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Laplace transform method for logistic growth in a population and predator models

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Abstract: We apply the Laplace transform method to logistic growth model in a population and prey-predator models.Among the techniques appearing in research literature, this method is very effective in terms of accuracy, stability and convergence of solutions. Analytical and numerical results are given and compared. We show that the Laplace transform method we have presented in this work gives accurate approximate solutions to nonlinear problems of physics, mathematics and biology.

Keywords: Laplace transform method, single species, interacting species, lotka-volterra systems

1 Introduction

Many scientists have tried to solve nonlinear differential equations appearing in the research literature [1,5,6,10-12]. However, most of the methods they use require a tedious analysis to solve their model equations. The Laplace transform method (LTM) is a particular integral transform invented by the French mathematician Pierre-Simon Laplace, and systematically developed by the British physicist Oliver Heaviside, to simplify the solutions of many differential equations that describe physical processes. LTM turns differential equations into polynomial equations, which are easier to handle. Once solved, use of the inverse Laplace transform method reverts to the time domain. However, for the nonlinear equations, LTM can not be used alone. Therefore, for the nonlinear case we use the series expansion of the dependent variable to get the solution.

In this paper we aim to give an application of this method to couple of nonlinear biological models that consist of a logistic growth model in a population and a prey-predator model. First, we study the logistic growth model in a population [9]

$$\frac{dU}{dt} = sU(1 - U/A), \tag{1}$$

where s and A are positive constants. Here $U = U(t)$ represents the population of the species at time t, and $sU(1 - U/A)$ is the per capita growth rate, and A is the carrying capacity of the environment. Non-dimensionalization of Eq.(1) by setting [9]

$$u(\tau) = \frac{U(t)}{A}, \quad \tau = st,$$

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results in

$$\frac{du}{d\tau} = u(1 - u). \tag{2}$$

If $U(0) = U_0$, then $u(0) = U_0/A$, and the analytical solution of Eq.(2) follows easily

$$u(\tau) = \frac{1}{1 + (A/U_0 - 1)e^{-\tau}}. \tag{3}$$

Next, we study the Predator-Prey Models: Lotka-Volterra systems as an interacting species model defined by [9]

$$\frac{dU}{d\tau} = U(e - Vf), \tag{4}$$

$$\frac{dV}{d\tau} = V(gU - h), \tag{5}$$

where e, f, g and h are constants. Here $U = U(t)$ is the prey population and $V = V(t)$ that of the predator at time t. We can non-dimensionalize the system (4)-(5) [9] by setting

$$u(\tau) = \frac{gU(t)}{h}, \ \ v(\tau) = \frac{fV(t)}{e}, \ \ \tau = et, \ \ \beta = h/e,$$

and it becomes

$$\frac{du}{d\tau} = u(1 - v), \tag{6}$$

$$\frac{dv}{d\tau} = \beta\left[g(u, v) - v\right]. \tag{7}$$

In section 2, we apply the LTM [2,3,7,8] to the models of the form Eq.(2) and Eqs.(6)-(7), respectively.

2 Laplace transform method (LTM)

2.1 LTM for logistic growth model

In this subsection we consider the model equation of the form

$$\frac{du}{d\tau} = u - f(u), \quad u(0) = u_0, \tag{8}$$

where f is a nonlinear function of u. We look for the solution u satisfying (8). Therefore, we assume the solution, u of (8) has an infinite series representation of the form

$$u = u(\tau) = \sum_{n=0}^{\infty} a_n \tau^n, \tag{9}$$

and it satisfies the required conditions for the existence of the Laplace transform, $\mathscr{L}$ defined by

$$\mathscr{L}(u(t)) = \int_0^\infty e^{-st} u(t)dt, \ t \geq 0, \tag{10}$$

provided that this integral exists. Applying the Laplace transform to the both sides of the differential equation in (8) we get

