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Reflexive Idempotent Property Skewed by Ring Endomorphism

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Abstract: The notion of an α-skew reflexive idempotent ring has been introduced in this paper to extend the concept of skew reflexive idempotent ring and that of an α-rigid ring. First basic properties of α-skew reflexive idempotent rings have been considered, including some examples needed in the process. It has been prove that for a ring R with an endomorphism α and $n \geq 2$, if R satisfies the condition "$eRfRfR = 0$ implies $eRf = 0$ "and R is a right α-skew RIP ring, then $V_n(R)$ is a right $\bar{\alpha}$-skew RIP ring. Also it has proven that if R is an algebra over a field K and D the Dorroh extension of R by K, where α is an endomorphism of R with $\alpha(1) = 1$, then R is a right α-skew RIP ring if and only if D is a right $\bar{\alpha}$-skew RIP ring. It's shown that if M is a multiplicative closed subset of a ring R consisting of central regular elements and α an automorphism of R, then R is right α-skew RIP if and only if $M^{-1}R$ is right $\bar{\alpha}$-skew RIP.

Keywords: reflexive ring, reflexive idempotent ring, α-skew RMI rings, matrix ring, α-rigid ring, Dorroh extension.

1 Introduction

Throughout this paper, all rings are associative with identity. We denote by $R[x]$ the polynomial ring with an indeterminate x over R. Let $\mathbb{Z}$ (resp., $\mathbb{Z}_n$) denotes the ring of integers (resp., the ring of integers modulo n). Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Denote $\{(a_{ij}) \in U_n(R) \mid \text{the diagonal entries of } (a_{ij}) \text{ are all equal}\}$ by $D_n(R)$. Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. Let $Id(R)$ be the set of all idempotent elements of a ring R.

Recall that a ring is *reduced* if it has no non-zero nilpotent elements. A ring R is called *reversible* [7] if $ab = 0$ implies $ba = 0$, for $a, b \in R$, and a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, *IFP* ring) [5] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. A ring R is called *Ablian* if every idempotent is central. Commutative rings and reduced rings are clearly reversible. A simple computation gives that reversible rings are IFP and IFP rings are Abelian, but the converses does not hold in general. We will freely use these facts without reference.

A ring R is called *semi-prime* if $aRa = 0$, for every $a \in R$ implies $a = 0$, every semiprime rings is reduced but clearly the converse is not true.

Generalized reduced rings were extended by ring endomorphisms. According to Krempa [17], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called α*-rigid* [12] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by [12, Proposition 5]. By Hashemi and Moussavi [9], a ring R is α*-compatible* if for each $a, b \in R$, $a\alpha(b) = 0$ if and only if $ab = 0$. Therefore every α-rigid ring is α-compatible, but the converse is not true.

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In [4, Definition 2.1], an endomorphism α of a ring R is called *right skew reversible* if whenever $ab = 0$ for $a,b \in R$, $b\alpha(a) = 0$, and the ring R is called *right α-skew reversible* if there exists a right skew reversible endomorphism α of R. Similarly, left α-skew reversible rings are defined. A ring R is called *α-skew reversible* if it is both left and right α-skew reversible. Note that R is an α-rigid ring if and only if R is semiprime and right α-skew reversible for a monomorphism α of R by [4, Proposition 2.5(iii)]. (We change over from "an α-reversible ring" in [4] to "an α-skew reversible ring" to cohere with other related definitions).

In [21, Definition 2.1], an endomorphism α of a ring R is called *right (resp., left) skew reflexive* if whenever $aRb = 0$ for $a,b \in R$, $bR\alpha(a) = 0$, and the ring R is called *right (resp., left) α-skew reflexive* if there exist a right (resp., left) skew reflexive endomorphism α of R. A ring R is *α-skew reflexive* if it is both right and left α-skew reflexive. A ring R is reflexive if R is 1_R-reflexive ring, where 1_R denotes the identity endomorphism of R. Any domain R is obviously α-skew reflexive for any endomorphism α of R, but the converse need not hold by the [21, Example 2.2].

In [20], a ring R is said to have *the reflexive-idempotents-property (simply, RIP)* if R satisfies the property that

$$eRf = 0 \text{ implies } fRe = 0 \text{ for any } e,f \in Id(R).$$

A ring R is called RIP if it satisfies the reflexive-idempotents-property. It can be easily checked that every one-sided idempotent reflexive ring is RIP,entailing that Abelian rings are RIP.

In [2, definition 3.1], an endomorphism α of a ring R is called a *right (resp., left) skew RMI* if whenever $eMf = 0$ for $e,f \in Id(R)$ and for a maximal ideal M of R, $fM\alpha(e) = 0$ (resp., $\alpha(f)Me = 0$). A ring R is called *right (resp., left) α-skew RMI* if there exists a right (resp., left) skew RMI endomorphism α of R. A ring R is called *α-skew RMI* if it is both left and right α-skew RMI. Both simple ring and domains are obviously α-skew RMI ring for any endomorphism α of given ring R, but the converses need not hold.

Motivated by the above facts, the concepts of α-skew RIP has been introduced , as a generalization of α-rigid rings. First basic examples and properties of α-skew RIP rings has been found. Also an α-skew RIP property of some kind of polynomials have been discussed. It has been prove that for a ring R with an endomorphism α and $n \geq 2$. If R satisfies the condition "$eRfRfR = 0$ implies $eRf = 0$ "and R is a right α-skew RIP ring, then $V_n(R)$ is a right $\bar{\alpha}$-skew RIP ring. Also it has proven that if R is an algebra over a field K, and D be the Dorroh extension of R by K where α be an endomorphism of R with $\alpha(1) = 1$ then R is a right α-skew RIP ring if and only if D is a right $\bar{\alpha}$-skew RMI ring. It shown that if M is a multiplicative closed subset of a ring R consisting of central regular elements and α an automorphism of R, then R is right α-skew RIP if and only if $M^{-1}R$ is right $\bar{\alpha}$-skew RIP.

Throughout this paper, α denotes a nonzero endomorphism of given rings, unless specified otherwise.

2 Basic properties and characterizations of right α-skew RIP rings

In this section basic properties, characterizations and basic extensions of α-skew RIP rings and related concepts have been observed, including some kind of examples needed in the process. We begin with the following definition.

Definition 1. An endomorphism α of a ring R is said to have the *right (resp., left) skew reflexive idempotent property* (simply, skew RIP) if α satisfy the property

$$eRf = 0 \text{ for } e,f \in Id(R),\ fR\alpha(e) = 0 \text{ (resp., } \alpha(f)Re = 0).$$

A ring R is called *right (resp., left) α-skew RIP* if there exists a right (resp., left) skew RIP endomorphism α of R. A ring R is called *α-skew RIP* if it is both left and right α-skew RIP.

A ring R is RIP if R is 1_R-RIP, where 1_R denotes the identity endomorphism of R. Any domains R is obviously α-skew RIP ring for any endomorphism α of given ring R, but the converses need not hold by the following example, which also shows that the α-skew RIP property is not left-right symmetric.

Example 1. Let S be an RIP ring. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in S \right\}.$$

(1) Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

For

$$E = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, F = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \text{ and } E, F \in Id(R),$$

assume that $ERF = 0$. Then for any $\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in R$, we have $aua' = 0$ and so $aSa' = 0$. Since S is RIP, $aSa' = 0$. This entails that $FR\alpha(E) = 0$ and hence R is a right α-skew RIP ring.

