(t)\| \le k(\varepsilon) \|x_\varepsilon(t_0)\| E_\alpha(\lambda(t-t_0)) + \rho(\varepsilon),\ \forall t \ge t_0 \ge 0, \tag{8}$$

with $\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$ *and there exists* K, λ_1, $\lambda_2 > 0$ such that $\lambda_1 \le \lambda(\varepsilon) \le \lambda_2$, $0 < K(\varepsilon) \le K$ for all $\varepsilon \in]0, \varepsilon^*]$.

Theorem 1. Let $p \ge 1$ *and* $\varepsilon^* > 0$. *Assume that for all* $\varepsilon \in]0, \varepsilon^*]$ *there exist a continuous function* $V_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, *a continuous function*: $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *and positive constants scalar* $a_1(\varepsilon)$, $a_2(\varepsilon)$, $a_3(\varepsilon)$, $\eta_1(\varepsilon)$ *and* $\eta_2(\varepsilon)$ *such that*

(1) $a_1(\varepsilon)\|x\|^p \le V_\varepsilon(t,x) \le a_2(\varepsilon)\|x\|^p + \eta_1(\varepsilon), \forall t \ge t_0, x \in \mathbb{R}^n$
(2) $V_\varepsilon(t,x)$ *has a conformable derivative of order* α *for all* $t > t_0$, $x \in \mathbb{R}^n$
(3) $T_{t_0}^{\alpha} V_\varepsilon(t, x_\varepsilon(t)) \le -a_3(\varepsilon)\|x\|^p + \eta_2(\varepsilon)\mu(t), \forall t \ge t_0, x \in \mathbb{R}^n$ *with*

$$\frac{a_3(\varepsilon)}{a_2(\varepsilon)} \ge \lambda,\ 0 < \frac{a_2(\varepsilon)}{a_1(\varepsilon)} \le k,\ \lambda, k > 0$$

There exists $M_1 \ge 0$ *such that*

$$\int_0^t (s-t_0)^{\alpha-1} E_\alpha(-\lambda(t-t_0)) E_\alpha(\lambda(s-t_0)) \mu(s)\, ds \le M_1, \forall t \ge 0$$

• $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$, *where*

$$C(\varepsilon) = \eta_1(\varepsilon)\frac{a_2(\varepsilon) + M a_3(\varepsilon)}{a_1(\varepsilon) a_2(\varepsilon)} + \eta_2(\varepsilon)\frac{M}{a_1(\varepsilon)} \tag{9}$$

with $M = M_1 + \dfrac{1}{\lambda}$.
Then, the system $(3.1)-(3.2)$ *is* ε^**-uniformly practically fractional exponentially stable.*

4 Stability for perturbed conformable systems

Let us consider the following perturbed system

$$T_{t_0}^{\alpha} x = Ax + Bu + g(t,x,u,\varepsilon), t > t_0 \tag{10}$$

$$x(t_0) = x_0 \tag{11}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $g(.,.,.,\varepsilon) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a given nonlinear function. Let us give the following assumptions:

(H_1) The perturbation term $g(t,x,u,\varepsilon)$ satisfies, for all $t \ge 0, \varepsilon > 0, x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$.

$$\|g(t,x,u,\varepsilon)\| \le \delta_1(\varepsilon)\nu(t) + \delta_2(\varepsilon)\|x\| + \delta_3(\varepsilon)\|u\|,$$

here $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon) > 0$, $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and ν is a nonnegative continuous function.

(H_2) There exists a constant, square, symmetric and positive definite matrix P, a constant matrix $K(q \times n)$ and a positive constant η such that

$$(A+BK)^T P + P(A+BK) + \eta I < 0, \tag{12}$$

and $t \to \int_{t_0}^{t} (s-t_0)^{\alpha-1} E_\alpha(-\lambda(t-t_0)) E_\alpha(\lambda(s-t_0)) v^2(s)\, ds$ is a bounded function.

Theorem 2. *Assume that* (H_1) *and* (H_2) *hold, then the feedback law*

$$u(x) = Kx \tag{13}$$

ε^**-uniformly practically fractional exponentially stabilizes the system* $(10)-(11)$

5 Stability of nonlinear conformable systems with uncertainties

We discuss the problem of stabilization for a class of nonlinear conformable systems with uncertainties. Consider the system

$$T_{t_0}^{\alpha} x = Ax + B(\Phi(x,u) + u) + g(x,u), \tag{14}$$

$$x(t_0) = x_0, \tag{15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $\Phi : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$.

