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Existence of Mild Solution for Neutral Functional Mixed Integrodifferential Evolution Equations with Nonlocal Conditions

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Abstract: In this paper, we explore the existence of the mild solution for nonlinear neutral functional mixed-Volterra integrodifferential evolution equations with the nonlocal condition. The findings are achieved by implementing the fractional power of operators and Sadovskii's fixed point theorem. As an application, a controllability problem is discussed for the considered systems.

Keywords: Mild solution; Neutral integrodifferential equations; Nonlocal conditions; semigroup theory; Sadovskiiâs fixed point theorem.

1 Introduction

Several authors [1–5] have investigated the theory of neutral differential equations in Banach spaces. A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are implied as well as the present state of the system. Neutral differential equations emerge in problems dealing with electric networks containing lossless transmission lines. Such networks appeared, for example, in high speed computers where lossless transmission lines are applied to interconnect switching circuits.

To make the use of the method of semigroups, Pazy [6] examined the existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations. Balachandran et al. [7] analyzed the existence of results for nonlinear abstract neutral differential equations with time varying delays of the form

$$\begin{gathered}\frac{d}{dx}\left[x(t)+F\left(t,\,x(t),\,x(b_1(t)),\ldots,\,x(b_m(t))\right)\right]=Ax(t)+G\left(t,\,x(t),\,x(a_1(t)),\,\ldots,\,x(a_n(t))\right),\\ x(0)=x_0,\end{gathered}\tag{1}$$

where $t\in J=[0,\,a]$, A is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$ in a Banach space X by employing Schaefer fixed point theorem. The same class of neutral equation with nonlocal condition is also investigated by Fu and Ezzinbi [8] by applying Sadovskii's fixed point theorem.

The nonlocal Cauchy problem was first evaluated by Byszewski [9]. The significance of the problem comprises in the fact that it is more general and has a finer effect than the classical initial condition. In the past many years, theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been worked by different authors [10, 11].

The authors [12, 13] studied the existence of solutions for neutral functional integrodifferential equations in Banach spaces. Recently, Manimaran et al. [14] examined the existence of solutions for neutral functional integrodifferential evolution equations with nonlocal conditions by applying the fractional power of operators and Sadovskii's fixed point theorem. **Kumar et.al [15] proved the mild solution of the impulsive fractional integrodifferential equation with nonlocal conditions in Banach spaces by using fractional calculus and fixed point theorem**. **Munusamy et. al [16] showed the existence of solutions for some functional integrodifferential equations with nonlocal conditions to**

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establish the results by using the resolvent operator theory and Sadovskii-Krasnosel'skii type of fixed point theorem. The authors [17, 18] established the existence as well as controllability results of different classes of integrodifferential equations in abstract spaces.

In the following paper, we investigate the existence of mild solution of the evolution equation of the form

$$\begin{gathered}\frac{d}{dx}\left[x(t)+F\left(t,\,x(t),\,x(b_1(t)),\ldots,\,x(b_m(t))\right)\right]+Ax(t)=G\left(t,\,x(t),\,x(a_1(t)),\,\ldots,\,x(a_n(t))\right)\\ +H\left(t,\,x(t),\,\int_0^t k(t,\,s,\,x(s))\,ds,\,\int_0^T g(t,\,s,\,x(s))\,ds\right),\,t\in J=[0,\,T],\\ x(0)+h(x)=x_0,\end{gathered} \tag{2}$$

where $-A$ generates an analytic semigroup and $F,\,G,\,H,\,k,\,g,\,h$ are given functions to be specified later. The findings are the generalization and continuation of some results (see [6–8]).

2 Preliminaries

All over the paper X will be a Banach space with norm$\|\,.\,\|$. Let -A be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $S(t)$. Assume that $0\in\rho(A)$, then define the fractional power A^{α}, for $0\le\alpha\le 1$, as a closed linear operator on its domain $D(A^{\alpha})$. Also, the subspace $D(A^{\alpha})$ is dense. Additionally, $D(A^{\alpha})$ is a Banach space under the norm

$$\|x\|_{\alpha}=\|A^{\alpha}x\|,\,x\in D(A^{\alpha}), \tag{3}$$

which is expressed by X_{α}. Then for every $0<\alpha\le 1$, $X_{\alpha}\to X_{\beta}$ for $0<\beta<\alpha\le 1$ and the imbedding is compact whenever the resolvent operator of A is compact. The properties will be used for the semigroup $S(t)$ which is given below.