However, R is not left α-skew RIP. Since if we take

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } E, F \in Id(R).$$

Then, $ERF = 0$,

$$0 \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^2\right) = \alpha(F)E^2 \in \alpha(F)RE,$$

showing that R is not left α-skew RIP.

(2) Let $\beta : R \to R$ be an endomorphism defined by

$$\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By the similar method to the proof (1), we can show that R is left β-skew RIP, but not right β-skew RIP.

Consider the reverse condition of a right α-skew RIP ring R,

$$(*) \qquad eR\alpha(f) = 0 \text{ for } e, f \in Id(R) \text{ implies } fRe = 0.$$

The reverse condition of a right α-skew RIP ring does not hold by the following example.

Example 2. Recall the right α-skew RIP

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in \mathbb{Z}_2 \right\},$$

with the endomorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

as in Example 1, R does not satisfy the condition (*): For

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} E, F \in Id(R), \text{ and } C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R,$$

we have $0 = EC\alpha(F) \in ER\alpha(F)$, but

$$0 \neq \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = ECF \in FRE.$$

Lemma 1. *Every α-rigid ring is a reduced α-skew RIP with α a monomorphism.*

Proof. Let R be α-rigid. Then R is reduced ring and α is a monomorphism. Assume that $eRf = 0$, for $e, f \in Id(R)$. Then $erf = 0$, for all $r \in R$ so $efr = 0$, since R is reduced, so we obtain $fr\alpha(e)\alpha(fr\alpha(e)) = fr\alpha(efr)\alpha^2(e) = 0$. This shows that $fr\alpha(e) = 0$ so $fR\alpha(e) = 0$, since R is α-rigid. Hence R is right α-skew RIP. Now by the similar argument to above from $eRf = 0$, we also have $ref = 0$, for all $r \in R$ since R is reduced. Thus we get $\alpha(\alpha(f)re)\alpha(f)re = \alpha^2(f)\alpha(ref)re = 0$. This shows that $\alpha(f)re = 0$, so $\alpha(f)Re = 0$, since R is α-rigid. This shows that R is left α-skew RIP. Whence R is α-skew RIP. □

The converse of Lemma 1 does not hold by the following example.

Example 3. Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a,b \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

It is clear that α is an automorphism, R is not semiprime and hence R is not α-rigid.

Now, for $E, F \in Id(R)$, let $EBF = 0$, for all $B \in R$ with

$$E = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \ F = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} m & n \\ 0 & m \end{pmatrix}.$$

Then we have $amc = 0$ and $amd + anc + bmc = 0$, for every m. This implies that $a = 0$ or $c = 0$. If $a = 0$, then $bmc = 0$ and hence $b = 0$ or $c = 0$. If $c = 0$, then $amd = 0$ for each $m \in R$, and hence $a = 0$ or $d = 0$. In both cases, we obtain $FB\alpha(E) = 0$. Hence R is right $\bar{\alpha}$-skew RIP. Similarly we obtain R is a left $\bar{\alpha}$-skew RIP. Therefore, R is $\bar{\alpha}$-skew RIP.

For a non-empty subset X of a ring R, we write $r_R(X) = \{c \in R \mid Xc = 0\}$, which is called the *right annihilator* of X in R. Similarly, $\ell_R(X)$ denotes the left annihilator of X in R. We have the basic equivalence for right α-skew RIP rings as follows.

Proposition 1. *For a ring R with an endomorphism α, the following are equivalent:*

(1) *R is a right α-skew RIP ring.*
(2) *For $e \in Id(R)$, $r_R(eR) \subseteq \ell_R(R\alpha(e))$.*
(3) *For non-empty subsets E and F of $Id(R)$, $ERF = 0$ implies $FR\alpha(E) = 0$.*
(4) *$IJ = 0$ implies $J\alpha(I) = 0$, where I and J are any right (or left) ideals of $Id(R)$.*
(5) *$IJ = 0$ implies $J\alpha(I) = 0$, where I and J are any ideals of $Id(R)$.*

Proof. (1)⇔(2) and (3)⇒(4)⇒(5) are straightforward.

(2)⇒(3): Assume that the condition (2) holds. Let E and F be two non-empty subsets of $Id(R)$ with $ERF = 0$. Then $eRf = 0$, for any $e \in E$, $f \in F$ and hence $fR\alpha(e) = 0$ by assumption. Thus, $FR\alpha(E) = \sum_{e\in E, f\in F} fR\alpha(e) = 0$.

(5)⇒(1): Assume that the condition (5) holds. Let $eRf = 0$, and $e, f \in Id(R)$. Then $ReRRfR = 0$ and so the condition (5) implies that $fR\alpha(e) \subseteq RfR\alpha(ReR) = 0$. We conclude that R is a right α-skew RIP ring. □

Proposition 2. *Let α be an endomorphism of a ring R.*

(1) *Let R be a reversible ring. Then R is right α-skew RIP if and only if R is left α-skew RIP.*
(2) *If R is an α-skew RIP ring, then $eRf = 0$ for $e, f \in Id(R)$ implies $eR\alpha^{2k}(f) = 0$ and $\alpha^{2k-1}(f)Re = 0$, for a positive integer k.*

Proof. (1) Let R be a reversible ring. Suppose that R is right α-skew RIP and $eRf = 0$ for $e, f \in Id(R)$. Then $ef = 0$ implies $fe = 0$ implies $fRe = 0$, so $eR\alpha(f) = 0$ then $e\alpha(f) = 0$, so $\alpha(f)e = 0$ hence $\alpha(f)Re = 0$, so R is left α-skew RIP. The converse can be shown by an almost same argument as above.

(2) Suppose that R is α-skew RIP and $eRf = 0$ for $e, f \in Id(R)$. Then $\alpha(f)Re = 0$ implies $eR\alpha^2(f) = 0$ implies $\alpha^3(f)Re = 0$, then $eR\alpha^4(f) = 0, \cdots$, so we get $eR^{2k}(f) = 0$ and $\alpha^{2k-1}(f)Re = 0$, for $k \geqslant 1$ inductively. The reminder of the proof is similar to above. □

Proposition 3. *Let α be an endomorphism of a ring R, S a ring and suppose that $\sigma : R \to S$ a ring isomorphism. Then R is a right α-skew RIP ring if and only if S is a right $\sigma\alpha\sigma^{-1}$-skew RIP ring.*

Proof. Suppose R is a right α-skew RIP. Let $e'Sf' = 0$, for $e', f' \in Id(S)$, then there exists $e, f \in Id(R)$ such that $\sigma(e) = e'$, $\sigma(f) = f'$ and $\sigma(R) = S$.

So $e'Sf' = 0$ implies $\sigma(e)\sigma(R)\sigma(f) = \sigma(0)$ and $\sigma(eRf) = \sigma(0)$, so $eRf = 0$, since σ is an isomorphism. Then $fR\alpha(e) = 0$ by hypothesis. This entails that

$$0 = \sigma(0) = \sigma(fR\alpha(e)) = \sigma(fR\alpha((\sigma^{-1}\sigma)(e))) = \sigma(f)\sigma(R)\sigma(\alpha(\sigma^{-1}(\sigma(e))) = f'S(\sigma\alpha\sigma^{-1})(e') = 0.$$

Thus S is a right $\sigma\alpha\sigma^{-1}$-skew RIP ring.

