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Expectiles for Fuzzy Variables and their applications to network flow problems with non-negative Fuzzy costs

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Abstract: Expectile Value at Risk (E-VaR) is conventionally used for measuring risk in random environments. In this study, we introduce Fuzzy E-VaR (FE-VaR) while focusing on the credibility distributions of fuzzy variables. In the proposed models, FE-VaR values are minimized to obtain risk-averse decisions for some fuzzy network optimization problems. The cost coefficients are assumed to be non-negative fuzzy variables with known credibility distributions. To validate the efficiency, some simulation studies are finally presented.

Keywords: Fuzzy variables, credibility distributions, value at risk criterion, expectiles, simulation, network flow problems.

1 Introduction

Owing to the incomplete information and subjective human judgments, the uncertain parameters of an optimization problem are often difficult to present exactly. In some uncertainty situations when probability distributions for costs cannot be obtained due to lack of historical data, these cost values for good, moderate and bad economic conditions (or the least, most possible, and the greatest amounts) can be provided from experts. Therefore, fuzzy numbers are described to better analyze and cope with the aforementioned imprecision issues. The credibility distributions have been used instead of probability distributions to extend the stochastic decision system characteristics to the fuzzy ones. To handle the risk caused by fuzzy uncertainty, we make use of credibility theory in this study.

Belonging to the family of generalized-quantiles, expectiles are the only one-parameter family of coherent, convex, and elicitable (robust or suitable for backtesting) monetary risk measures, and can be used for loss minimization problems, and provide a financial interpretation in terms of capital requirements, see [1] for details. FE-VaR can be seen as an alternative way of measuring risk and this paper introduces some credibilistic E-VaR minimization models. This generalization is easy to apply any other optimization problems where have non-negative fuzzy cost coefficients in the objective functions.

The literature survey, which includes existing Fuzzy VaR (abbreviated F-VaR) criterion models, is as follows. Peng [2] proposed the F-VaR and the credibilistic average VaR (which is also known as fuzzy conditional VaR or expected shortfall). Two-stage fuzzy decision-making problems under the VaR criteria are considered in [3,4,5]. Yang et al. [6] developed a new fuzzy $p-$hub center problem with VaR criterion, where the travel times are characterized by fuzzy variables with known possibility distributions. For portfolio optimization applications, see [7,8,9,10,11,12,13,14,15,16,17,18]. Yuan et al. [19] introduced the F-VaR into crop production planning and proposed a risk-based decision-making approach.

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In this paper, we focused on the expectile-based VaR minimization when the underlying mathematical programming models have vague cost parameters. To the best of our knowledge, optimization problems with FE-VaR criteria have not yet been examined in the literature. So, this paper aims at the investigation of the credibilistic E-VaR which provides a way of quantifying and handling fuzzy risk. A risk-averse decision is to minimize the expected total cost while controlling the total fuzzy cost variability. E-VaR minimization models in stochastic environments can be found in [20] and [21].

In the simulation phase, to generate a single value of a fuzzy variable, we use Procedure 1 in [22], that is, we first generate a uniformly distributed value of α over the interval $(0,1)$, we then generate another random number from the uniform distribution on $\alpha-$cut interval.
The rest of the paper is organized into four sections as follows. Section 2 is devoted to present some necessary preliminary concepts, including credibility theory, and fuzzy sample generation. In Section 3, quantiles and expectiles for fuzzy variables are described analogously to random variables. Next, some applications to network optimization problems, namely fuzzy shortest path problem and fuzzy transportation problem, and corresponding simulation studies are given in Section 4. The last section contains a brief conclusion.

2 Preliminaries

The basic concepts of the credibility theory are given in this section, see [23] for details.

Definition 1. *Let Θ be a nonempty set, and 2^{Θ} be the power set of Θ. Each element of 2^{Θ} is called a fuzzy event. Let ξ be a fuzzy variable with the membership function μ, and t be a real number. The credibility of the fuzzy event $\{\xi \leq t\}$ can be given as:*

$$Cr\{\xi \leq t\} = \frac{1}{2}(\sup_{u\leq t}\mu(u) + 1 - \sup_{u>t}\mu(u)). \tag{1}$$

Thus, $(\Theta, 2^{\Theta}, Cr)$ is called a credibility space.

In this study, we only consider continuous fuzzy variables in which the membership functions are continuous. Although there are many types of fuzzy variables, the following two types of fuzzy variables $-$ triangular and trapezoidal $-$ are commonly used in applications.