$$sU(s) - u_0 = U(s) - F(s), \tag{11}$$

where $U(s) = \mathscr{L}(u(t))$ and $F(s) = \mathscr{L}\{f(u(t))\}$ are the Laplace transforms of the functions $u(t)$ and $f(u(t))$, respectively. Solving (11) for $U(s)$ we obtain

$$U(s) = u_0 \frac{1}{s-1} - \frac{F(s)}{s-1}. \tag{12}$$

Therefore, assuming the inverse Laplace transform $\mathscr{L}^{-1}$ exists and applying it to (12) we end up with

$$u(t) = u_0 \exp(t) - \mathscr{L}^{-1}\left(\frac{F(s)}{s-1}\right). \tag{13}$$

2.2 LTM for prey-predator model

In this subsection we study the system of non-linear differential equations of the form

$$\frac{du}{d\tau} = u - f(u,v), \tag{14}$$

$$\frac{dv}{d\tau} = \beta\left[g(u,v) - v\right], \tag{15}$$

with initial conditions

$$u(0) = u_0, \quad v(0) = v_0, \tag{16}$$

where f and g are nonlinear functions of u and v. Here β is some positive constant. We look for the solutions (u,v) satisfying (14)-(16). We assume the solutions u and v of the system (14)-(15) have the following infinite series expansions of the form

$$u = u(\tau) = \sum_{n=0}^{\infty} a_n \tau^n, \quad v = v(\tau) = \sum_{n=0}^{\infty} b_n \tau^n, \tag{17}$$

and they satisfy the required conditions for the existence of their Laplace transforms. Applying the Laplace transform to the system (14)-(15) and using (16) we get

$$sU(s) - u_0 = U(s) - F(s),$$
$$sV(s) - v_0 = \beta(G(s) - V(s)), \tag{18}$$

where $U(s) = \mathscr{L}(u(t))$, $F(s) = \mathscr{L}\{f(u(t),v(t))\}$, $V(s) = \mathscr{L}(v(t))$, $G(s) = \mathscr{L}\{g(u(t),v(t))\}$ are the Laplace transforms of the functions $u(t)$, $f(u(t),v(t))$, $v(t)$ and $g(u(t),v(t))$, respectively.

Solving (18) for $U(s)$ and $V(s)$ we get

$$U(s) = u_0 \frac{1}{s-1} - \frac{F(s)}{s-1},$$
$$V(s) = v_0 \frac{1}{s+\beta} + \beta \frac{G(s)}{s+\beta}. \tag{19}$$

If we assume the inverse Laplace transforms exist and apply them to the system (19) we find

$$u(t) = u_0 \exp(t) - \mathscr{L}^{-1}\left(\frac{F(s)}{s-1}\right),$$
$$v(t) = v_0 \exp(-\beta t) + \beta \mathscr{L}^{-1}\left(\frac{G(s)}{s+\beta}\right), \tag{20}$$

which are the desired solutions of the initial value problem (14)-(16).

3 Applications and results

Example 1. We consider the initial value problem (8). For numerical computations we take $N_0 = 2$ and $A = 1$. Therefore, $u_0 = N_0/A = 2$. We let $f(u) = u^2$ as in (2) so that one has

$$f(u) = \left(\sum_{n=0}^{\infty} a_n t^n\right)^2 = a_0^2 + 2a_0 a_1 t + (2a_0 a_2 + a_1^2)t^2 + (2a_0 a_3 + 2a_1 a_2)t^3 + \cdots.$$

We then obtain

$$F(s) = \frac{a_0^2}{s} + \frac{2a_0 a_1}{s^2} + \frac{4a_0 a_2 + 2a_1^2}{s^3} + \frac{12a_0 a_3 + 12a_1 a_2}{s^4} + \cdots.$$