1. There is a $M\ge 1$ such that $\|S(t)\|\le M$ for each $0\le t\le T$;
2. For any $\alpha>0$, there exists a positive constant C_{α} such that

$$\|A_{\alpha}S(t)\|\le\frac{C_{\alpha}}{t^{\alpha}}\,,\;0<t\le T. \tag{4}$$

We assume the conditions which are given below:

(A_1) $F:[0,\,T]\times X^{m+1}\to X$ is a continuous function, and there exists $\beta\in(0,\,1)$ and $L_1,\,L_2>0$ such that the function $A^{\beta}F$ satisfies the Lipschitz condition:

$$\left\|A^{\beta}F(s_1,\,x_0,\,x_1,\ldots,\,x_m)-A^{\beta}F(s_2,\,\bar{x}_0,\,\bar{x}_1,\ldots,\,\bar{x}_m)\right\|\le L_1\left(|s_1-s_2|+\max_{i=0,\,1,\ldots,\,m}\|x_i-\bar{x}_i\|\right), \tag{5}$$

for any $0\le s_1,\,s_2\le T,\,x_i,\,\bar{x}_i\in X,\;\;i=0,\,1,\,2,\,\ldots,\,m$; and the inequality

$$\left\|A^{\beta}F(t,\,x_0,\,x_1,\ldots,\,x_m)\right\|\le L_2(\max\{\|x_i\|:i=0,1,\ldots,m\}\,+1\,), \tag{6}$$

holds for any $(t,\,x_0,\,x_1,\ldots,\,x_m)\in[0,\,T]\times X^{m+1}$.

(A_2) The function $G:[0,\,T]\times X^{n+1}\to X$ fulfills the following conditions:

1. For each $t\in[0,\,T]$, the function $G(t,\,.):X^{n+1}\to X$ is continuous and for each $(x_0,\,x_1,\ldots,\,x_n)\in X^{n+1}$ the function $G(.,\,x_0,\,x_1,\ldots,\,x_n):[0,\,T]\to X$is strongly measurable;
2. For each positive number $p\in\mathbb{N}$, there is a positive function $g_p\in L^1([0,\,T])$ such that

$$\begin{gathered}\sup_{\|x_0\|,\,\ldots,\,\|x_n\|\le p}\|G(t,\,x_0,\,x_1,\ldots,\,x_n)\|\;\le g_p(t)\\ \text{and}\\ \liminf_{p\to\infty}\frac{1}{p}\int_0^T g_p(s)\,ds\;=\gamma_1<\infty.\end{gathered} \tag{7}$$

(A_3)The function $H:[0,\,T]\times X\times X\times X\to X$ satisfies the following conditions:

1. For each $t\in[0,\,T]$, the function $H(t,.,.,.):X\times X\times X\to X$ and for all $x,\,y,\,z\in X$,$H(.,\,x,\,y,\,z):[0,\,T]\to X$is strongly measurable.

2.For each positive number $r \in \mathbb{N}$, there exists a positive function $q_r \in L^1([0,\ T])$ such that

$$\sup_{\|x\| \le r} \left\| H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right) \right\| \le q_r(s)$$
$$\text{and} \tag{8}$$
$$\lim_{r\to\infty} \inf \frac{1}{r}\int_0^T q_r(s)\,ds\ = \gamma_2 < \infty.$$

$(A_3)\, a_j,\ b_j \in C([0,\ T];\ [0,\ T]),\ i = 1,\ 2,\ \ldots, n,\ j = 1,\ 2,\ \ldots,\ m; h \in C(E;X)$. Here and hereafter $E = C([0,\ T];X)$, and h satisfies that

1.There exist positive constants L_3 and $L_3{}'$ such that $\|h(x)\| \le L_3\|x\| + L_3{}'$ for all $x \in E$;
2.h is a completely continuous map.

