New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2021.417

On (k,μ)-paracontact metric spaces satisfying some conditions on the $w_0^\star$-curvature tensor

Pakize Uygun[1] *and Mehmet Atçeken*[2]

[1]Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, Aksaray, Turkey

Received: 13 February 2021, Accepted: 12 March 2021
Published online: 18 April 2021.

Abstract: The object of the present paper is to study (k,μ)-Paracontact metric manifold. We introduce the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $W_0^\star(X,Y)\cdot P=0$, $W_0^\star(X,Y)\cdot R=0$, $W_0^\star(X,Y)\cdot \widetilde{Z}=0$, $W_0^\star(X,Y)\cdot S=0$ and $W_0^\star(X,Y)\cdot \widetilde{C}=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. In my opinion some exciting results on a (k,μ)-Paracontact metric manifold are obtained.

Keywords: (k,μ)-Paracontact manifold, η-Einstein manifold, $W_0^\star$ curvature tensor, Riemannian curvature tensor.

1 Introduction

In the modern geometry, the geometry of paracontact manifolds has turn into a subject of growing interest for its substantial applications in applied mathematics and physics. Paracontact manifolds are smooth manifolds of dimension $(2n+1)$ equipped with a $(1,1)-$tensor ϕ, a vector field ξ, and a 1-form η satisfying $\eta(\xi)=1$, $\phi^2=I-\eta\otimes\xi$ and ϕ induces an almost paracomplex structure on each fibre of $D=\ker(\eta)$[1]. Moreover if the manifold is equipped with a pseudo-Riemannian metric g so that

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y),\ \ g(X,\phi Y)=d\eta(X,Y),$$

for $X,Y\in\chi(M)$ and (M,ϕ,ξ,η,g) is called to be an almost paracontact metric manifold. Any such pseudo-Riemannian metric manifold is of signature $(n+1,n)$. In 1985, Kaneyuki and Williams started the view of paracontact geometry[7]. Zamkovoy achieved a systematic research on paracontact metric manifolds[15]. Recently, B. Cappeletti-Montano, I. Kupeli Erken and C. Murathan introduced a new type of paracontact geometry socalled paracontact metric $(k,\mu)-$space, where k and μ are constant[5].

K. Yano and S. Sawaki introduced the idea of quasi-conformal curvature tensor which is generalization of conformal curvature tensor[11]. It plays an important role in differential geometry as well as in theory of relativity. M. Atçeken studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor[2,13,14]. G.P. Pokhariyal and R. S. Mishra researched curvature tensors and their relativistic significance[8].

Motivated by the above authors, in this paper we investigate (k,μ)-paracontact manifolds, which satify the curvature conditions $W_0^\star(X,Y)\cdot P=0$, $W_0^\star(X,Y)\cdot R=0$, $W_0^\star(X,Y)\cdot \widetilde{Z}=0$, $W_0^\star(X,Y)\cdot S=0$ and $W_0^\star(X,Y)\cdot \widetilde{C}=0$ where P is the

* Corresponding author e-mail: pakizeuygun@hotmail.com

weyl curvature tensor, R is the Riemannian curvature tensor, $\widetilde{Z}$ is the concircular curvature tensor, S is the Ricci tensor, $\widetilde{C}$ is the quasi-conformal curvature tensor and $W_0^{\star}$ is the $W_0^{\star}-$ curvature tensor.

2 Preliminaries

A contact manifold is a $C^{\infty}-(2n+1)$ dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M^{2n+1}. Given such a form η, it is well known that there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\phi^2X=X-\eta(X)\xi, \qquad \eta(\xi)=1, \qquad \eta\circ\phi=0, \qquad \phi\xi=0, \tag{1}$$

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y), \quad g(X,\xi)=\eta(X) \tag{2}$$

for all vector fields X,Y on M. Then the structure (ϕ,ξ,η,g) on M is called a paracontact metric structure and the manifold equipped with such a structure is called a almost paracontact metric manifold[7].

We now define a $(1,1)$ tensor field h by $h=\frac{1}{2}L_{\xi}\phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi=-\phi h, \qquad h\xi=0, \qquad Tr.h=Tr.\phi h=0. \tag{3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\widetilde{\nabla}_X\xi=-\phi X+\phi hX \tag{4}$$

for any $X\in\chi(M)$[15]. For a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if ξ is a killing vector field or equivalently, $h=0$, then it is called a K-paracontact manifold.

A paracontact metric structure (ϕ,ξ,η,g) is normal, that is, satisfies $[\phi,\phi]+2d\eta\otimes\xi=0$, which is equivalent to

$$(\widetilde{\nabla}_X\phi)Y=-g(X,Y)\xi+\eta(Y)X$$

for all $X,Y\in\chi(M)$[15]. If an almost paracontact metric manifold is normal, then it called paracontact metric manifold. Any para-Sasakian manifold is K-paracontact, and the converse holds when $n=1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y) \tag{5}$$

for all $X,Y\in\chi(M)$, but this is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true[4].