Conversely, let S be a right $\sigma\alpha\sigma^{-1}$-skew RIP and let $eRf = 0$ for $e, f \in Id(R)$. Set $\sigma(e) = e', \sigma(f) = f'$ and $\sigma(R) = S$. Then $e', f' \in Id(S)$ such that $e'Sf' = \sigma(eRf) = 0$. Since S is a right $\sigma\alpha\sigma^{-1}$-skew RIP, we have $0 = f'S(\sigma\alpha\sigma^{-1})(e') = 0$, so $\sigma(fR\alpha(e)) = 0$. Since σ is an isomorphism $fR\alpha(e) = 0$. Shows that R is a right α-skew RIP ring. □

For an endomorphism α and an idempotent e of a ring R let $\alpha(e)=e$. Then we have an endomorphism $\bar{\alpha}: eRe \to eRe$ defined by $\bar{\alpha}(ere)=e\alpha(r)e$.

Proposition 4. *Let R be a ring with an endomorphism α such that $\alpha(e)=e$, for $e^2=e\in R$.*

(1) *If R is a right α-skew RIP ring, then eRe is a right $\bar{\alpha}$-skew RIP ring , for all $e\in Id(R)$.*
(2) *If e is central for $e\in Id(R)$, then R is a right α-skew RIP ring if and only if eR and $(1-e)R$ are right $\bar{\alpha}$-skew RIP ring.*

Proof. (1) Let R be a right α-skew RIP ring and $e\in Id(R)$. Suppose $e'Rf'=0$ for $e',f'\in eRe$. Note that $ee'e=e'$ and $ef'e=f'$. By claim, hence, $0=e'Rf'=e'eRef'=ee'e(eRe)ef'e=ee'eRefe=e'Rf'=f'R\alpha(e')$, since R is a right α-skew RIP ring. Thus, $0=f'R\alpha(e')=ef'e(eRe)e\alpha(e')e=ef'eR\bar{\alpha}(ee'e)=f'R\alpha(e')$. Showing that eRe is a right $\bar{\alpha}$-skew RIP ring.

(2) Let $e\in Id(R)$ be central. Suppose that eR and $(1-e)R$ are right $\bar{\alpha}$-skew RIP rings, let $e'Rf'=0$ for $e',f'\in R$. Recall that eR and $(1-e)R$, respectively. By the proof of (1). Then $ee'(eR)ef'=0$ and $(1-e)e'((1-e)R)(1-e)f'=0$. By hypothesis, we have $0=ef'(eR)f'e'r\alpha(ee')=ef'(eR)e\alpha(e')=ef'R\alpha(e')$ and $0=(1-e)f'((1-e)R)f'e'r\alpha((1-e)e')=(1-e)f'((1-e)R)((1-e)\alpha(e'))=(1-e)f'R\alpha(e')$.

Thus, $f'R\alpha(e')=ef'R\alpha(e')+(1-e)f'R\alpha(e')=0$, showing that R is a right α-skew RIP ring.

The converse part follows from (1) directly. □

The next Theorem gives the relationship between RIP and $\alpha-$skew RIP rings under the condition that R is an α-compatible ring.

Theorem 1. *Let R be an α-compatible ring. Then R is an RIP ring if and only if R is α-skew RIP ring.*

*Proof.*Let R be an RIP ring and $eRf=0$ for $e,f\in Id(R)$, then $fRe=0$. So $erf=0$, for all $r\in R$ and $fre=0$ by hypothesis and $fc\alpha(e)=0$ for all $c\in R$ (by α-compatibility), hence $fR\alpha(e)=0$, and so R is right α-skew RIP ring. By the same method in above we get R is left α-skew RIP ring. Therefore R is an α-skew RIP ring.

Conversely, let R be an α-skew RIP ring and $eRf=0$ for $e,f\in Id(R)$, so $ecf=0$ for all $c\in R$. Then $fc\alpha(e)=0$ by hypothesis and $fce=0$ for all $c\in R$(by α-compatibility), so $fRe=0$. Therefore R is an RIP ring. □

According the following Example, we see that the condition "α-compatibility "is not superfluous in Theorem 1.

Example 4. Consider the ring $R=\mathbb{Z}_2\oplus\mathbb{Z}_2$ with endomorphism $\alpha: R\to R$ defined by $\alpha((a,b))=(b,a)$ with the usual addition and multiplication α is not compatible, for $a=(0,1), b=(1,0)\in Id(R)$, we have $ab=0$, but $(0,0)\neq(0,1)^2=a\alpha(b)$. The ring R is a commutative semiprime ring hence it is RIP. However, R is not an α-skew RIP. Indeed, for $a=(0,1), b=(1,0)\in Id(R)$ and $(1,1)\in R$, we have $(0,1)(1,1)(1,0)=(0,0)$, but $(1,0)(1,1)(1,0)=(1,0)\neq(0,0)\in bR\alpha(a)$.

Recall that for a ring R with an endomorphism α and an ideal I of R, if I is an α-ideal (i.e., $\alpha(I)\subseteq I$) of R, then $\bar{\alpha}: R/I\to R/I$ defined by $\bar{\alpha}(a+I)=\alpha(a)+I$, for $a\in R$ is an endomorphism of a factor ring R/I. The class of right α-skew RIP ring is not closed under homomorphic images and vice versa in general, by help of [21, Example 2.8 and Example 2.9].

Proposition 5.*Let R be a ring with an automorphism α and I an α-ideal of R. If R/I is a right $\bar{\alpha}$-skew RIP and I is α-rigid as a ring without identity, then R is a right α-skew RIP ring.*

Proof. Suppose that R/I is a right $\bar{\alpha}$-skew RIP and I is α-rigid as a ring without identity. Let $eRf = 0$ for $e, f \in Id(R)$. Then $(e+I)(f+I) = I$ and $e+I, f+I \in Id(R/I)$. Since R/I is right $\bar{\alpha}$-skew RIP, $fR\alpha(e) \subseteq I$. Hence $fR\alpha(e)R\alpha(fR\alpha(e)R) = fR\alpha(eRf)\alpha(R\alpha(e)R) = 0$ implies $fR\alpha(e) = 0$ since I is an α-rigid ring. Thus, R is right α-skew RIP. □

The next example illuminates the condition "I is α-rigid as a ring without identity "of Proposition 5 cannot be weakened by the condition "I is a right α-skew RIP as a ring without identity ".

Example 5. Consider $R = U_3(F)$ over a division ring F and an automorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

For

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } E, F \in Id(R),$$

we have $ERF = 0$, but $0 \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = F^2\alpha(E) = FR\alpha(E)$, showing that R is not right α-skew RIP.

Clearly, the ideal $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of R is right α-skew RIP, but not α-rigid (as a ring without identity), and the factor ring

$$R/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \mid a, c \in F \right\}.$$

Thus R/I is right $\bar{\alpha}$-skew RIP.

Proposition 6. *Let A be a commutative ring satisfying a condition that $ef = 0$ for $e, f \in Id(A)$ implies $e = -e$ or $f = -f$. Then the ring*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in A \right\}$$

is α-skew RIP, where α is an automorphism of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Proof. Let $EAF = 0$, for non-zero

$$E = \begin{pmatrix} e & f \\ 0 & e \end{pmatrix}, F = \begin{pmatrix} e_1 & f_1 \\ 0 & e_1 \end{pmatrix} \in Id(R).$$

Then $EF = 0$, so we have $ee_1 = 0$ and $ef_1 + fe_1 = 0$.

Case 1. Either $e = 0$ or $e_1 = 0$. Let $e = 0$. Then $f \neq 0$ and $fe_1 = 0$, entailing $e_1 r(-f) = 0$ for all $r \in R$. This yields

$$\begin{pmatrix} e_1 & f_1 \\ 0 & e_1 \end{pmatrix}\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} 0 & -f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_1 rf \\ 0 & 0 \end{pmatrix} = 0$$

for every $\begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \in R$, and hence $FR\alpha(E) = 0$. The computation for the case of $e_1 = 0$ is similar, also obtaining $FR\alpha(E) = 0$.