Theorem 2.1. (Sadovskii's Fixed Point Theorem [19])
Let P be a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(D)) \le \alpha(D)$ for every bounded set D of x with $\alpha(D) > 0$. If $P(E) \subset E$ for convex, closed and bounded set E of X, then P has a bounded point in E (where $\alpha(.)$ denotes the Kuratowski's measures of non-compactness).

3 Existence of the Mild Solution

Definition 3.1. A continuous function $x(.) : [0,\ T] \to X$ is said to be a mild solution of the nonlocal Cauchy problem (2), if the function $AS(t-s)F(s,\ x(s),\ x(b_1(s)),\ \ldots,\ x(b_m(s))),\ s \in [0,\ T)$is integrable on $[0,\ T)$ and the following integral equation is verified:

$$\begin{aligned} x(t) = S(t)\left[x_0 + F(0,\ x(0), x(b_1(0)),\ \ldots,\ x(b_m(0))) - h(x)\right] - F(t,\ x(t),\ x(b_1(t)),\ \ldots,\ x(b_m(t))) \\ + \int_0^t AS(t-s)F(s,\ x(s),\ x(b_1(s)),\ \ldots,\ x(b_m(s)))\,ds + \int_0^t S(t-s)G(s,\ x(s),\ x(a_1(s)),\ \ldots,\ x(a_n(s)))\,ds \\ + \int_0^t S(t-s)\left[H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right]ds. \end{aligned} \tag{9}$$

Theorem 3.1. If assumptions (A_1)-(A_4) are satisfied and $x_0 \in X$, then the nonlocal Cauchy problem (2) has a mild solution provided that

$$L_0 := \left[(M+1)M_0 + \frac{C_{1-\beta}T^\beta}{\beta}\right]L_1 < 1 \tag{10}$$

and

$$M_0L_2 + M(L_3 + M_0L_2 + \gamma_1 + \gamma_2) + L_2\frac{C_{1-\beta}T^\beta}{\beta} < 1, \tag{11}$$

where $M_0 = \left\|A^{-\beta}\right\|$.
Proof : For the sake of brevity, we rewrite that

$$\begin{gathered} (t,\ x(t),\ x(b_1(t)),\ \ldots,\ x(b_m(t))) = (t,\ v(t)) \\ \textit{and} \\ (t,\ x(t),\ x(a_1(t)),\ \ldots,\ x(a_n(t))) = (t,\ u(t)). \end{gathered} \tag{12}$$

Define the operator ψ on E by the formula

$$\begin{aligned} (\psi x)(t) = S(t)\left[x_0 + F(0,\ v(0)) - h(x)\right] - F(t,\ v(t)) + \int_0^t AS(t-s)F(s,\ v(s))\ ds\ + \int_0^t S(t-s)G(s,\ u(s))\,ds \\ + \int_0^t S(t-s)\left[H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right]ds,\ 0 \le t \le T. \end{aligned} \tag{13}$$

For each positive integer p, let

$$D_p = \{x \in E : \|x(t)\| \leq p,\ 0 \leq t \leq T\}. \tag{14}$$

Then for each p, D_p is obviously a bounded closed convex set in E. Since by (4) and (5) the following relation holds:

$$\|AS(t-s)F(s,\ v(s))\| \leq \left\|A^{1-\beta}S(t-s)A^{\beta}F(s,\ v(s))\right\| \leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_2(p+1), \tag{15}$$

then from Bocher's theorem [20] it follows that $AS(t-s)F(s,\ v(s))$ is integrable on $[0,\ T]$, so ψ is well defined on D_p. We claim that there exists a positive integer p such that $\psi D_p \subseteq D_p$. If it is not true, then for each positive integer p, there is a function $x_p(.) \in D_p$, but $\psi x_p \notin D_p$, that is $\|\psi x_p(t)\| > p$ for some $t(p) \in [0,\ T]$, where $t(p)$ denotes t is independent of p.On the other hand, we have