Definition 1. *A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)$,where a,b are smooth functions on M. If $b=0$, then the manifold is also called Einstein[23].*

Definition 2. *A paracontact metric manifold is said to be a $(k,\mu)-$paracontact manifold if the curvature tensor $\widetilde{R}$ satisfies*

$$\widetilde{R}(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY] \tag{6}$$

for all $X,Y \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$[16].

In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)-$manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{7}$$

for all $X,Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k,\mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are also some common results for $k < -1$ and $k > -1$.

Lemma 1. *There does not exist any paracontact $(k,\mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$[5].*

In a paracontact metric $(k,\mu)-$manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n > 1$, the following relation hold :

$$h^2 = (k+1)\phi^2, \text{ for } k \neq -1, \tag{8}$$

$$(\widetilde{\nabla}_X \phi)Y = -g(X - hX,Y)\xi + \eta(Y)(X - hX), \tag{9}$$

$$S(X,Y) = [2(1-n)+n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) + [2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \tag{10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{11}$$

$$QY = [2(1-n)+n\mu]Y + [2(n-1)+\mu]hY + [2(n-1)+n(2k-\mu)]\eta(Y)\xi, \tag{12}$$

$$Q\xi = 2nk\xi, \tag{13}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{14}$$

for any vector fields X,Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1},g), respectively[5].

The concept of quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki[11]. Quasi-conformal curvature tensor of a $(2n+1)$-dimensional Riemannian manifold is defined as

$$\widetilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \tag{15}$$

where a and b are arbitrary scalars, and r is the scalar curvature of the manifold, Q, S and r denote the Ricci operator, Ricci tensor and scalar curvature of manifold, respectively.

Let (M,g) be an $(2n+1)$-dimensional Riemannian manifold. Then the concircular curvature tensor $\widetilde{Z}$ is defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{16}$$

for all $X,Y,Z \in \chi(M)$[10]. Then the projective curvature tensor P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{17}$$

for all $X,Y,Z \in \chi(M)$, where r is the scalar curvature of M and Q is the Ricci operator given by $g(QX,Y) = S(X,Y)$[10].

Then the curvature tensor $W_0^\star$ is defined by

$$W_0^\star(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[S(Y,Z)X - g(X,Z)QY], \tag{18}$$

for all $X,Y,Z \in \chi(M)$[8].

3 A $(k,\mu)-$ paracontact manifold satisfying certain conditions on the $W_0^\star$-curvature tensor

In this section, we will give the main results for this paper.

Let M be $(2n+1)-$dimensional $(k,\mu)-$paracontact metric manifold and we denote $W_0^\star$-curvature tensor from (18), we have for later

$$W_0^\star(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY) + \frac{1}{2n}(S(Y,Z)\xi - \eta(Z)QY). \tag{19}$$

In (19), choosing $X = \xi$, we obtain

$$W_0^\star(\xi,Y)\xi = k(2\eta(Y)\xi - Y) - \mu hY - \frac{1}{2n}QY. \tag{20}$$

Setting $X = \xi$, in (6) it follows

$$R(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{21}$$

In the same way, choosing $Z = \xi$ in (15) and using (6), we have

$$\begin{aligned}\widetilde{C}(X,Y)\xi =& (ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)(\eta(Y)X - \eta(X)Y) + a\mu(\eta(Y)hX - \eta(X)hY) \\ &+ b(\eta(Y)QX - \eta(X)QY)\end{aligned} \tag{22}$$

In (22), choosing $X = \xi$ and using (11), we obtain

$$\widetilde{C}(\xi,Y)\xi = (ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)(\eta(Y)\xi - Y) - a\mu hY + b(2nk\eta(Y)\xi - QY). \tag{23}$$

In same way from (6) and (16), we get

$$\widetilde{Z}(X,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{24}$$

from which

$$\widetilde{Z}(\xi,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)\xi - Y) - \mu hY. \tag{25}$$

From (6) and (17), we have

$$P(X,Y)\xi = \mu(\eta(Y)hX - \eta(X)hY). \tag{26}$$

Choosing $Z=\xi$ in (26), we obtain

$$P(\xi,Y)\xi=-\mu hY. \tag{27}$$

Next, we suppose that (k,μ)-paracontact manifold M is a $W_0^{\star}-$flat. From (18), we have

$$2nR(X,Y)Z=S(Y,Z)X-g(X,Z)QY=0.$$

For $Z=\xi$, it follows

$$2nR(X,Y)\xi=S(Y,\xi)X-\eta(X)QY=0.$$

By using (6) and (11), we have

$$2n\{k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]\}+2nk\eta(Y)X-\eta(X)QY=0$$

or

$$4nk\eta(Y)g(X,Z)-2nk\eta(X)g(Y,Z)-\eta(X)S(Y,Z)+\mu[\eta(Y)g(hX,Z)-\eta(X)g(hY,Z)]=0,$$

for any $Z\in\chi(M)$. It follows for $Y=\xi$

$$4nkg(X,Z)-4nk\eta(X)\eta(Z)+\mu g(hX,Z)=0. \tag{28}$$

Substituting hX into X, we have

$$4nkg(hX,Z)+\mu g(h^2X,Z)=4nkg(hX,Z)+\mu(1+k)g(\phi^2X,Z)=0. \tag{29}$$

From (28) and (29), we conclude that

$$\mu^2(1+k)-16n^2k^2=0.$$

This tell us that $(k,\mu)-$ paracontact manifold is not $W_0^{\star}-$flat provided $(k,\mu)\neq 0$.