Case 2. If $e \neq 0$ and $e_1 \neq 0$, then we have $e = -e$ or $e_1 = -e_1$ by the condition of A. Since A is commutative, then we get

$$e_1re = 0, e_1rf + f_1re = (-e_1)r(-f) + f_1re = 0,$$

and

$$e_1r(-f) + f_1r(-e) = (-e_1)rf + f_1re = 0,$$

for all $r \in R$. If $e = -e$, then we get

$$0 = e_1r(-f) + f_1r(-e) = e_1r(-f) + f_1re = 0.$$

If $e_1 = -e_1$, then we have

$$0 = -((-e_1)rf + f_1r(-e)) = -(-e_1)rf - f_1r(-e) = (-e_1)r(-f) + f_1re = e_1r(-f) + f_1re = 0.$$

Consequently, we get $FR\alpha(A)$ in any case. □

3 Extensions of right α-skew RIP ring

In this section several kinds of ring extensions which have role in ring theory are extended , being concerned with right α-skew RIP rings.

Given a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used. Note that $T(R,R) = D_2(R)$ and For an endomorphism α of a ring R and the trivial extension $T(R,R)$ of R, $\bar{\alpha} : T(R,R) \to T(R,R)$ defined by

$$\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix},$$

is an endomorphism of $T(R,R)$. Since $T(R,0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ by $T(R,0)$ to α. We have the following.

Proposition 7. *If R is a division ring, then $T(R,R)$ is a right $\bar{\alpha}$-skew RIP.*

Proof. Let R be a division ring. Suppose that $ET(R,R)F = 0$, for

$$E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, F = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \text{ and } E,F \in Id(T(R,R)).$$

Then, we have

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = 0,$$

implies $erf = 0$. Since R is a division ring, so R is a domain and $erf = 0$ implies $e = 0$ or $rf = 0$, so we have $e = 0$ or $r = 0$ or $f = 0$, so from

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha(e) & 0 \\ 0 & \alpha(e) \end{pmatrix} = 0,$$

we get $fr\alpha(e) = 0$ and this implies that $FT(R,R)\bar{\alpha}(E) = 0$. Thus $T(R,R)$ is a right $\bar{\alpha}$-skew RIP ring. □

For $n \geq 2$, let $Mat_n(R)$ (resp., $U_n(R)$) denote the $n \times n$ full matrix (rep., upper triangular matrix ring) over a ring R. For an endomorphism α of R, the map $\bar{\alpha} : Mat_n(R) \to Mat_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ is an endomorphism of $Mat_n(R)$. Note that the extended map $\bar{\alpha}$ of any subring S with $\alpha(S) \subseteq S$ of $Mat_n(R)$ for $n \geq 2$ is similarly defined component-wise. Use e_{ij} for the matrix (i, j)-entry 1 and elsewhere 0.

Proposition 8. *Let R be a ring with endomorphism α. Then R is a right α-skew RIP ring if and only if $Mat_n(R)$ is a right $\bar{\alpha}$-skew RIP ring, for all $n \geq 2$.*

Proof. Suppose that R is a right α-skew RIP ring. Let $S = Mat_n(R)$ and $EF = 0$ for ideals E, F of $Id(S)$. Using an elementary ring theoretic argument, there exists ideals I and J and of $Id(R)$ such that $E = Mat_n(I), F = Mat_n(J)$. Then

$$Mat_n(IJ) = Mat_n(I)Mat_n(J) = EF = 0,$$

implies $IJ = 0$. Since R is a right α-skew RIP ring, then $J\alpha(I) = 0$ by Proposition 1 (5). This yields $F\bar{\alpha}(E) = 0$, and so S is a right $\bar{\alpha}$-skew RIP ring for all $n \geq 2$ by Proposition 1 (5).

Conversely, Suppose that $Mat_n(R)$ is right $\bar{\alpha}$-skew RIP for all $n \geq 2$. Let $eRf = 0$ for $e, f \in Id(R)$. For $E = e\sum_{i=1}^{n} E_{ii}$, $F = f\sum_{i=1}^{n} E_{ii} \in Id(Mat_n(R))$, we have $EMat_n(R)F = 0$, and so $FMat_n(R)\bar{\alpha}(E) = 0$ by hypothesis. This implies that $fR\alpha(e) = 0$, showing that R is right α-skew RIP ring. □

The following example shows that both $U_n(R)$ and $D_m(R)$ over any ring R cannot be right $\bar{\alpha}$-skew RIP ring for any $n \geq 2$ and $m \geq 3$ respectively, and hence the class of right α-skew RIP rings is not closed under subrings, noting that $Mat_n(R)$ over right α-skew RIP ring for $n \geq 2$ is right $\bar{\alpha}$-skew RIP.

Example 6. Let α be an endomorphism of any non-zero ring R with $\alpha(1) = 1$(e.g., R is an α-rigid ring).

(1) Consider a ring $S = U_n(R)$ for $n \geq 2$. For $e = E_{22},\ f = E_{11} \in Id(U_n(R))$, we have $eSf = 0$. But $fS\bar{\alpha}(e) = RE_{12} \neq 0$. This shows that $U_n(R)$ is not right $\bar{\alpha}$-skew RIP for $n \geq 2$.

(2) Consider a ring $S = D_n(R)$ for $n \geq 3$. For $e = E_{(n-1)n}, f = E_{(n-2)(n-1)} \in Id(D_n(R))$, we have $eSf = 0$, but $fS\bar{\alpha}(e) \neq 0$, showing that $D_n(R)$ is not right $\bar{\alpha}$-skew RIP for $n \geq 3$.

Note that if R is an α-rigid ring. Then the ring $S = D_3(R)$ is not $\bar{\alpha}$-skew RIP by (2). For $e = E_{23} = f \in Id(D_3(R))$, we have $eSf = 0$ and for any $n \geq 0$, we get $eS\bar{\alpha}^n(f) = 0$. This illuminates that the converse of the Proposition 2(2) does not hold in general.

The converse of Proposition 4(1) does not hold in general. For the non right α-skew RMI R, $R = U_2(Z)$ in Example 6(1) but for $e = e_{11} \in Id(R), eRe \cong Z$ is a right α-skew RIP clearly for any endomorphism α.

Proposition 9. *Let α be an endomorphism of a ring R. If R is a right α-skew reversible ring, then R is right α-skew RIP.*

Proof. Assume that R is α-skew RIP and $eRf = 0$ for $e, f \in Id(R)$. Then $efr = 0$ for any $r \in R$ and so $(fr)\alpha(e) = 0$. Thus $fR\alpha(e) = 0$, showing that R is α-skew RIP.

The converse of Proposition 9 is not hold by the following example.

Example 7. Consider a ring $R = M_2(\mathbb{Z})$ with an automorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

R is a right α-skew RIP ring by Proposition 8 but, it is not a right α-skew reversible ring For

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } A, B \in R,$$

we have $AB = 0$, but $0 \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B\alpha(A)$, showing that R is not a right α-skew reversible ring.

For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$.

Lemma 2. *If R is a reduced ring, then $eRfRf = 0$ if and only if $eRf = 0$ for $e, f \in Id(R)$.*

Proof. For $e, f \in Id(R)$, $eRfRf = 0$ implies that $eRfRfR = 0$, and so $(eRfR)^2 = eRfReRfR \subseteq eRfRfR = 0$. Since R is reduced $eRfR = 0$ and so $eRf = 0$. The converse is obvious. □

The converse of Lemma 2 does not hold by the following Example.