$$\begin{aligned}
p < \left\|(\psi x_p)(t)\right\| = \Bigg\| & S(t)[x_0 - h(x_p) + F(0,\ v_p(0))] - F(t,\ v_p(t)) + \int_0^t AS(t-s)F(s,\ v_p(s))\ ds \\
& + \int_0^t S(t-s)G(s,\ u_p(s))ds + \int_0^t S(t-s)\left[H\left(s,\ x_p(s),\ \int_0^s k(s,\ \tau,\ x_p(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x_p(\tau))d\tau\right)\right]ds\Bigg\| \\
\leq & \left\|S(t)[x_0 - h(x_p) + F(0,\ v_p(0))]\right\| + \left\|A^{-\beta}A^{\beta}F(t,\ v_p(t))\right\| + \left\|\int_0^t A^{1-\beta}S(t-s)A^{\beta}F(s,\ v_p(s))\ ds\right\| \\
& + \left\|\int_0^t S(t-s)G(s,\ u_p(s))ds\right\| + \left\|\int_0^t S(t-s)\left[H\left(s,\ x_p(s),\ \int_0^s k(s,\ \tau,\ x_p(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x_p(\tau))d\tau\right)\right]ds\right\| \\
\leq & M\left[\|x_0\| + L_3 p + L_3^{'+M_0L_2(p+1)}\right] + M_0L_2(p+1) + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_2(p+1)\ ds + M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds \\
\leq & M\left[\|x_0\| + L_3 p + L_3'\right] + M_0L_2(p+1)(M+1) + \frac{T^{\beta}}{\beta}C_{1-\beta}L_2(p+1) + M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds.
\end{aligned} \tag{16}$$

Dividing into both sides by p and taking the lower limit as $p \to +\infty$, we get

$$\begin{aligned}
& ML_3 + M_0L_2M + M_0L_2 + \frac{T^{\beta}}{\beta}C_{1-\beta}L_2 + M\gamma_1 + M\gamma_2 \geq 1, \\
& \Rightarrow M_0L_2 + M(L_3 + M_0L_2 + \gamma_1 + \gamma_2) + \frac{T^{\beta}}{\beta}C_{1-\beta}L_2 \geq 1.
\end{aligned} \tag{17}$$

This contradicts (11). Hence, for some positive integer p, we must have $\psi D_p \subseteq D_p$.

Next, we will show that the operator ψ has a fixed point on D_p, which implies equation (2) has a mild solution. To this end, we decompose ψ as $\psi = \psi_1 + \psi_2$, where the operators $\psi_1,\ \psi_2$ are defined on D_p, respectively, by

$$\begin{aligned}
(\psi_1 x)(t) = S(t)F(0,\ v(0)) - F(t,\ v(t)) + \int_0^t AS(t-s)F(s,\ v(s))ds \\
\text{and} \\
(\psi_2 x)(t) = S(t)[x_0 - h(x)] \\
+ \int_0^t S(t-s)G(s,\ u(s))ds + \int_0^t S(t-s)\left[H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))d\tau\right)\right]ds,
\end{aligned} \tag{18}$$

for $0 \leq t \leq T$, and we will verify that ψ_1 is a contraction while ψ_2 is a compact operator.

To prove that ψ_1 is a contraction, we take $x_1,\ x_2 \in D_p$. Then for each $t \in [0,\ T]$ and by condition (A_1) and (10), we have

$$\begin{aligned}\|(\psi_1 x_1)(t)-(\psi_1 x_2)(t)\| &\le \|S(t)[F(0,\ v_1(0))-F(0,\ v_2(0))]\|+\|F(t,\ v_1(t))-F(t,\ v_2(t))\| \\ &+\left\|\int_0^t AS(t-s)[F(s,\ v_1(s))-F(s,\ v_2(s))]\,ds\right\| \\ &\le (M+1)M_0L_1\sup_{0\le s\le T}\|x_1(s)-x_2(s)\| + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1\sup_{0\le s\le T}\|x_1(s)-x_2(s)\|\,ds \\ &\le L_1\left[(M+1)M_0+\frac{C_{1-\beta}T^{\beta}}{\beta}\right]\sup_{0\le s\le T}\|x_1(s)-x_2(s)\| \le L_0\sup_{0\le s\le T}\|x_1(s)-x_2(s)\|.\end{aligned} \tag{19}$$