Example 1. A triangular fuzzy variable ξ can be determined by a triplet (a,b,c) with $a<b<c$, it is easy to obtain that

$$Cr\{\xi \leq t\} = \begin{cases} 0, & t<a \\ \frac{t-a}{2(b-a)}, & a \leq t < b \\ \frac{t-2b+c}{2(c-b)}, & b \leq t < c \\ 1, & t \geq c \end{cases}$$

from the identity (1).

Example 2. For a trapezoidal fuzzy variable (a,b,c,d) with $a<b<c<d$, credibility of the fuzzy event $\{\xi \leq t\}$ is:

$$Cr\{\xi \leq t\} = \begin{cases} 0, & t<a \\ \frac{t-a}{2(b-a)}, & a \leq t < b \\ \frac{1}{2}, & b \leq t < c \\ \frac{t-2c+d}{2(d-c)}, & c \leq t < d \\ 1, & t \geq d \end{cases}.$$

2.1 Expected value of a Fuzzy variable

Definition 2. *Let ξ be a fuzzy variable. Its expected value can be defined as:*

$$E[\xi] = \int_0^\infty Cr\{\xi \geq t\}dt - \int_{-\infty}^0 Cr\{\xi \leq t\}dt \tag{2}$$

provided that at least one of the two integrals is finite.

Note that, a credibility measure possesses self-duality property, so:

$$Cr\{\xi \geq t\} = 1 - Cr\{\xi < t\}. \tag{3}$$

Also, we only consider non-negative fuzzy variables, which means that $\mu(x) = 0$ for all $x < 0$, where μ is the membership function. So, $a \geq 0$ in Example 1 and 2, and second integral in the identity (2) vanishes.

Example 3. Based on their credibility measures, expected values for Example 1-2 are: $(a+2b+c)/4$ and $(a+b+c+d)/4$, respectively.

Definition 3. **[2]** *The credibility distribution function (cdf) $F_\xi : \mathbb{R} \to [0,1]$ of a fuzzy variable ξ can be defined as:*

$$F_\xi(t) = Cr\{\theta \in \Theta | \xi(\theta) \leq t\}. \tag{4}$$

Note that, for a non-negative fuzzy variable ξ, $F_\xi(t) = 0$ for all $t < 0$.

Definition 4. **[2]** *The credibility density function $f_\xi : \mathbb{R} \to [0,\infty]$ of a fuzzy variable ξ is a function such that:*

$$F_\xi(t) = \int_{-\infty}^t f_\xi(y)dy, \tag{5}$$

where F_ξ is the cdf of the fuzzy variable ξ.

Note that, $\int_{-\infty}^{\infty} f_\xi(y)dy = 1$.

Example 4. Credibility density functions for Example 1-2 are:

$$f_\xi(t) = \begin{cases} \frac{1}{2(b-a)}, & a \leq t < b \\ \frac{1}{2(c-b)}, & b \leq t < c \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and

$$f_\xi(t) = \begin{cases} \frac{1}{2(b-a)}, & a \leq t < b \\ \frac{1}{2(d-c)}, & c \leq t < d \\ 0, & \text{otherwise} \end{cases}, \tag{7}$$

respectively.

2.2 Generating Fuzzy numbers

For a triangular fuzzy variable $\xi_1 = (a_1, b_1, c_1)$ and trapezoidal fuzzy variable $\xi_2 = (a_2, b_2, c_2, d_2)$, samples can be respectively generated as following:

$$\xi_1 = u_2\left(u_1(b_1 - a_1) + a_1\right) + (1 - u_2)\left(c_1 - u_1(c_1 - b_1)\right), \tag{8}$$

$$\xi_2 = u_2\left(u_1(b_2 - a_2) + a_2\right) + (1 - u_2)\left(d_2 - u_1(d_2 - c_2)\right), \tag{9}$$

where $u_1, u_2 \sim U(0,1)$ and independent. Note that, $E[\xi_1] = (a_1 + 2b_1 + c_1)/4$ and $E[\xi_2] = (a_2 + b_2 + c_2 + d_2)/4$.