Example 8. Let $\mathbb{Z}_2$ be the ring of integer modulo 2. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\},$$

It is clear that R is not reduced ring. But R satisfies the relation $ERF = 0$ if and only if $ERFRFR = 0$. For

$$E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, F = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \in Id(R).$$

Indeed if $ERF = 0$, then $ERFRFR = 0$. Now let $ERFRFR = 0$, so we have

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ 0 & r_1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = 0.$$

We have $er_1fr_2f = 0$ implies $er_1f^2r_2 = 0$, $r_1, r_2 \in \mathbb{Z}_2$. Now if $r_2 = 0$, then it is trivial, so $r_2 = 1$ and $er_1f^2 = 0$ implies $er_1f = 0$ since $\mathbb{Z}_2$ is reduced. Therefore $ERF = 0$.

Theorem 2. (1) *Let R be a ring with an endomorphism α and $n \geq 2$. If R satisfies the condition "$eRfRfR = 0$ implies $eRf = 0$ "and R is a right α-skew RIP ring, then $V_n(R)$ is a right $\bar{\alpha}$-skew RIP ring.*

(2) *If $V_n(R)$ is a right $\bar{\alpha}$-skew RIP, then R is right α-skew RIP.*

Proof. (1) Suppose that R satisfies the condition "$eRfRfR = 0$ implies $eRf = 0$ "and R is a right α-skew RIP ring. We use $(a_1, a_2, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix},$$

Let $EV_n(R)F = 0$ for $E = (e_1, e_2, e_3, \ldots, e_n)$, $F = (f_1, f_2, f_3, \ldots, f_n)$ and $E, F \in Id(V_n(R))$. For any $r \in R$, $E(r, 0, \cdots, 0)B = 0$. Thus we have the following equations:

$$e_1 r f_1 = 0 \tag{1}$$

$$e_1 r f_2 + e_2 r f_1 = 0 \tag{2}$$

$$e_1 r f_3 + e_2 r f_2 + e_3 r f_1 = 0 \tag{3}$$

$$\vdots$$

$$e_1 r f_{n-1} + e_2 r f_{n-2} + \cdots + e_{n-1} r f_1 = 0 \tag{4}$$

$$e_1 r f_n + e_2 r f_{n-1} + \cdots + e_{n-1} r f_2 + e_n r f_1 = 0. \tag{5}$$

From Eq. (I), we see

$$e_1 R f_1 = 0 \text{ and } f_1 R\alpha(e_1) = 0. \tag{6}$$

If we multiply Eq. (II) on the right-hand side by sf_1 for any $s \in R$, then $e_1 r f_2 s f_1 + e_2 r f_1 s f_1 = 0$ and hence $e_2 R f_1 = 0$ by Lemma 2 and Eq.(VI), and $e_1 R f_2 = 0$. Thus

$$f_1 R\alpha(e_2) = 0 \text{ and } f_2 R\alpha(e_1) = 0. \tag{7}$$

If we multiply Eq. (III) on the right-hand side by sf_1 for any $s \in R$, then

$$e_1 r f_3 s f_1 + e_2 r f_2 s f_1 + e_3 r f_1 s f_1 = 0,$$

so $e_3 r f_1 = 0$ by Lemma 2 and the above. Then Eq. (III) becomes

$$e_1 r f_3 + e_2 r f_2 = 0 \tag{8}$$

If we multiply Eq. (VIII) on the right-hand side by sf_2 for any $s \in R$, then $e_2 r f_2 = 0$ and $e_1 r f_3 = 0$ by the similar argument to above. Thus, we have

$$e_i R f_j, \text{ and } f_j R\alpha(e_i) = 0 \text{ for all } 2 \leq i + j \leq 4.$$

Inductively, we assume that

$$e_iRf_j, \text{ and } f_jR\alpha(e_i) = 0 \text{ for all } i+j \leq n.$$

If we multiply Eq. (V) on the right-hand side by $s_1f_1, s_2f_2, ..., s_{n-1}f_{n-1}$ for any $s_1, s_2, ..., s_{n-1} \in R$, in turn, then

$$e_nRf_1 = 0, e_{n-1}Rf_2 = 0, ..., e_2Rf_{n-1} = 0 \text{ and } e_1Rf_n = 0,$$

by the similar computation to above, and so

$$f_iR\alpha(e_j) = 0 \text{ for all } i+j = n+1.$$

Consequently, we get $FR\bar{\alpha}(E) = 0$ and therefore $V_n(R)$ is a right $\bar{\alpha}$-skew RIP ring.

(2) It follows from the similar computation to the proof of the sufficient condition in Proposition 8 □

Corollary 1. *Let R be a ring with an endomorphism α and $n \geq 2$. If R is an α-rigid, then $V_n(R)$ is a right $\bar{\alpha}$-skew RIP ring* .

Corollary 2. *Let R be a semiprime ring with an endomorphism α. Then the following are equivalent:*

(1) *R is right α-skew RIP.*
(2) *The trivial extension $T(R,R)$ of R is right $\bar{\alpha}$-skew RIP.*
(3) *$R[x]/(x^n)$ is right $\bar{\alpha}$-skew RIP, for $n \geq 2$.*

Theorem 3. *A ring R is right α-skew RIP if and only if $D_n(R)$ is right α-skew RMI, for $n \geq 2$.*

Proof. Assume that R is right α-skew RIP. Let $\mathscr{M}$ be a maximal ideal of $D_n(R)$. Then there exists a maximal ideal M of R such that $\mathscr{M} = \left\{ \begin{pmatrix} m & R & \cdots & R \\ 0 & m & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix} \mid m \in M \right\}$. Suppose that $E\mathscr{M}F = 0$, for $E = (e_{ij})$, $F = (f_{kl})$ and $E, F \in Id(D_n(R))$. Let $A = (a_{uv})$ be any in $\mathscr{M}$. Set $e_{ii} = e = e^2 \neq 0$, $a_{uu} = a$, $f_{kk} = f = f^2 \neq 0$, for all $i, u, k = 1, \ldots, n$. Then $e_{ij} \in ReR$ and $f_{kl} \in RfR$, for all i, j, k, l by help of the proof of [19, Theorem 3.9].

Note that a (resp., a_{12}) runs over M (resp., R). From $EAF = 0$, we get

$$0 = \begin{pmatrix} e & e_{12} \\ 0 & e \end{pmatrix} \begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \begin{pmatrix} f & f_{12} \\ 0 & f \end{pmatrix} = \begin{pmatrix} eaf & eaf_{12} + ea_{12}f + e_{12}af \\ 0 & eaf \end{pmatrix},$$

entailing $eaf = 0$ and $eaf_{12} + ea_{12}f + e_{12}af = 0$. But eaf implies $eMf = 0$ because a is an arbitrary in M. So we get $eaf_{12} + e_{12}af = 0$ because $e_{ij} \in ReR$ and $f_{kl} \in RfR$. Thus $ea_{12} = 0$, so $eRf = 0$ because a_{12} runs over R. Since R is right α-skew RIP, $eRf = 0$ gives $fR\alpha(e) = 0$. This result yields $F\mathscr{M}\bar{\alpha}(E) = 0$ by using again the fact that $e_{ij} \in ReR$ and $f_{kl} \in RfR$ for all i, j, k, l. Therefore $D_n(R)$ is right α-skew RMI.