Thus $\|\psi x_1-\psi x_2\| \le L_0\|x_1-x_2\|$.
So by assumption $0<L_0<1$, we see that ψ_1 is a contraction.
To prove that ψ_2 is compact, firstly we prove that ψ_2 is continuous on D_p. Let $\{x_n\} \subseteq D_p$ with $x_n \to x$in D_p, then by$(A_2)(i)$, we have

$$G(s,\ u_n(s)) \to G(s,\ u(s)),\ \text{as } n\to\infty. \tag{20}$$

$$\begin{gathered}H\left(t,\ x_n(t),\ \int_0^t k(t,\ s,\ x_n(s))\,ds,\ \int_0^T g(t,\ s,\ x_n(s))\,ds\right) \to H\left(t,\ x(t),\ \int_0^t k(t,\ s,\ x(s))\,ds,\ \int_0^T g(t,\ s,\ x(s))\,ds\right), \\ \text{as } n\to\infty.\end{gathered} \tag{21}$$

Since $\|G(s,\ u_n(s))-G(s,\ u(s))\| \le 2g_p(s)$ and

$$\left\|H\left(t,\ x_n(t),\ \int_0^t k(t,\ s,\ x_n(s))\,ds,\ \int_0^T g(t,\ s,\ x_n(s))\,ds\right)-H\left(t,\ x(t),\ \int_0^t k(t,\ s,\ x(s))\,ds,\ \int_0^T g(t,\ s,\ x(s))\,ds\right)\right\| \le 2q_r(s), \tag{22}$$

by the dominated convergence theorem, we have

$$\begin{gathered}\|\psi_2 x_n-\psi_2 x\| = \sup_{0\le s\le T}\left\|S(t)[h(x)-h(x_n)]+\int_0^t S(t-s)[G(s,\ u_n(s))-G(s,\ u(s))]\,ds + \int_0^t S(t-s)\times\right. \\ \left.\left[H\left(s,\ x_n(s),\ \int_0^s k(s,\ \tau,\ x_n(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x_n(\tau))\,d\tau\right)-H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right]ds\right\| \\ \to 0,\ \text{as } n\to\infty\ i.e.\ \psi_2 \text{is continuous.}\end{gathered} \tag{23}$$

Next, we prove that $\{\psi_2 x : x\in D_p\}$ is a family of equicontinuous functions. To see this we fix $t_1>0$ and let $t_2>t_1$ and $\varepsilon>0$ be enough small. Then

$$\begin{aligned}\|(\psi_2 x)(t_2)-(\psi_2 x)(t_1)\| &\le \|S(t_2)-S(t_1)\|\,\|x_0-h(x)\|+\int_0^{t_1-\varepsilon}\|S(t_2-s)-S(t_1-s)\|\,\|G(s,\ u(s))\|\,ds \\ &+\int_{t_1-\varepsilon}^{t_1}\|S(t_2-s)-S(t_1-s)\|\,\|G(s,\ u(s))\|\,ds+\int_{t_1}^{t_2}\|S(t_2-s)\|\,\|G(s,\ u(s))\|\,ds \\ &+\int_0^{t_1-\varepsilon}\|S(t_2-s)-S(t_1-s)\|\left\|H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right\|ds \\ &+\int_{t_1-\varepsilon}^{t_1}\|S(t_2-s)-S(t_1-s)\|\left\|H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right\|ds \\ &+\int_{t_1}^{t_2}\|S(t_2-s)\|\left\|H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))\,d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))\,d\tau\right)\right\|ds.\end{aligned} \tag{24}$$