Theorem 1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $W_0^{\star}(X,Y)\cdot\widetilde{C}=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_0^{\star}(X,Y)\cdot\widetilde{C}=0$. This implies that

$$(W_0^{\star}(X,Y)\widetilde{C})(U,W)Z=W_0^{\star}(X,Y)\widetilde{C}(U,W)Z-\widetilde{C}(W_0^{\star}(X,Y)U,W)Z-\widetilde{C}(U,W_0^{\star}(X,Y)W)Z-\widetilde{C}(U,W)W_0^{\star}(X,Y)Z=0, \tag{30}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=Z=\xi$ in (30), making use of (19), (20) and (22), for $A=[ak+2nkb-\frac{r}{2n(2n+1)}(\frac{a}{2n}+2b)]$, we have

$$\begin{aligned}(W_0^{\star}(\xi,Y)\widetilde{C})(U,W)\xi &= W_0^{\star}(\xi,Y)(A(\eta(W)U-\eta(U)W)+a\mu(\eta(W)hU-\eta(U)hW)+b(\eta(W)QU-\eta(U)QW))\\ &-\widetilde{C}(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY)+\frac{1}{2n}(S(Y,U)\xi-\eta(U)QY),W)\xi\\ &-\widetilde{C}(U,k(g(Y,W)\xi-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)+\frac{1}{2n}(S(Y,W)\xi-\eta(W)QY))\xi\\ &-\widetilde{C}(U,W)(k(2\eta(Y)\xi-Y)-\mu hY-\frac{1}{2n}QY)=0.\end{aligned} \tag{31}$$

Taking into account (19), (23) and inner product both sides of (31) by $Z \in \chi(M)$, we obtain

$$\begin{aligned}
&2nkg(\widetilde{C}(U,W)Y,Z)+2n\mu g(\widetilde{C}(U,W)hY,Z)+g(\widetilde{C}(U,W)QY,Z)+2nk\mu a(\eta(W)\eta(Z)g(Y,hU)-\eta(U)\eta(Z)g(Y,hW))\\
&+2na\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(Y,W))+a\mu(\eta(W)\eta(Z)S(Y,hU)-\eta(U)\eta(Z)S(Y,hW))\\
&+2nkA(g(Y,U)g(W,Z)-g(Y,W)g(U,Z))+2nA\mu(g(hY,U)g(W,Z)-g(hY,W)g(U,Z))\\
&+2nka\mu(g(Y,U)g(hW,Z)-g(Y,W)g(hU,Z))+2na\mu^2(g(hY,U)g(hW,Z)-g(hY,W)g(hU,Z))\\
&+2nkb(g(Y,U)S(W,Z)-g(Y,W)S(U,Z))+2n\mu b(g(hY,U)S(W,Z)-S(U,Z)g(hY,W))\\
&+2n\mu b(\eta(W)\eta(Z)S(hY,U)-\eta(U)\eta(Z)S(hY,W))+b(S(Y,U)S(Z,W)-S(Y,W)S(U,Z))\\
&+b(\eta(W)\eta(Z)S(Y,QU)-\eta(U)\eta(Z)S(Y,QW))+4n^2k^2b(g(Y,W)\eta(U)\eta(Z)-g(Y,U)\eta(W)\eta(Z))\\
&+4n^2kb\mu(g(hY,W)\eta(U)\eta(Z)-g(hY,U)\eta(W)\eta(Z))+a\mu(S(Y,U)g(Z,hW)-S(Y,W)g(hU,Z))\\
&+A(S(Y,U)g(Z,W)-S(Y,W)g(U,Z))=0
\end{aligned} \tag{32}$$

Using (1),(12) and (15) choosing $W=Y=e_i$, ξ in (32), $1 \le i \le n$, for orthonormal basis of $\chi(M)$, we arrive

$$\begin{aligned}
&(2nk-b+A-ak-4nkb+a[2(1-n)+n\mu])S(U,Z)+(2na\mu-2nb\mu+[2(n-1)+\mu](a-b))S(U,hZ)\\
&+(2nkbr+2nk(2n+1)(A-ak-2nkb)+2nk(A-ak-2nkb)+4n^2b\mu(1+k)[2(n-1)+\mu]\\
&+ak[2(n-1)+n(2k-\mu)]+br[2(1-n)+n\mu]+2nb(1+k)[2(1-n)+n\mu]^2-r(ak+2nkb)\\
&-4n^2kA+2na\mu^2(1+k))g(U,Z)+(2n\mu(A-ak-2nkb)+a\mu 2(1-n)+n\mu]-ar\mu-4n^2ka\mu)g(U,hZ)\\
&+(-ak[2(n-1)+n(2k-\mu)]+8(nk)^2b-2na\mu^2(1+k)(2n+1)-2na\mu(1+k)[2(n-1)+\mu]\\
&+[2(1-n)+n\mu](2nkb-br)-2nb(1+k)[2(n-1)+\mu]^2-4n^2b\mu(1+k)[2(n-1)+\mu])\eta(U)\eta(Z)=0.
\end{aligned} \tag{33}$$