Conversely, assume that $D_n(R)$ is right α-skew RMI for $n \geq 2$ and let $eRf = 0$ for $e, f \in Id(R)$. Then $E = e\sum_{i=1}^{n} E_{ii}, F = f\sum_{i=1}^{n} E_{ii}$ and $E, F \in Id(D_n(R))$, and note that

$$\mathscr{M} = \left\{ \begin{pmatrix} m & R & \cdots & R \\ 0 & m & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix} \mid m \in M \right\}$$

is a maximal ideal of $D_n(R)$ for any maximal ideal M of R. From $eRf = 0$ we obtain $E\mathscr{M}F = 0$. Since $D_n(R)$ is right α-skew RMI, $F\mathscr{M}\bar{\alpha}(E) = 0$ and this yields

$$0 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} m & r \\ 0 & m \end{pmatrix} \begin{pmatrix} \alpha(e) & 0 \\ 0 & \alpha(e) \end{pmatrix} = \begin{pmatrix} fm\alpha(e) & fr\alpha(e) \\ 0 & fm\alpha(e) \end{pmatrix},$$

where r is an arbitrary in R. So we have $fR\alpha(e) = 0$, thus R is right α-skew RIP. □

Theorem 4. *Let R be a ring with an endomorphism α.*

(1) *If $D_2(R)$ over a ring R is a right $\bar{\alpha}$-skew RMI, then R is a right α-skew RIP.*
(2) *If R has two or more maximal ideals and R is a right α-skew RMI ring, then R is a right α-skew RIP.*

Proof. (1) Note first that a maximal ideal of $D_2(R)$ is of the form $\left\{ \begin{pmatrix} m & r \\ 0 & m \end{pmatrix} \mid m \in M, r \in R \right\}$, where M is a maximal ideal of R. Let $D_2(R)$ be a right $\bar{\alpha}$-skew RMI ring and suppose that $eRf = 0$, for $e, f \in Id(R)$. Then

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} m & r \\ 0 & m \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = 0,$$

for all $m \in M$ and $r \in R$. Since $D_2(R)$ is a right $\bar{\alpha}$-skew RMI, we have

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} m & r \\ 0 & m \end{pmatrix} \begin{pmatrix} \alpha(e) & 0 \\ 0 & \alpha(e) \end{pmatrix} = 0.$$

This yields $fR\alpha(e) = 0$, so R is a right α-skew RIP ring.

(2) Let R be a ring and M_1, M_2 two distinct maximal ideals of R. Then $M_1 + M_2 = R$, say $1 = m_1 + m_2$ for $m_i \in M_i$. Assume that $eRf = 0$ for $e, f \in Id(R)$. Then $eM_1 f = 0$ and $eM_2 f = 0$. Here R is a right α-skew RMI so $fM_1\alpha(e) = 0$ hence $fM_2\alpha(e) = 0$. Therefore $fr\alpha(e) = f(rm_1 + rm_2)\alpha(e) = 0$ for every $r \in R$, entailing that $fR\alpha(e) = 0$. □

Theorem 5. *A ring R is right α-skew RIP if and only if $D_n(R)$ is right α-skew RIP for $n \geq 2$.*

*Proof.*Assume that R is a right α-skew RIP. Suppose that $ED_n(R)F = 0$ for $E = (e_{ij})$, $F = (f_{kl})$ and $E, F \in Id(D_n(R))$. Let $A = (a_{uv})$ be any in $D_n(R)$. Set $e_{ii} = e = e^2 \neq 0$, $a_{uu} = a$, $f_{kk} = f = f^2 \neq 0$, for all $i, u, k = 1, \ldots, n$. Then $e_{ij} \in ReR$ and $f_{kl} \in RfR$, for all i, j, k, l by help of the proof of [19, Theorem 3.9].

Note that a (resp., a_{12}) runs over R (resp., R). From $EAF = 0$, we get

$$0 = \begin{pmatrix} e & e_{12} \\ 0 & e \end{pmatrix} \begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \begin{pmatrix} f & f_{12} \\ 0 & f \end{pmatrix} = \begin{pmatrix} eaf & eaf_{12} + ea_{12}f + e_{12}af \\ 0 & eaf \end{pmatrix},$$

entailing $eaf = 0$ and $eaf_{12} + ea_{12}f + e_{12}af = 0$. But eaf implies $eRf = 0$ because a is an arbitrary in R. So we get $eaf_{12} + e_{12}af = 0$ because $e_{ij} \in ReR$ and $f_{kl} \in RfR$. Thus $ea_{12} = 0$, so $eRf = 0$ because a_{12} runs over R. Since R is right α-skew RIP, $eRf = 0$ gives $fR\alpha(e) = 0$. This result yields $FD_n(R)\bar{\alpha}(E) = 0$ by using again the fact that $e_{ij} \in ReR$ and $f_{kl} \in RfR$ for all i, j, k, l. Therefore $D_n(R)$ is right α-skew RIP.

Conversely, assume that $D_n(R)$ is right α-skew RIP for $n \geq 2$ and suppose $eRf = 0$ for $e, f \in Id(R)$. Then $E = e\sum_{i=1}^{n} E_{ii}, F = f\sum_{i=1}^{n} E_{ii}$ and $E, F \in Id(D_n(R))$, and note that From $eRf = 0$ we obtain $ED_n(R)F = 0$. Since $D_n(R)$

is right α-skew RIP, $FD_n(R)\bar{\alpha}(E) = 0$ and this yields

$$0 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} m & r \\ 0 & m \end{pmatrix} \begin{pmatrix} \alpha(e) & 0 \\ 0 & \alpha(e) \end{pmatrix} = \begin{pmatrix} fm\alpha(e) & fr\alpha(e) \\ 0 & fm\alpha(e) \end{pmatrix},$$

where r is an arbitrary in R. So we have $fR\alpha(e) = 0$, thus R is right α-skew RIP. □

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *a left regular* is defined, and *regular* means if it is both left and right regular (and hence not a zero divisor). A multiplicatively closed (m.c., for short) subset M of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in M$, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is shown, by [24, Theorem 2.1.12], that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring of R with respect to S exists. Suppose that the right quotient ring Q of R exists. For an automorphism α of R and any $au^{-1} \in Q(R)$ where $a \in R$ and $u \in S$, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(au^{-1}) = \alpha(a)\alpha(u)^{-1}$ is also an endomorphism. Note that the right quotient ring Q of an α-rigid ring R is $\bar{\alpha}$-rigid, where α is an automorphism of R. As a parallel result to this, we have the following result whose proof is modified from the proof of [19, Theorem 2.11]. Let R be a ring with the classical right quotient ring $Q(R)$. Then each automorphism α of R extends to $Q(R)$ by setting $\bar{\alpha}(ab^{-1}) = \alpha(a)(\alpha(b))^{-1}$ for $a, b \in R$, assuming that $\alpha(b)$ is regular for each regular element $b \in R$.