Noting that $\|G(s,\, u(s))\| \le g_p(s)$ and $g_p(s) \in L'$, we see that $\|(\psi_2 x)(t_2) - (\psi_2 x)(t_1)\|$ tends to zero independent of $x \in D_p$ as $t_2 - t_1 \to 0$ since the compactness of $S(t)\,(t>0)$ implies the continuity of $S(t)\,(t>0)$ in t in the uniform operators topology. We can prove that the functions $\psi_2 x,\ x \in D_p$ are equicontinuous at $t = 0$. Hence,ψ_2 maps D_p into a family of equicontinuous functions.
It remains to prove that $V(t) = \left\{(\psi_2 x)(t) :\ x \in D_p\right\}$ is relatively compact in X. $V(0)$ is relatively compact in X. Let $0 \le t \le T$ be fixed and $0 < \varepsilon < t$. For $x \in D_p$, we define

$$\begin{aligned}(\psi_{2,\,\varepsilon} x)(t) &= S(t)\left[x_0 - h(x)\right] + \int_0^{t-\varepsilon} S(t-s)\, G(s,\, u(s))\, ds \\ &+ \int_0^{t-\varepsilon} S(t-s)\left[H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\, d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\, d\tau\right)\right] ds \\ &= S(t)\left[x_0 - h(x)\right] + S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)\, G(s,\, u(s))\, ds \\ &+ S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)\left[H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\, d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\, d\tau\right)\right] ds.\end{aligned} \tag{25}$$

Then from the compactness of $S(\varepsilon)\,(\varepsilon > 0)$, we obtain $V_\varepsilon(t) = \left\{(\psi_{2,\varepsilon} x)(t) :\ x \in D_p\right\}$ is relatively compact in X for every $\varepsilon,\ 0 < \varepsilon < t$. Moreover, for every $x \in D_p$, we have

$$\begin{aligned}&\|(\psi_2 x)(t) - (\psi_{2,\,\varepsilon} x)(t)\| \\ &\le \int_{t-\varepsilon}^t \|S(t-s)\, G(s,\, u(s))\|\, ds + \int_{t-\varepsilon}^t \left\|S(t-s)\left[H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\, d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\, d\tau\right)\right]\right\| ds \\ &\le M\int_{t-\varepsilon}^t g_p(s)\, ds + M\int_{t-\varepsilon}^t q_r(s)\, ds.\end{aligned} \tag{26}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in X.
Thus ψ_2 is a compact operator by the Arzela-Ascoli theorem. These logics allow us to deduce that $\psi = \psi_1 + \psi_2$ is a condensing map on D_p, and by the fixed point theorem of Sadovskii there exists a fixed point $x(.)$ for ψ on D_p. Therefore, the equation (2) has a mild solution, and the proof is completed.

4 Application

As an application of Theorem3.1, we shall consider the system (2) with control parameter such as:

$$\begin{aligned}\frac{d}{dx}\left[x(t) + F(t,\, x(t),\, x(b_1(t)), \dots,\, x(b_m(t)))\right] + Ax(t) &= Cw(t) + G(t,\, x(t),\, x(a_1(t)),\, \dots,\, x(a_n(t))) \\ &+ H\left(t,\, x(t),\, \int_0^t k(t,\, s,\, x(s))\, ds,\, \int_0^T g(t,\, s,\, x(s))\, ds\right),\ t \in J = [0,\, T], \\ x(0) + h(x) &= x_0,\end{aligned} \tag{27}$$

where the control function $w(\,.\,)$ is given in $L^2(J,\, W)$- the Banach space of admissible control function with W as a Banach space and C is a bounded linear operator from W into X. The mild solution of the system (27) is given by

$$\begin{aligned}x(t) &= S(t)\left[x_0 + F(0,\, v(0)) - h(x)\right] - F(t,\, v(t)) + \int_0^t AS(t-s)\, F(s,\, v(s))\, ds \\ &+ \int_0^t S(t-s)\left[Cw(s) + G(s,\, u(s)) + H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\, d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\, d\tau\right)\right] ds.\end{aligned} \tag{28}$$

Definition 4.1. The system (27) is said to be controllable on the interval J if for every $x_0,\ x_1 \in X$, there exists a control $w \in L^2(J,\, W)$such that the mild solution $x(\,.\,)$ of (27) satisfies

$$x(0) + h(x) = x_0 \text{ and } x(T) = x_1. \tag{29}$$

To establish the result, we need the following additional condition:

(A_5) The linear operator $Q : L^2(J, W) \to X$ defined by

$$Qw = \int_0^T S(T-s)Cw(s)\,ds, \tag{30}$$

has an induced inverse operator $\tilde{Q}^{-1}$,which takes values in $L^2(J, W)/\ker Q$ and there exists a positive constant M_1 such that $\left\|C\tilde{Q}^{-1}\right\| \le M_1$.