Using (8) and replacing hZ of Z in (33), we get

$$\begin{aligned}
&(2nk-b+A-ak-4nkb+a[2(1-n)+n\mu])S(U,hZ)+(1+k)(2na\mu-2nb\mu+[2(n-1)+\mu](a-b))S(U,Z)\\
&-2nk(1+k)(2na\mu-2nb\mu+[2(n-1)+\mu](a-b))\eta(U)\eta(Z)+(2nkbr+2nk(2n+1)(A-ak-2nkb)\\
&+2nk(A-ak-2nkb)+4n^2b\mu(1+k)[2(n-1)+\mu]+ak[2(n-1)+n(2k-\mu)]+br[2(1-n)+n\mu]\\
&+2nb(1+k)[2(1-n)+n\mu]^2-r(ak+2nkb)-4n^2kA+2na\mu^2(1+k))g(U,hZ)\\
&+(1+k)(2n\mu(A-ak-2nkb)+a\mu[2(1-n)+n\mu]-ar\mu-4n^2ka\mu)g(U,Z)\\
&-(1+k)(2n\mu(A-ak-2nkb)+a\mu[2(1-n)+n\mu]-ar\mu-4n^2ka\mu)\eta(U)\eta(Z)=0.
\end{aligned} \tag{34}$$

From (33), (34) and also using (10), for the sake of brevity, we set

$$\begin{aligned}
c &= (2nk-b+A-ak-4nkb+a[2(1-n),+n\mu])\\
d &= (2na\mu-2nb\mu+[2(n-1)+\mu](a-b))\\
e &= (2nkbr+2nk(2n+1)(A-ak-2nkb)+2nk(A-ak-2nkb)+ak[2(n-1)+n(2k-\mu)]+br[2(1-n)+n\mu]\\
&\quad +4n^2b\mu(1+k)[2(n-1)+\mu]+2nb(1+k)[2(1-n)+n\mu]^2-r(ak+2nkb)-4n^2kA+2na\mu^2(1+k)),\\
f &= (2n\mu(A-ak-2nkb)+a\mu[2(1-n)+n\mu]-ar\mu-4n^2ka\mu),\\
t &= (2n\mu(A-ak-2nkb)+a\mu[2(1-n)+n\mu]-ar\mu-4n^2ka\mu)g(U,hZ)+(-ak[2(n-1)+n(2k-\mu)]\\
&\quad +8(nk)^2b-2na\mu^2(1+k)(2n+1)-2na\mu(1+k)[2(n-1)+\mu]+[2(1-n)+n\mu](2nkb-br)\\
&\quad -2nb(1+k)[2(n-1)+\mu]^2-4n^2b\mu(1+k)[2(n-1)+\mu])
\end{aligned}$$

and

$$E = (fd(1+k)-ec)[2(n-1)+\mu]+(fc-ed)[2(1-n)+n\mu],$$
$$D = (c^2-d^2(1+k))[2(n-1)+\mu]+(fc-de),$$
$$F = (fc-de)[2(n-1)+n(2k-\mu)]-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu],$$

we conclude

$$DS(U,Z) = Eg(U,Z)+F\eta(U)\eta(Z).$$

So, M is an $\eta-$Einstein manifold. The converse is obvious. This completes of the proof.

Theorem 2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $W_0^{\star}(X,Y)\cdot P=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_0^{\star}(X,Y)\cdot P=0$. This yields to

$$(W_0^{\star}(X,Y)P)(U,W)Z = W_0^{\star}(X,Y)P(U,W)Z-P(W_0^{\star}(X,Y)U,W)Z-P(U,W_0^{\star}(X,Y)W)Z-P(U,W)W_0^{\star}(X,Y)Z=0, \tag{35}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=Z=\xi$ in (35) and using (19), (20), (26), we obtain

$$\begin{aligned}(W_0^{\star}(\xi,Y)P)(U,W)\xi = {} & W_0^{\star}(\xi,Y)(\mu(\eta(W)hU-\eta(U)hW)-P(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY)\\ & +\frac{1}{2n}(S(Y,U)\xi-\eta(U)QY),W)\xi-P(U,k(g(Y,W)\xi-\eta(W)Y+\mu(g(hY,W)\xi-\eta(W)hY)\\ & +\frac{1}{2n}(S(Y,W)\xi-\eta(W)QY)\xi+P(U,W)(k(2\eta(Y)\xi-Y)-\mu hY-\frac{1}{2n}QY)=0.\end{aligned} \tag{36}$$