3 Quantiles and expectiles for Fuzzy variables

By using the identity (2), fuzzy quantiles or expectiles can be defined in a similar manner as in random variables. A financial position L is defined by a monetary fuzzy variable representing possible losses. In this paper, a positive value of L denotes a loss. The F-VaR is the greatest loss which is not exceeded with a given high confidence level α, so:

$$F - VaR_\alpha(L) = \inf\{l \in \mathbb{R} \,|\, Cr\{L \le l\} \ge \alpha\} = F_L^{-1}(\alpha) \tag{10}$$

where $\alpha \in (0,1)$[2]. In other words, α-quantile can be defined as the minimizer of the expectation of the following score, that is:

$$F - VaR_\alpha(L) = \arg\min_l E[\alpha (L - l)^+ + (1 - \alpha)(L - l)^-] \tag{11}$$

where $(L - l)^+ = \max\{L - l, 0\}$ and $(L - l)^- = \max\{l - L, 0\}$. Similarly, ω-fuzzy expectile $Fe_\omega(L)$ or FE-VaR can be defined as the minimizer of the expectation of the following score, so:

$$Fe_\omega(L) = \arg\min_l E\left[\omega\left((L - l)^+\right)^2 + (1 - \omega)\left((L - l)^-\right)^2\right] \tag{12}$$

for $\omega \in (0,1)$. From the first order of optimality condition, the following identity holds:

$$\omega E[(L - Fe_\omega(L))^+] = (1 - \omega) E[(L - Fe_\omega(L))^-]. \tag{13}$$

Alternatively, one can calculate ω-expectile from the following identity:

$$Fe_\omega(L) = E[L] + \frac{2\omega - 1}{1 - \omega} \int_{Fe_\omega(L)}^{\infty} (t - Fe_\omega(L)) f_L(t)\,dt. \tag{14}$$

From the identity (13), it can be seen as an additional capital value to be held in reserve to handle risky situations and to have a sufficiently high gain-loss ratio $\frac{\omega}{1-\omega}$ for a specified prudentiality level ω. In this study, it is chosen that $\omega \in (0.5, 1)$ for more prudent and risk averse decisions.

If we take $\omega = 0.5$ in the Eq. (14), $Fe_\omega(L)$ reduces to the expected loss. To simplify the identity (14), $\frac{2\omega-1}{1-\omega} = \theta$ such that $\theta \in [0, \infty)$. Risk aversity increases when the risk parameter ω or θ increases.

4 Illustrative examples

As stated previously, we consider fuzzy network optimization examples which include non-negative fuzzy cost coefficients. In this paper, we assume that the right-hand side values in the optimization problems are precisely known, the coefficients of the objective functions are the same type of mutually independent fuzzy numbers.

Example 5. A fuzzy shortest path problem with non-negative triangular fuzzy arc lengths is considered in [24]. The objective or loss function Z is a triangular fuzzy number since the coefficients are triangular. Let us assume that $Z = (a,b,c)$. Two separate models should be considered. They are as follows:

$$\begin{aligned}
&\min \ \ Fe_{\omega}(Z),\ subject\ \ to\\
\\
&a = 19x_1+15x_2+58x_3+38x_4+54x_5+12x_6+8x_7+70x_8+65x_9+20x_{10},\\
&b = 25x_1+20x_2+63x_3+41x_4+57x_5+15x_6+9x_7+75x_8+75x_9+25x_{10},\\
&c = 29x_1+25x_2+68x_3+46x_4+62x_5+18x_6+10x_7+80x_8+85x_9+30x_{10},\\
&E[Z] = 24.5x_1+20x_2+63x_3+41.5x_4+57.5x_5+15x_6+9x_7+75x_8+75x_9+25x_{10},\\
\\
&Fe_{\omega}(Z) = E[Z]+\theta\left[\tfrac{(b-Fe_{\omega}(Z))^2}{4(b-a)}+\tfrac{b+c-2Fe_{\omega}(Z)}{4}\right],\\
&\left(Fe_{\omega}(Z) = E[Z]+\theta\left[\tfrac{(c-Fe_{\omega}(Z))^2}{4(c-b)}\right],\ \text{for Model 2}\right)\\
&Fe_{\omega}(Z)\geq a,\ (Fe_{\omega}(Z)\geq b,\ \text{for Model 2})\\
&Fe_{\omega}(Z)\leq b,\ (Fe_{\omega}(Z)\leq c,\ \text{for Model 2})\\
\\
&x_1 = 1,\\
&x_2+x_3-x_1 = 0,\\
&x_2-x_4-x_5 = 0,\\
&x_3+x_4-x_6-x_7 = 0,\\
&x_6-x_8 = 0,\\
&x_5+x_7-x_9 = 0,\\
&x_8+x_9-x_{10} = 0,\\
&x_{10} = 1,\\
&x_j = 0 \text{ or } 1,\ j = 1,2,\ldots,10,
\end{aligned} \tag{15}$$

Model 1 gives infeasible solutions except for the expected value minimization problem for $\theta = 0$. Optimal solutions for Model 2 are given in Table 1.