For the construction of the operator Q and its inverse, see [21].

Theorem 4.1. If the assumptions (A_1) - (A_5) are satisfied then the system (27) is controllable on J if

$$L_0 := \left[(M+1)M_0 + \frac{C_{1-\beta}T^{\beta}}{\beta}\right]L_1 < 1. \tag{31}$$

$$(1+MM_1T)\left(ML_3 + M_0L_2M + M_0L_2 + \frac{C_{1-\beta}L_2T^{\beta}}{\beta} + M\gamma_1 + M\gamma_2\right) < 1. \tag{32}$$

Proof: Using the assumption(A_5), for an arbitrary function $x(\,.)$, define the control

$$\begin{aligned} w(t) = \tilde{Q}^{-1}\Bigg[&x_1 - S(T)\{x_0 + F(0,\, v(0)) - h(x)\} + F(T,\, v(T)) - \int_0^T AS(T-s)F(s,\, v(s))\,ds \\ &- \int_0^T S(T-s)\left\{G(s,\, u(s)) + H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\,d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\,d\tau\right)\right\}ds\Bigg](t). \end{aligned} \tag{33}$$

We shall show that when using this control the operator φ defined by

$$\begin{aligned} (\varphi x)(t) &= S(t)[x_0 + F(0,\, v(0)) - h(x)] - F(t,\, v(t)) + \int_0^t AS(t-s)F(s,\, v(s))\,ds \\ &+ \int_0^t S(t-s)\left[Cw(s) + G(s,\, u(s)) + H\left(s,\, x(s),\, \int_0^s k(s,\, \tau,\, x(\tau))\,d\tau,\, \int_0^T g(s,\, \tau,\, x(\tau))\,d\tau\right)\right]ds, t \in J, \end{aligned} \tag{34}$$

has a fixed point $x(\,.)$. Then this fixed point $x(\,.)$ is a mild solution of the problem (27), and we can easily verify that$x(T) = (\varphi x)(T) = x_1$. This means that the control w steers the system from the initial state x_0 to x_1 in time T, which implies that the system is controllable.

Our aim is to prove that there exists a positive integer p such that$\varphi D_p \subseteq D_p$. If possible, for each positive integer p, there is a function $x_p(\,.) \in D_p$, but $\varphi x_p \notin D_p$, that is $\left\|\varphi x_p(t)\right\| > p$ for $t \in [0,\ T]$, from

$$\begin{aligned}
p < \left\|(\varphi x_p)(t)\right\| = &\bigg\| S(t)\left\{x_0 - h(x_p) + F(0,\ v_p(0))\right\} - F(t,\ v_p(t)) + \int_0^t AS(t-s)F(s,\ v_p(s))\ ds \\
&+ \int_0^t S(t-s)G(s,\ u_p(s))ds + \int_0^t S(t-s)\left\{H\left(s,\ x_p(s),\ \int_0^s k(s,\ \tau,\ x_p(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x_p(\tau))d\tau\right)\right\}ds \\
&+ \int_0^t S(t-s)C\tilde{Q}^{-1}\left[x_1 - S(T)\left\{x_0 + F(0,\ v_p(0)) - h(x_p)\right\} + F(T,\ v_p(T)) - \int_0^T AS(T-s)F(s,\ v_p(s))ds\right. \\
&\left. - \int_0^T S(T-s)\left\{G(s,\ u_p(s)) + H\left(s,\ x_p(s),\ \int_0^s k(s,\ \tau,\ x_p(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x_p(\tau))d\tau\right)\right\}\right](s)ds\bigg\| \\
&\le M\left\{\|x_0\| + L_3 p + L_3' + \left\|A^{-\beta}A^{\beta}F(0,\ v_p(0))\right\|\right\} + \left\|A^{-\beta}A^{\beta}F(t,\ v_p(t))\right\| \\
&+ \left\|\int_0^t A^{1-\beta}S(t-s)A^{\beta}F(s,\ v_p(s))\ ds\right\| + M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds \\
&+ M_1 M\int_0^t \left[\|x_1\| + M\left\{\|x_0\| + L_3 p + L_3' + \left\|A^{-\beta}A^{\beta}F(0,\ v_p(0))\right\|\right\} + \left\|A^{-\beta}A^{\beta}F(T,\ v_p(T))\right\|\right. \\
&\left. + \left\|\int_0^T A^{1-\beta}S(T-s)A^{\beta}F(s,\ v_p(s))ds\right\| + +M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds\right]ds \\
&\le M\left[\|x_0\| + L_3 p + L_3'\right] + M_0 L_2(p+1)(M+1) + \frac{C_{1-\beta}L_2(p+1)T^{\beta}}{\beta} + M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds \\
&+ MM_1\left[\|x_1\| + M\left\{\|x_0\| + M_0 L_2(p+1) + L_3 p + L_3'\right\} + M_0 L_2(p+1) + \int_0^T \frac{C_{1-\beta}L_2(p+1)}{(T-s)^{1-\beta}}ds\right. \\
&\left. + M\int_0^T g_p(s)ds + M\int_0^T q_r(s)ds\right]T.
\end{aligned} \tag{35}$$