Taking into account that (19), (26), (27), putting $U=\xi$ and inner product both sides of in (36) by $\xi\in\chi(M)$, we get

$$2nk^2g(Y,W)+2n\mu kg(Y,hW)-\frac{1}{2n}S(QY,W)-\mu S(Y,hW)=0. \tag{37}$$

Using (1) and (12), in (37) we get

$$\begin{aligned}&(b[2(1-n)+n\mu])S(Y,W)+(2n\mu+b[2(n-1)+\mu])S(Y,hW)-4nk^2g(Y,W)-4nk^2g(Y,hW)+(2nk)^2[2(n-1)\\ &+n(2k-\mu)]\eta(Y)\eta(W)=0.\end{aligned} \tag{38}$$

Replacing hZ of Z in (38) and making use of (8), we get

$$\begin{aligned}&(b[2(1-n)+n\mu])S(Y,hW)+(1+k)(2n\mu+b[2(n-1)+\mu])S(Y,W)-2nk(1+k)(2n\mu+b[2(n-1)+\mu])\eta(Y)\eta(W)\\ &-4nk^2g(Y,hW)-4nk(1+k)g(Y,W)+(1+k)(4nk)\eta(Y)\eta(W)=0.\end{aligned} \tag{39}$$

From (38), (39) and using (10), for the sake of brevity, we put

$$c = (b[2(1-n)+n\mu]),$$
$$d = (2n\mu+b[2(n-1)+\mu]),$$
$$e = -4nk^2,$$
$$f = -4nk^2,$$
$$t = (2nk)^2[2(n-1)+n(2k-\mu)]),$$

and

$$E = (fd(1+k)-ec)[2(n-1)+\mu]+(fc-ed)[2(1-n)+n\mu],$$
$$D = (c^2-d^2(1+k))[2(n-1)+\mu]+(fc-de),$$
$$F = (fc-de)[2(n-1)+n(2k-\mu)]-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu],$$

that is,

$$DS(Y,W) = Eg(Y,W)+F\eta(Y)\eta(W).$$

Thus, M is an η-Einstein manifold. The converse is obvious.

Theorem 3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $W_0^{\star}(X,Y)\cdot R=0$ if and only if M is an $\eta-$Einstein manifold..*

Proof. Suppose that $W_0^{\star}(X,Y)\cdot R=0$. This implies that

$$(W_0^{\star}(X,Y)R)(U,W)Z = W_0^{\star}(X,Y)R(U,W)Z-R(W_0^{\star}(X,Y)U,W)Z-R(U,W_0^{\star}(X,Y)W)Z-R(U,W)W_0^{\star}(X,Y)Z=0, \quad (40)$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (40) and making use of (6), (19), (20), we obtain

$$\begin{aligned}(W_0^{\star}(\xi,Y)R)(U,W)\xi = {} & W_0^{\star}(\xi,Y)(k(\eta(W)U-\eta(U)W)+\mu(\eta(W)hU-\eta(U)hW)-R(k(g(Y,U)\xi-\eta(U)Y)\\ & +\mu(g(hY,U)\xi-\eta(U)hY)+\frac{1}{2n}(S(Y,U)\xi-\eta(U)QY),W)\xi-R(U,k(g(Y,W)\xi-\eta(W)Y)\\ & +\mu(g(hY,W)\xi-\eta(W)hY)\xi+\frac{1}{2n}(S(Y,W)\xi-\eta(W)QY))\xi-R(U,W)(k(2\eta(Y)\xi-Y)\\ & -\mu hY-\frac{1}{2n}QY)=0. \end{aligned} \quad (41)$$

Inner product both sides of (41) by $Z\in\chi(M)$ and using of (19), (20) and (21) we get

$$\begin{aligned}& 2nkg(R(U,W)Y,Z)+2n\mu g(R(U,W)hY,Z)+g(R(U,W)QY,Z)+2nk\mu(\eta(W)\eta(Z)g(Y,hU)-\eta(U)\eta(Z)g(Y,hW))\\ & +2n\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(Y,W))+\mu(\eta(W)\eta(Z)S(Y,hU)-\eta(U)\eta(Z)S(Y,hW))\\ & +2nk^2(g(Y,U)g(W,Z)-g(Y,W)g(U,Z))+2n\mu k(g(Y,U)g(hZ,W)-g(Y,W)g(hU,Z))\\ & +2nk\mu(g(hY,U)g(W,Z)-g(hY,W)g(U,Z))+2n\mu^2(g(hY,U)g(hW,Z)-g(hY,W)g(hU,Z))\\ & +k(S(Y,U)g(W,Z)-S(Y,W)g(U,Z))+\mu(g(hW,Z)S(Y,U)-S(Y,W)g(hU,Z))=0. \end{aligned} \quad (42)$$