Using (12) one gets

$$U(s) = \frac{2}{s-1} - \frac{F(s)}{s-1},$$
$$= \frac{2}{s-1} - \left\{\frac{a_0^2}{s(s-1)} + \frac{2a_0 a_1}{s^2(s-1)} + \frac{4a_0 a_2 + 2a_1^2}{s^3(s-1)} + \frac{12a_0 a_3 + 12a_1 a_2}{s^4(s-1)} + \cdots\right\}.$$

If we apply the inverse Laplace transform to this equation and use (9) we get

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots = 2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)$$
$$- a_0^2 t - (\frac{a_0^2}{2} + a_0 a_1)t^2 - \left(\frac{a_0^2}{6} + \frac{a_0 a_1}{3} + \frac{2a_0 a_2}{3} + \frac{a_1^2}{3}\right)t^3 - \cdots$$
$$= 2 + (2 - a_0^2)t + \left(1 - \frac{a_0^2}{2} - a_0 a_1\right)t^2$$
$$+ \left(\frac{1}{3} - \frac{a_0^2}{6} - \frac{a_0 a_1}{3} - \frac{2a_0 a_2}{3} - \frac{a_1^2}{3}\right)t^3 + \cdots. \tag{21}$$

Equating coefficients of power t yields

$$a_0 = 2, \quad a_1 = 2 - a_0^2 \implies a_1 = -2,$$
$$a_2 = 1 - \frac{a_0^2}{2} - a_0 a_1 \implies a_2 = 3,$$
$$a_3 = \frac{1}{3} - \frac{a_0^2}{6} - \frac{a_0 a_1}{3} - \frac{2 a_0 a_2}{3} - \frac{a_1^2}{3} \implies a_3 = -\frac{13}{3},$$
$$\vdots$$

As a result, one obtains the solution $u(t)$ from (9) as follows

$$u(t) = 2 - 2t + 3t^2 - \frac{13}{3} t^3 + \cdots, \tag{22}$$

which is the exactly solution obtained in Eq.(3) in the closed form. This solution is precisely the same as the one we obtained in [4].

Let $\phi_n(t)$ be the nth partial sums of the series (9), i.e,

$$\phi_n(t) = \sum_{k=0}^{n} a_k t^k. \tag{23}$$

As one sees from Fig.1 that we have obtained a very close approximation to the exact solution of logistic growth model in the time interval [0,0.15] by calculating only 4 terms of the series in (23). This tells us that the speed of convergence of LTM is very fast. Of course, the total errors can be made small, for $t > 0.15$ by finding new terms of the related series.
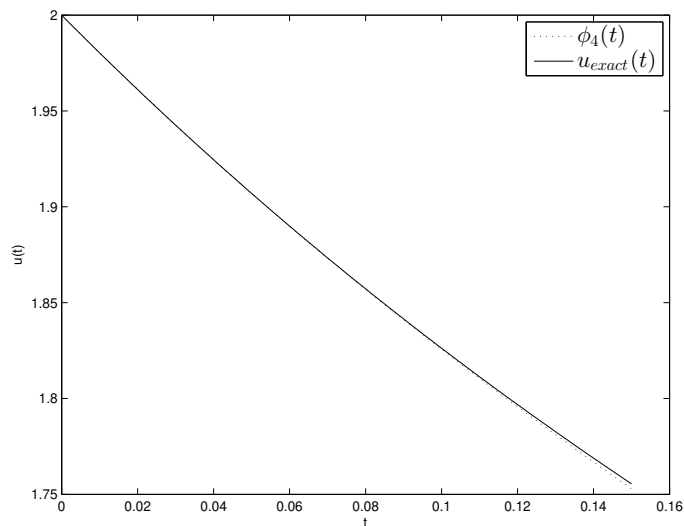

Fig. 1: Solution to the logistic growth model in a population.