Dividing into both sides by p and taking the lower limit as $p \to \infty$, we get

$$\begin{aligned}
1 \le\ & ML_3 + M_0 L_2 M + M_0 L_2 + \frac{C_{1-\beta}L_2 T^{\beta}}{\beta} + M\gamma_1 + M\gamma_2 \\
&+ MM_1 T\left(M_0 L_2 M + L_3 M + M_0 L_2 + \frac{C_{1-\beta}L_2 T^{\beta}}{\beta} + M\gamma_1 + M\gamma_2\right) \\
\Rightarrow 1 \le\ & (1 + MM_1 T)\left(ML_3 + M_0 L_2 M + M_0 L_2 + \frac{C_{1-\beta}L_2 T^{\beta}}{\beta} + M\gamma_1 + M\gamma_2\right).
\end{aligned} \tag{36}$$

However, this contradicts (32). Hence for positive integer p, $\varphi D_p \subseteq D_p$.
In order to apply Sadovskii's fixed point theorem, we decompose φ as $\varphi = \varphi_1 + \varphi_2$, where the operators $\varphi_1,\ \varphi_2$ are defined on D_p, by

$$\begin{aligned}
(\varphi_1 x)(t) &= S(t)F(0,\ v(0)) - F(t,\ v(t)) + \int_0^t AS(t-s)F(s,\ v(s))ds, \\
(\varphi_2 x)(t) &= S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)G(s,\ u(s))ds + \int_0^t S(t-s)\left[H\left(s,\ x(s),\ \int_0^t k(t,\ \tau,\ x(\tau))d\tau,\ \int_0^T g(t,\ \tau,\ x(\tau))d\tau\right)\right]ds \\
&+ \int_0^t S(t-s)C\tilde{Q}^{-1}\left[x_1 - S(T)\left\{x_0 + F(0,\ v(0)) - h(x)\right\} + F(T,\ v(T)) - \int_0^T AS(T-s)F(s,\ v(s))ds\right. \\
&\left. - \int_0^T S(T-s)\left(G(s,\ u(s)) + H\left(s,\ x(s),\ \int_0^s k(s,\ \tau,\ x(\tau))d\tau,\ \int_0^T g(s,\ \tau,\ x(\tau))d\tau\right)\right)ds\right](s)ds,
\end{aligned} \tag{37}$$

for $t \in J$. We have already proved that φ_1 verify a contraction condition. The proof that φ_2 is a compact operator can be completed by a similar manner as we have done it in Theorem 3.1, and hence it is omitted.

5 Conclusion

In this article, the existence of the mild solution for neutral functional mixed integrodifferential evolution equations with nonlocal conditions in general Banach spaces is discussed. We have applied fractional power of operators and Sadovskiiâs fixed point theorem to establish the result. An application is provided to illustrate the obtained result.

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