Making use of (8), (12) and choosing $W=Y=e_i,\ \xi\ 1\le i\le n$, for orthonormal basis of $\chi(M)$ in (42), we have

$$\begin{aligned}& (k(2n+1)+[2(1-n)+n\mu])S(U,Z)+(\mu(2n+1)+[2(n-1)+\mu])S(U,hZ)+(k[2(n-1)+(2k-\mu)]\\ & -kr+2\mu^2(1+k)-(2nk)^2)g(U,Z)+(\mu[2(n-1)+n(2k-\mu)]-\mu r+2nk\mu-(2n)^2k)g(U,hZ)\\ & +(-k[2(n-1)+n(2k-\mu)]-2n\mu^2(1+k)(2n+1)-2n\mu(1+k)[2(n-1)+\mu]\eta(U)\eta(Z)=0. \end{aligned} \quad (43)$$

Replacing hZ of Z in (43) and taking into account (8), we get

$$\begin{aligned}& (k(2n+1)+[2(1-n)+n\mu])S(U,hZ)+(1+k)(\mu(2n+1)+[2(n-1)+\mu])S(U,Z)-2nk(1+k)(\mu(2n+1)\\ & +[2(n-1)+\mu])\eta(U)\eta(Z)+(k[2(n-1)+(2k-\mu)]-kr+2\mu^2(1+k)-(2nk)^2)g(U,hZ)+(1+k)(\mu[2(n-1)\\ & +n(2k-\mu)]-\mu r+2nk\mu-(2n)^2k)g(U,Z)-(1+k)(\mu[2(n-1)+n(2k-\mu)]\\ & -\mu r+2nk\mu-(2n)^2k)\eta(U)\eta(Z)=0. \end{aligned} \quad (44)$$

From (43), (44) and by using (10), for the sake of brevity, we set

$$\begin{aligned}
c &= (k(2n+1)+[2(1-n)+n\mu]),\\
d &= (\mu(2n+1)+[2(n-1)+\mu]),\\
e &= (k[2(n-1)+(2k-\mu)]-kr+2\mu^2(1+k)-(2nk)^2),\\
f &= (\mu[2(n-1)+n(2k-\mu)]-\mu r+2nk\mu-(2n)^2k),\\
t &= (-k[2(n-1)+n(2k-\mu)]-2n\mu^2(1+k)(2n+1)-2n\mu(1+k)[2(n-1)+\mu]
\end{aligned}$$

and

$$\begin{aligned}
E &= (fd(1+k)-ec)[2(n-1)+\mu]+(fc-ed)[2(1-n)+n\mu],\\
D &= (c^2-d^2(1+k))[2(n-1)+\mu]+(fc-de),\\
F &= (fc-de)[2(n-1)+n(2k-\mu)]-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu],
\end{aligned}$$

we conclude

$$DS(U,Z)=Eg(U,Z)+F\eta(U)\eta(Z),$$

which verifies our assertion. The converse is obvious.

Theorem 4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $W_0^\star(X,Y)\cdot\widetilde{Z}=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_0^\star(X,Y)\cdot\widetilde{Z}=0$. This means that

$$(W_0^\star(X,Y)\widetilde{Z})(U,W,Z)=W_0^\star(X,Y)\widetilde{Z}(U,W)Z-\widetilde{Z}(W_0^\star(X,Y)U,W)Z-\widetilde{Z}(U,W_0^\star(X,Y)W)Z-\widetilde{Z}(U,W)W_0^\star(X,Y)Z=0,\quad(45)$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (45) and making use of (19), (24) for $A=k-\frac{r}{2n(2n+1)}$, we obtain

$$\begin{aligned}
&(W_0^\star(\xi,Y)\widetilde{Z})(U,W)\xi=W_0^\star(\xi,Y)(A(\eta(W)U-\eta(U)W)+\mu(\eta(W)hU-\eta(U)hW)-\widetilde{Z}(k(g(Y,U)\xi-\eta(U)Y)\\
&+\mu(g(hY,U)\xi-\eta(U)hY)+\frac{1}{2n}(S(Y,U)\xi-\eta(U)QY,W))\xi-\widetilde{Z}(U,k(g(Y,W)\xi-\eta(W)Y)\\
&+\mu(g(hY,W)\xi-\eta(W)hY)+\frac{1}{2n}(S(Y,W)\xi-\eta(W)QY))\xi-\widetilde{Z}(U,W)(k(2\eta(Y)\xi-Y)\\
&-\mu hY-\frac{1}{2n}QY)=0. \qquad (46)
\end{aligned}$$