Theorem 6. *Let S be an m.c. subset of a ring R and α an automorphism of R. Suppose that S satisfies the right Ore condition and S consists of regular elements. If R is right α-skew RIP, then the right quotient ring Q of R with respect to S is right $\bar{\alpha}$-skew RIP.*

Proof. Suppose that R is a right α-skew RIP. Let $E = eu^{-1}$, $F = fv^{-1}$ with $e, f \in Id(R)$ and $u, v \in S$. Then we have $0 = EQF = eQ(fv^{-1})$, since $Q = u^{-1}Q$. Thus $e(rs^{-1})(fv^{-1}) = 0$ for any $rs^{-1} \in Q$. By hypothesis, there exists $c \in Id(R)$ and $w \in S$ such that $s^{-1}f = cw^{-1}$. Hence, $0 = e(rs^{-1})(fv^{-1}) = ercw^{-1}v^{-1}$ for any $r \in R$, so we have $eRc = 0$ and $cR\alpha(e) = 0$. From $eRc = 0$ and $fw = sc$, we get $0 = ersc = erfw$, for any $r \in R$ and hence $eRf = 0$ and $fR\alpha(e) = 0$. Since $v^{-1}Q = Q$, $FQ\bar{\alpha}(E) = fQ(\alpha(e)\alpha^{-1}(u))$. Consider $f(rt^{-1})\alpha(e)\alpha^{-1}(u)$ for any $rt^{-1} \in Q$. For $\alpha(e)$ and t, there exist $d \in R$ and $l \in S$ such that $\alpha(e)l = td$ and $t^{-1}\alpha(e) = dl^{-1}$. Since α is an automorphism, there exist $l', t', d' \in Id(R)$ such that $l = \alpha(l'), t = \alpha(t')$ and $d = \alpha(d')$, and hence $el' = t'd'$. The facts that $eRf = 0$ and $el' = t'd'$ imply $0 = el'rf = t'd'rf$ for any $r \in R$, so $d'Rf = 0$ and hence $fR\alpha(d') = fRd = 0$. Since $fRd = 0$, $0 = frdl^{-1}\alpha^{-1}(u) = f(rt^{-1})\bar{\alpha}(eu^{-1})$ for any $rt^{-1} \in Q$ and thus $FQ\bar{\alpha}(E) = 0$. Therefore Q is right $\bar{\alpha}$-skew RIP. □

The following proposition is obtained by applying the method in the proof of Theorem 6

Proposition 10. *Let M be an m.c. subset of a ring R consisting of central regular elements and α an automorphism of R. Then R is right α-skew RIP if and only if $M^{-1}R$ is right $\bar{\alpha}$-skew RIP.*

Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We still denote the extended maps $R[x] \to R[x]$ by $\bar{\alpha}$. The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} r_i x_i$ with the usual addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers. We denote this by $R[x; x^{-1}]$. For an endomorphism α of R, we denote the map $R[x; x^{-1}] \to R[x; x^{-1}]$ by the same endomorphism as in the polynomial ring $R[x]$ above. The following result extends the class of right α-skew RIP rings.

Corollary 3. *For a ring R with an automorphism α, $R[x]$ is a right $\bar{\alpha}$-skew RIP if and only if $R[x; x^{-1}]$ is a right $\bar{\alpha}$-skew RIP.*

Proof. It directly follows from Proposition 6. For, let $M = \{1, x, x^2, \cdots\}$, where M is a multiplicatively closed subset of $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$. □

Let R be a ring with an endomorphism α. Suppose that $R[x]$ is a right $\bar{\alpha}$-skew RIP ring and $eRf = 0$ for $e, f \in Id(R)$. Then $eR[x]f = 0$ by [10, Lemma 2.1], so $fR[x]\bar{\alpha}(e) = 0$ and $fR\alpha(e) = 0$. Thus R is right α-skew RIP.

Recall that a ring R is called *quasi-Armendariz* [10] provided that $a_iRb_j = 0$ for all i, j whenever $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ and $f(x), g(x) \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. Semiprime rings are quasi-Armendariz by [10, Corollary 3.8], but the converse does not hold in general.

Proposition 11. *Let R be a quasi-Armendariz ring with an endomorphism α. Then the following are equivalent:*

(1) *R is right α-skew RIP.*
(2) *$R[x]$ is right $\bar{\alpha}$-skew RIP.*
(3) *$R[x;x^{-1}]$ is right $\bar{\alpha}$-skew RIP.*

Proof. It suffices to show (1) $\Rightarrow$ (2) by Corollary 3 and the above argument. Assume that R is right α-skew RIP. Let $f(x)R[x]g(x) = 0]$ for $f(x) = \sum_{i=0}^{m} e_ix^i$, $g(x) = \sum_{j=0}^{n} f_jx^j$ and $f(x), g(x) \in Id(R[x])$. Since R is quasi-Armendariz and right α-skew RIP, we have $e_iRf_j = 0$ for all i, j and hence $f_jR\alpha(e_i) = 0$.. This entails that $g(x)R[x]\bar{\alpha}(f(x)) = 0$ and so $R[x]$ is right $\bar{\alpha}$-skew RIP. □

Let R be an algebra over a commutative ring S. Due to Dorroh [8], the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$. We use D to denote the Dorroh extension of R by S.

For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

In the following, we give some other example of right α-skew RIP rings.

Theorem 7. *Let R be an algebra over a commutative ring S and α an endomorphism of R with $\alpha(1) = 1$. Then R is a right α-skew RIP ring if and only if The Dorroh extension D of R by S is a right $\bar{\alpha}$-skew RIP ring.*

Proof. First, note that $s \in S$ is identified with $s1 \in R$, so $R = \{r + s \mid (r, s) \in D\}$. Suppose that R is a right α-skew RIP ring with $\alpha(1) = 1$ and suppose that $(e_1, s_1)D(e_2, s_2) = 0$ for $(e_1, s_1), (e_2, s_2) \in Id(D)$. Since $(e_i, s_i) = (e_i^2 + 2s_ie_i, s_i^2)$, we have that $(e_i + s_i1)^2 = e_i^2 + 2s_ie_i + s_i^2 1 = e_i + s_i1$ is an idempotent in R. Since

$$(e_1, s_1)(r, 0)(e_2, s_2) = (e_1re_2 + s_1re_2 + s_2e_1r + s_1s_2r, 0) = (0, 0),$$

and

$$(e_1 + s_11)r(e_2 + s_21) = e_1re_2 + s_1re_2 + s_2e_1r + s_1s_2r,$$

for $(e_1 + s_11), (e_2 + s_21) \in Id(R)$ and $r \in R$, we have $(e_1, s_1)R(e_2, s_2) = 0$. Since R is a right α-skew RIP ring, we get

$$0 = (e_2 + s_21)R\alpha(e_1 + s_11) = (e_2 + s_21)R(\alpha(e_1) + s_1).$$

Hence

$$e_2r\alpha(e_1) + s_2r\alpha(e_1) + s_1e_2r + s_1s_2r = 0,$$

for all $r \in R$. Let $(r,0) \in D$. Then

$$(e_2,s_2)(r,0)(\alpha(e_1),s_1) = ((e_2r+s_2r)\alpha(e_1)+s_1(e_2r+s_2r),0) = (e_2r\alpha(e_1)+s_2r\alpha(e_1)+s_1e_2r+s_1s_2r,0) = (0,0),$$

showing that $(e_2,s_2)D\bar{\alpha}(e_1,s_1) = 0$. Therefore D is a right $\bar{\alpha}$-skew RIP ring.