Example 2. We now solve the problem (14)-(16) with initial data $u(0) = 1.3$, $v(0) = 0.6$. We proceed as in section 2.2. We take $\beta = 1, f(u,v) = g(u,v) = uv$ in (14)-(15) so that we have

$$f(u,v) = g(u,v) = \left(\sum_{n=0}^{\infty} a_n t^n \right) \left(\sum_{n=0}^{\infty} b_n t^n \right)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)t + (a_0 b_2 + a_1 b_1 + a_2 b_0)t^2 + \cdots,$$

and the corresponding Laplace transforms of these functions become

$$F(s) = G(s) = \frac{a_0 b_0}{s} + \frac{a_0 b_1 + a_1 b_0}{s^2} + \frac{2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0}{s^3} + \cdots.$$

Using (19) one gets

$$U(s) = \frac{1.3}{s-1} - \frac{F(s)}{s-1},$$

$$= \frac{1.3}{s-1} - \left\{ \frac{a_0 b_0}{s(s-1)} + \frac{a_0 b_1 + a_1 b_0}{s^2(s-1)} + \frac{2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0}{s^3(s-1)} + \cdots \right\},$$

$$V(s) = \frac{0.6}{s+1} + \frac{G(s)}{s+1},$$

$$= \frac{0.6}{s+1} + \left\{ \frac{a_0 b_0}{s(s+1)} + \frac{a_0 b_1 + a_1 b_0}{s^2(s+1)} + \frac{2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0}{s^3(s+1)} + \cdots \right\}.$$

If we apply the inverse Laplace transform to these equation and use (17) we get

$$
\begin{aligned}
a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots = {}& 1.3 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) \\
& - a_0 b_0 t - (a_0 b_0 + a_0 b_1 + a_1 b_0)\frac{t^2}{2!} \\
& - (a_0 b_0 + a_0 b_1 + a_1 b_0 + 2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0)\frac{t^3}{3!} - \cdots \\
= {}& 1.3 + (1.3 - a_0 b_0)t + (1.3 - a_0 b_0 - a_0 b_1 - a_1 b_0)\frac{t^2}{2!} \\
& + (1.3 - a_0 b_0 - a_0 b_1 - a_1 b_0 - 2a_0 b_2 - 2a_1 b_1 - 2a_2 b_0)\frac{t^3}{3!} + \cdots,
\end{aligned}
$$

(24)

and

$$
\begin{aligned}
b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots = {}& 0.6 \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \right) + a_0 b_0 t + (a_0 b_1 + a_1 b_0 - a_0 b_0)\frac{t^2}{2!} \\
& + (a_0 b_0 - a_0 b_1 - a_1 b_0 + 2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0)\frac{t^3}{3!} - \cdots \\
= {}& 0.6 + (a_0 b_0 - 0.6)t + (0.6 - a_0 b_0 + a_0 b_1 + a_1 b_0)\frac{t^2}{2!} \\
& + (-0.6 + a_0 b_0 - a_0 b_1 - a_1 b_0 + 2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0)\frac{t^3}{3!} + \cdots.
\end{aligned}
$$

(25)

Equating coefficients of power t yields

$$a_0 = 1.3, \quad a_1 = 1.3 - a_0 b_0 \implies a_1 = 0.52$$
$$a_2 = \frac{1}{2!}(1.3 - a_0 b_0 - a_0 b_1 - a_1 b_0) \implies a_2 = -0.013$$
$$a_3 = \frac{1}{3!}(1.3 - a_0 b_0 - a_0 b_1 - a_1 b_0 - 2a_0 b_2 - 2a_1 b_1 - 2a_2 b_0) \implies a_3 = -0.1122$$
$$\vdots \tag{26}$$

$$b_0 = 0.6, \quad b_1 = a_0 b_0 - 0.6 \implies b_1 = 0.18$$
$$b_2 = \frac{1}{2!}(0.6 + a_0 b_1 + a_1 b_0 - a_0 b_0) \implies b_2 = 0.183$$
$$b_3 = \frac{1}{3!}(-0.6 + a_0 b_0 - a_0 b_1 - a_1 b_0 + 2a_0 b_2 + 2a_1 b_1 + 2a_2 b_0) \implies b_3 = 0.0469$$
$$\vdots \tag{27}$$