Using (19), (24), (25) and inner product both sides of (46) by $Z\in\chi(M)$, we get

$$\begin{aligned}
&2nkg(\widetilde{Z}(U,W)Y,Z)+2n\mu g(\widetilde{Z}(U,W)hY,Z)+g(\widetilde{Z}(U,W)QY,Z)+2nk\mu(\eta(W)\eta(Z)g(Y,hU)-\eta(U)\eta(Z)g(Y,hW))\\
&+2n\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(Y,W))+\mu(\eta(W)\eta(Z)S(Y,hU)-\eta(U)\eta(Z)S(Y,hW))\\
&+2nkA(g(Y,U)g(W,Z)-g(Y,W)g(U,Z))+2n\mu k(g(Y,U)g(hZ,W)-g(Y,W)g(hU,Z))\\
&+2nA\mu(g(hY,U)g(W,Z)-g(hY,W)g(U,Z))+2n\mu^2(g(hY,U)g(hW,Z)-g(hY,W)g(hU,Z))\\
&+A(S(Y,U)g(W,Z)-S(Y,W)g(U,Z))+\mu(g(hW,Z)S(Y,U)-S(Y,W)g(hU,Z))=0. \qquad (47)
\end{aligned}$$

Making use of (12), (16) and choosing $W=Y=e_i$, ξ $1\le i\le n$, for orthonormal basis of $\chi(M)$ in (47), we have

$$\begin{aligned}
&(k(2n+1)+[2(1-n)+n\mu])S(U,Z)+(\mu(2n+1)+[2(n-1)+\mu])S(U,hZ)+(k[2(n-1)+n(2k-\mu)]\\
&-(2nk)^2-rk+2n\mu^2(1+k))g(U,Z)+(2n\mu k(1-2n)+\mu 2(n-1)+n(2k-\mu)]-\mu r)g(U,hZ)\\
&+(-k[2(n-1)+n(2k-\mu)]-2n\mu^2(1+k)(2n+1)-2n\mu(1+k)[2(n-1)+\mu])\eta(U)\eta(Z)=0. \qquad (48)
\end{aligned}$$

Replacing hZ of Z in (48) and taking into account (8), we arrive

$$(k(2n+1)+[2(1-n)+n\mu])S(U,hZ)+(1+k)(\mu(2n+1)+[2(n-1)+\mu])S(U,Z)-2nk(1+k)(\mu(2n+1)$$
$$+[2(n-1)+\mu])\eta(U)\eta(Z)+(k[2(n-1)+n(2k-\mu)]-(2nk)^2-rk+2n\mu^2(1+k))g(U,hZ)+(1+k)(2n\mu k(1-2n)$$
$$+\mu 2(n-1)+n(2k-\mu)]-\mu r)g(U,Z)-(1+k)(2n\mu k(1-2n)+\mu[2(n-1)+n(2k-\mu)]-\mu r)\eta(U)\eta(Z)=0. \quad (49)$$

From (48), (49) and by using (10), for the sake of brevity, we set

$$c=(k(2n+1)+[2(1-n)+n\mu]),$$
$$d=(\mu(2n+1)+[2(n-1)+\mu]),$$
$$e=(k[2(n-1)+n(2k-\mu)]-(2nk)^2-rk+2n\mu^2(1+k)),$$
$$f=(2n\mu k(1-2n)+\mu[2(n-1)+n(2k-\mu)]-\mu r),$$
$$t=(-k[2(n-1)+n(2k-\mu)]-2n\mu^2(1+k)(2n+1)-2n\mu(1+k)[2(n-1)+\mu]),$$

and

$$E=[fd(1+k)-ec][2(n-1)+\mu]+(fc-de)[2(1-n)+n\mu],$$
$$D=(c^2-d^2(1+k))[2(n-1)+\mu]+(fc-ed),$$
$$F=(fc-de)[2(n-1)+n(2k-\mu)]-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu],$$

we have

$$DS(U,Z)=Eg(U,Z)+F\eta(U)\eta(Z).$$

This tell us, M is an η-Einstein manifold. The converse is obvious.

Theorem 5. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $W_0^{\star}(X,Y)\cdot S=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_0^{\star}(X,Y)\cdot S=0$. This means that

$$S(W_0^{\star}(X,Y)U,W)+S(U,W_0^{\star}(X,Y),W)=0, \quad (50)$$

for all $X,Y,U,W\in\chi(M)$. Setting $X=\xi$ in (50) and making use of (19), we obtain

$$S(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi-\eta(U)hY)+\frac{1}{2n}(S(Y,U)\xi-\eta(U)QY),W)+S(U,k(g(Y,W)\xi$$
$$-\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY)+\frac{1}{2n}(S(Y,W)\xi-\eta(W)QY)=0. \quad (51)$$

Using (8), (12) and setting $U=\xi$ in (51), we have

$$[2(1-n)+n\mu]S(Y,W)+(2n\mu[2(n-1)+\mu])S(Y,hW)-4nk^2g(Y,W)-4nk\mu g(hY,W)$$
$$+2nk[2(n-1)+n(2k-\mu)]\eta(Y)\eta(W)=0. \quad (52)$$

Putting (8) and replacing hW of W in (52), we get

$$[2(1-n)+n\mu]S(Y,hW)+(1+k)(2n\mu[2(n-1)+\mu])S(Y,W)-2nk(1+k)(2n\mu[2(n-1)+\mu])\eta(Y)\eta(W)$$
$$-4nk^2g(Y,hW)-4nk\mu(1+k)g(Y,W)-4nk\mu(1+k)\eta(Y)\eta(W)=0. \quad (53)$$