Theorem 3. *Assume that the following assumptions are satisfied.*

(H_3) *There exists a square, symmetric and a positive definite matrix* P *and* $\eta > 0$ *such that the following inequality holds:*

$$A^T P + PA + \eta I < 0. \tag{16}$$

(H_4) *There exists a nonnegative continuous function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *such that:*

$$\|\Phi(x,u)\| \le \Psi(x), \ \forall x \in \mathbb{R}^n, u \in \mathbb{R}^q. \tag{17}$$

(H_5) *There exists a constant* $k > 0$, *with* $2k\|P\| < \eta$ *and such that*

$$\|g(t,u)\| \le k\|x\|, \forall x \in \mathbb{R}^n, u \in \mathbb{R}^q$$

Suppose that the assumptions (H_1), (H_3), (H_4) *and* (H_5) *hold, then the feed-back law*

$$u(\varepsilon, x) = -\frac{B^T P x \Psi(x)^2}{\|B^T P x\| \Psi(x) + \rho(\varepsilon)}, \tag{18}$$

where $\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$, $\rho(\varepsilon) > 0$, $\forall \varepsilon > 0$, ε^**-uniformly practically fractional exponentially stabilizes the system* $(16)-(17)$.

6 Examples

Example 1.
Consider the following conformable fractional-order system:

$$\begin{cases} T_{t_0}^{\alpha}x_1(t) = -x_1 + x_2 + \varepsilon e^{-t}(x_1+u) + \varepsilon^2 \dfrac{1}{1+t^2}, \\ T_{t_0}^{\alpha}x_2(t) = x_1 - 2x_2 + \varepsilon e^{-t}x_2 + \varepsilon^2 \dfrac{2t}{1+t^2} + u, \\ T_{t_0}^{\alpha}x_3(t) = 3x_3 + \dfrac{\varepsilon^2}{1+t^2} + 1.5u, \end{cases} \tag{19}$$

where, $0<\alpha<1$ and $x(t) = \left(x_1(t), x_2(t), x_3(t)\right) \in \mathbb{R}^3$. This system has the same form as (10) with

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 1.5 \end{pmatrix}$$

and

$$g(t,x,u,\varepsilon) = \varepsilon e^{-t}(x_1,x_2,x_3) + \left(\frac{\varepsilon^2}{1+t^2}, \frac{2\varepsilon^2 t}{1+t^2}, \frac{\varepsilon^2}{1+t^2}\right).$$

The perturbation term $g(t,x,u,\varepsilon)$ satisfies (H_1) and $(H_2)$with $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \delta_3(\varepsilon) = \varepsilon$ and $\nu(t) = \sqrt{3}$.
Select $P = 2I$, since

$$A^T P + PA + I = \begin{pmatrix} -3 & 4 & 0 \\ 4 & -7 & 0 \\ 0 & 0 & 13 \end{pmatrix}.$$

After solving the inequality (3.8) via the Matlab LMI toolbox, we can obtain $\eta = 2.0381$, and

$$K = (-0.2844 \quad 1.8377 \quad -3.3003)$$

The chosen gain K confirm the ε^*-uniformly practical fractional exponential stability of the closed-loop system (2.15).

Example 2.
Consider the following conformable fractional-order system:

$$\begin{cases} T_{t_0}^{\alpha}x_1(t) = -3x_1 + x_2 + \dfrac{1}{4}\sin(u)x_1, \\ T_{t_0}^{\alpha}x_2(t) = x_1 - 3x_2 + 1.5u + \dfrac{x_1 x_2}{1+u^2}, \end{cases} \tag{19}$$

where $0<\alpha<1$ and $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. This system has the same form as $(14)-(15)$ with

$$B = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix},$$

$$\Phi(x,u) = \frac{x_1 x_2}{1+u^2}.$$

and

$$g(x,u) = \frac{1}{4}\sin(u)\,x.$$

Choose $P = I$ and $\eta = 1$, then

$$A^T P + PA + I < 0.$$

The perturbation term g satisfies (H_1) with $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \varepsilon$ and $\nu(t) = \sqrt{2}$. Select $P = 2I$, since

$$A^T P + PA + I = \begin{pmatrix} -7 & 4 \\ 4 & -3 \end{pmatrix} < 0.$$

Then, the assumptions of Theorem 1 are satisfied. Hence, we obtain the ε^*-practical fractional exponential stability of the closed loop fractional-order system (19) for some $\varepsilon^* > 0$ with

$$u(\varepsilon, x_1, x_2) = \frac{1.5x_2^3 x_1^2}{1.5x_2^2 |x_1| + \varepsilon}.$$

Conclusion

The stability problem for integer order systems depending on a parameter has been subject to several research works. However, to the best of our, no work is given to the new generalized mathematical representation, namely nonlinear systems with conformable derivative. In this paper, stability of nonlinear systems depending on a parameter using conformable derivative is described. Finally, some simulation results are given to validate theoretical results.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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