The remaining terms of the series may be obtained in this manner. Writing these terms into Eq.(17) we get the approximate solutions to the problem (14)-(16):

$$u(t) = 1.3 + 0.52t - 0.013t^2 - 0.1122t^3 - \cdots,$$
$$v(t) = 0.6 + 0.18t + 0.183t^2 + 0.0469t^3 + \cdots, \tag{28}$$

which are exactly the same as the ones we obtained in [4].

Figures (2) & (3) show the comparisions of the numerical solutions and Laplace transform solutions of the system (14)-(16). Numerical solutions of the system are obtained with the built-in Matlab ordinary differential equation solver (Ode45) while Laplace transform solutions are obtained by using only four terms of the series.

It is seen from the last two figures that the numerical solutions and Laplace transform solutions of the system (14)-(16) are very close to each other in the time interval [0,0.6]. A further approximation may be achieved by calculating new terms to the Laplace transform solutions for $t \geq 0.6$.

As result, the Laplace transform method studied in this work gives very accurate approximate solutions to nonlinear problems of physics, mathematics, biology, etc. It also does not require linearization and biologically unrealistic assumptions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

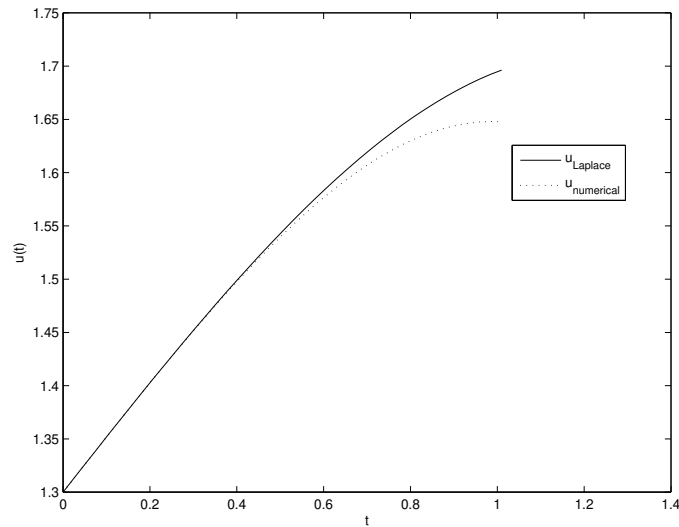All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

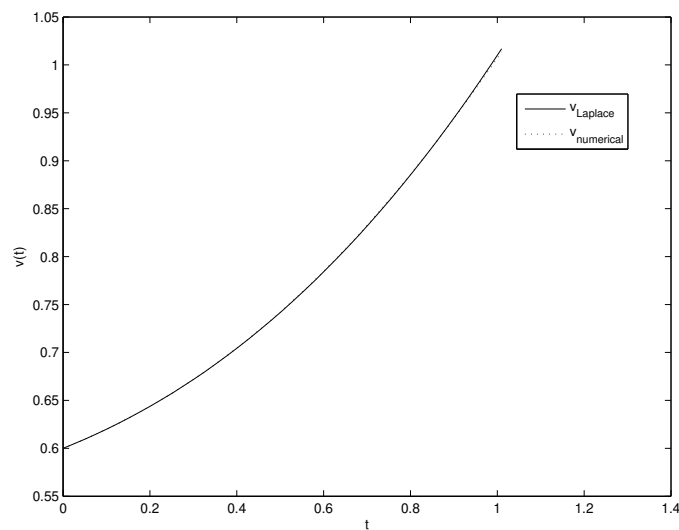**Fig. 2:** Solutions to the system (14)-(16) showing the Prey Population.

Fig. 3: Solutions to the system (14)-(16) showing the Predator Population.

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