From (52), (53) and by using (10), for the sake of brevity, we set

$$c = [2(1-n)+n\mu],$$
$$d = (2n\mu[2(n-1)+\mu]),$$
$$e = -4nk^2,$$
$$f = -4nk\mu,$$
$$t = 2nk[2(n-1)+n(2k-\mu)]$$

and

$$E = [fd(1+k)-ec][2(n-1)+\mu]+(fc-de)[2(1-n)+n\mu],$$
$$D = (c^2-d^2(1+k))[2(n-1)+\mu]+(fc-ed),$$
$$F = (fc-de)[2(n-1)+n(2k-\mu)]-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu],$$

then we have

$$DS(Y,W) = Eg(Y,W)+F\eta(Y)\eta(W).$$

Thus, M is an η-Einstein manifold. The converse is obvious.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] DV. Alekseevski, C. Medori and A. Tomassini, Maximally homogeneous para-CRmanifolds. Ann. Glob. Anal. Geom. 30 (2006), 1-27.
[2] M. Atçeken , On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor, Bulletin of Mathematical Analysis and Applications, 6(1)2014, 1-8.
[3] M. Atçeken and P. Uygun, Characterizations for totally geodesic submanifolds of (k,μ)-paracontact metric manifolds, Korean J. Math. 28(2020), 555-571.
[4] G. Calvaruso, Homogeneous paracontact metric three-manifolds, Illinois J. Math., 55(2011), 697-718.
[5] B. Cappelletti-Montano, I. Küpeli Erken and C. Murathan, Nullity conditions in paracontact geometry, Differential Geom. Appl., 30(2012), 665-693.
[6] U.C. De, J.B. Jun and A.K. Gazi, Sasakian manifolds with quasi-conformal curvature tensor, Bull. Korean Math. Soc. 45(2008), No.2, pp. 313-319.
[7] S. Kaneyuki and F.L. Williams, Almost paracontact and parahodge structures on manifolds, Nagoya Math. J., 99(1985), 173-187.
[8] G.P. Pokhariyal and R.S.Mishra, Curvature tensors and their relativistic significance II, Yokohama Math. J., 19 (1971), No. 2, 97-103.
[9] M.M. Tripathi and P. Gupta, $T-$curvature tensor on a semi-Riemannian manifold, 4 (2011), No. 1, 117-129.
[10] K. Yano and M. Kon, Structures manifolds, Singapore, World Scientific, (1984).
[11] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom. 2(1968), 161-184.

[12] Ü. Yıldırım, M. Atçeken and S. Dirik, A Normal Paracontact Metric Manifold Satisfying Some Conditions on the M-Projective Curvature Tensor. Konuralp Journal of Mathematics , vol: 7(1)(2019), 217-221.

[13] Ü. Yıldırım, M. Atçeken and S. Dirik, Pseudo projective curvature tensor satisfying some properties on a normal paracontact metric manifold, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2019),997-1006.

[14] Ü. Yıldırım, M. Atçeken and S. Dirik, On the pseudo-quasi conformal tensor of a normal paracontact metric manifold, Turk. J. Comput. Sci., 12(1), (2020), 49-55.

[15] S. Zamkovoy, Canonical connections on paracontact manifolds, Ann. Global Anal. Geom., 36(2009), 37-60.

[16] S.Zamkovoy and V. Tzanov, Non-existence of flat paracontact metric structures in dimension greater than or equal to five, Annuaire Univ. Sofia Fac. Math. Inform., 100(2011), 27-34.

[17] I. Küpeli Erken, Generalized $(\widetilde{k}\neq -1,\widetilde{\mu})-$paracontact metric manifolds with $\xi(\widetilde{\mu})=0$, Int. Electron. J. Geom. 8 (2015),no.1, 77-93.

[18] U.C. De and A. Sarkar, On the Projective curvature tensor of Generalized Sasakian-space forms, Quaestiones Mathematicae, 33(2010), 245-252.

[19] B. Cappeletti-Montano and L. Di. Terlizzi, Geometric structure associated to a contact metric (k,μ)-space, Pasific J. Math. 246 (2010),no.2, 257-292.

[20] D.E. Blair, T. Koufogiorgos and B.J. Papatoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math., 91 (1995), 189-214.

[21] G. Yüca, Y. Yaylı, A dual transformation between S$\widehat{O}(3)$ and S$\widehat{O}(2,1)$ and its geometric applications, Proc. Natl. Acad. Sci., India, Sect. A. Phys. Sci., 88-2 (2018) 267-273.

[22] G. Yüca, and Y. Yaylı, Dual transformations in Galilean Spaces, International Electronic Journal of Geometry, 13-(2) (2020), 52-61.

[23] W.M. Boothby, *An Introduction to Differentiable Manifolds and Riemanniann Geometry*, Academic Press, Inc. London, (1986).