Conversely, suppose that D is a right $\bar{\alpha}$-skew RIP and suppose $eRf = 0$ for $e,f \in Id(R)$. Then $e(r+s)f = 0$, for any $(r,s) \in D$. This implies $(e,0)(r,s)(f,0) = 0$, for any $(r,s) \in D$. Since D is a right $\bar{\alpha}$-skew RIP, we have $(f,0)(r,s)\bar{\alpha}(e,0) = 0$ and hence $f(r+s)\alpha(e) = 0$, proving that $fR\alpha(e) = 0$. Thus R is a right α-skew RIP ring. □

Corollary 4. *Let R be an algebra over a commutative ring S, and D be the Dorroh extension of R by S. If R is RIP and S is domain, then D is RIP.*

Theorem 8. *Let R be an algebra over a field K, and D the Dorroh extension of R by K. Let α be an endomorphism of R with $\alpha(1) = 1$. Then R is a right α-skew RIP ring if and only if D is a right $\bar{\alpha}$-skew RMI ring.*

Proof. First, note that $\mathscr{M} = R \oplus \{0\}$ is the unique maximal ideal of D, since every $(r,s) \in D$ is a unit when $s \neq 0$ by the proof of [14, Proposition 1.5] and that $s \in K$ is identified with $s1 \in R$ and so $R = \{r+s \mid (r,s) \in D\}$.

Suppose that R is a right α-skew RIP ring with $\alpha(1) = 1$ and let $(e_1,s_1)\mathscr{M}(e_2,s_2) = 0$ for $(e_1,s_1),(e_2,s_2) \in Id(D)$. Since $(e_i,s_i) = (e_i^2+2s_ie_i,s_i^2)$, we have that $(e_i+s_i1)^2 = e_i^2+2s_ie_i+s_i^2 1 = e_i+s_i1$ is an idempotent in R. Since

$$(e_1,s_1)(r,0)(e_2,s_2) = (e_1re_2+s_1re_2+s_2e_1r+s_1s_2r,0) = (0,0),$$

and

$$(e_1+s_11)r(e_2+s_21) = e_1re_2+s_1re_2+s_2e_1r+s_1s_2r,$$

for $(e_1+s_11),(e_2+s_21) \in Id(R)$ and $r \in R$. For any $(r,0) \in \mathscr{M}$, we have $(e_1,s_1)R(e_2,s_2) = 0$. Since R is a right α-skew RIP ring, we get

$$0 = (e_2+s_21)R\alpha(e_1+s_11) = (e_2+s_21)R(\alpha(e_1)+s_1).$$

Hence

$$e_2r\alpha(e_1)+s_2r\alpha(e_1)+s_1e_2r+s_1s_2r = 0,$$

for all $r \in R$. Let $(r,0) \in \mathscr{M}$. Then

$$(e_2,s_2)(r,0)(\alpha(e_1),s_1) = ((e_2r+s_2r)\alpha(e_1)+s_1(e_2r+s_2r),0) = (e_2r\alpha(e_1)+s_2r\alpha(e_1)+s_1e_2r+s_1s_2r,0) = (0,0),$$

showing that $(e_2,s_2)\mathscr{M}\bar{\alpha}(e_1,s_1) = 0$. Therefore, D is a right $\bar{\alpha}$-skew RMI ring.

Conversely, suppose that D is a right $\bar{\alpha}$-skew RMI ring and let $eRf = 0$ for $e,f \in Id(R)$. Then

$$(e,0)\mathscr{M}(f,0) = (eRf,0) = (0,0).$$

Since D is a right $\bar{\alpha}$-skew RMI ring,

$$(f,0)\mathcal{M}\bar{\alpha}(e,0)=(f,0)\mathcal{M}(\alpha(e),0)=0,$$

and it implies that $fR\alpha(e)=0$. Thus R is a right α-skew RIP ring. □

Lemma 3. *Let R be a right α-skew RIP ring with $\alpha(1)=1$. Then for any $e,f\in Id(R)$, $eRf=0$ implies $fR\alpha^n(e)=0$ for any positive integer n.*

Proof. Let $eRf=0$ with $e,f\in Id(R)$, then we have $fR\alpha^n(e)=0$ since R is right α-skew RIP. Note that $fR\alpha(e)=(1).fR.\alpha(e)=0$, this implies that $\alpha(e).fR.\alpha(1)=\alpha(e).fR.(1)=0$. It follows that $(1).fR.\alpha^2(e)=0$ by hypothesis. Continuting this process, we have $fR\alpha^n(e)=0$ for any positive integer n. □

We note that if R is a right α-skew RIP ring with $eRf=0$ for $e,f\in Id(R)$. Then we have $fR\alpha(e)=0$, and so $\alpha(e)R\alpha(f)=0$. By induction hypothesis, we can obtain $\alpha^n(e)R\alpha^n(f)=0$ for any positive integer n. However, we have the following corollary.

Corollary 5. *Let R be a right α-skew RIP ring with $\alpha(1)=1$. Then for any $e,f\in Id(R)$, $eRf=0$ implies $\alpha^m(e)R\alpha^n(f)=0$ for any positive integer n,m.*

Let R be a ring and α a monomorphism of R. Now we consider the Jordan's construction of an over-ring of R by α (see [15] for more details). Let $A(R,\alpha)$ be the subset $\{x^{-i}rx^i \mid r\in R \text{ and } i\geq 0\}$ of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$. Note that for $j\geq 0$, $x^jr=\alpha^j(r)x^j$ implies $rx^{-j}=x^{-j}\alpha^j(r)$ for $r\in R$. This yields that for each $j\geq 0$, we have $x^{-i}rx^i=x^{-(i+j)}\alpha^j(r)x^{i+j}$. It follows that $A(R,\alpha)$ forms a subring of $R[x,x^{-1};\alpha]$ with the following natural operations: $x^{-i}rx^i+x^{-j}sx^j=x^{-(i+j)}(\alpha^j(r)+\alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j)=x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $r,s\in R$ and $i,j\geq 0$. Note that $A(R,\alpha)$ is an over-ring of R, and the map $\bar{\alpha}:A(R,\alpha)\to A(R,\alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i)=x^{-i}\alpha(r)x^i$ is an automorphism of $A(R,\alpha)$. Jordan showed, with the use of left localization of the skew polynomial $R[x;\alpha]$ with respect to the set of powers of x, that for any pair (R,α), such an extension $A(R,\alpha)$ always exists in [15]. This ring $A(R,\alpha)$ is usually said to be the *Jordan extension* of R by α.

Finally, we give the following Proposition:

Proposition 12. *For a ring R with a monomorphism α, R is right α-skew RIP if and only if the Jordan extension $A=A(R,\alpha)$ of R by α is right $\bar{\alpha}$-skew RIP.*

*Proof.*It is enough to show the necessity. Suppose that R is right α-skew RIP and $eAf=0$, for $e=x^{-i}rx^i, f=x^{-j}sx^j\in Id(A)$, for $i,j\geq 0$ and for $c=x^{-k}tx^k\in A=A(R,\alpha)$. Then, $r,s\in Id(R)$ obviously. From $ecf=0$, we get

$$(x^{-i}rx^i)(x^{-k}tx^k)(x^{-j}sx^j)=0,$$

then we have

$$x^{-(i+k+j)}(\alpha^{j+k}(r)\alpha^{j+i}(t)\alpha^{k+i}(s))x^{(i+k+j)}=0.$$

This implies that

$$\alpha^{j+k}(r)\alpha^{j+i}(t)\alpha^{k+i}(s)=0.$$

Since R is right α-skew RIP, we have

$$\alpha^{k+i}(s)\alpha^{j+i}(t)\alpha^{j+k+1}(r)=0.$$

Therefore, we obtain

$$fc\alpha(e) = (x^{-j}sx^{j})(x^{-k}tx^{k})(x^{-i}\alpha(r)x^{i}) = x^{-(i+k+j)}(\alpha^{i+k}(s)\alpha^{j+i}(k)\alpha^{j+k+1}(r))x^{(i+k+j)} = 0.$$

Thus, the Jordan extension A of R by α is a right $\bar{\alpha}$-skew RIP ring. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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