Example 6. A fuzzy transportation problem with non-negative trapezoidal fuzzy costs is considered in [25]. It has two suppliers and three receivers. Each one of the suppliers S1 and S2 can deliver 70 units of the product. The demands of the receivers are as follows: R1 - 30, R2 - 30, R3 - 80. The objective or loss function Z is a trapezoidal fuzzy number since the coefficients are trapezoidal. Let us assume that $Z = (a,b,c,d)$. Three separate models should be considered. They are as follows:

$$
\begin{aligned}
&\min \ Fe_\omega(Z), \ subject \ to \\
&\\
&a = x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 5x_{22} + 6x_{23}, \\
&b = 2x_{11} + 4x_{12} + 6x_{13} + 8x_{21} + 10x_{22} + 7x_{23}, \\
&c = 10x_{11} + 20x_{12} + 20x_{13} + 10x_{21} + 20x_{22} + 7x_{23}, \\
&d = 100x_{11} + 80x_{12} + 60x_{13} + 40x_{21} + 30x_{22} + 10x_{23}, \\
&E[Z] = 28.25x_{11} + 26.5x_{12} + 22.25x_{13} + 15.5x_{21} + 16.25x_{22} + 7.5x_{23}, \\
&\\
&Fe_\omega(Z) = E[Z] + \theta\left[\frac{(b-Fe_\omega(Z))^2}{4(b-a)} + \frac{c+d-2Fe_\omega(Z)}{4}\right], \\
&\left(Fe_\omega(Z) = E[Z] + \theta\left[\frac{c+d-2Fe_\omega(Z)}{4}\right], \text{ for Model 2}\right) \\
&\left(Fe_\omega(Z) = E[Z] + \theta\left[\frac{(d-Fe_\omega(Z))^2}{4(d-c)}\right], \text{ for Model 3}\right) \\
&Fe_\omega(Z) \geq a, \ (Fe_\omega(Z) \geq b \text{ (or } c) \text{ for Model 2 (or Model 3))} \\
&Fe_\omega(Z) \leq b, \ (Fe_\omega(Z) \leq c \text{ (or } d) \text{ for Model 2 (or Model 3))} \\
&\\
&x_{11} + x_{21} = 30, \\
&x_{12} + x_{22} = 30, \\
&x_{13} + x_{23} = 80, \\
&x_{11} + x_{12} + x_{13} = 70, \\
&x_{21} + x_{22} + x_{23} = 70, \\
&x_{ij} \geq 0, \ i = 1,2 \ j = 1,2,3,
\end{aligned} \tag{16}
$$

Model 1-2 give infeasible solutions. Optimal solutions for Model 3 are given in Table 2.

Table 1: Optimal solutions for Model 2, Example 5

θ	$Fe_\omega(Z)$	(a,b,c)	$E[Z]$	x^*
0	201.000	(174,201,228)	201	$(1,1,0,1,0,1,0,1,0,1)^T$
0.5	198.031	(165,195,225)	195	$(1,1,0,1,0,0,1,0,1,1)^T$
1	200.147	(165,195,225)	195	$(1,1,0,1,0,0,1,0,1,1)^T$
2	203.038	(165,195,225)	195	$(1,1,0,1,0,0,1,0,1,1)^T$
3	205.000	(165,195,225)	195	$(1,1,0,1,0,0,1,0,1,1)^T$
8	209.334	(170,197,222)	196.5	$(1,0,1,0,0,0,1,0,1,1)^T$
18	212.556	(170,197,222)	196.5	$(1,0,1,0,0,0,1,0,1,1)^T$

In the simulation phase of both Examples 5 and 6, by using the MATLAB function *intlinprog*, we solve 5000 different optimization problems in which all fuzzy coefficients of the objective functions are generated independently via the identities (8) or (9). For each θ and sample problem, we subtract the corresponding expectile from the optimal objective value of the total cost. If this difference is negative (or positive), it corresponds to a gain (or a loss). Because financially speaking, the expectile value is a capital to be held in reserve to lower the fuzzy risk. We obtain simulated gain-loss ratios (see Table 3) by dividing the expected values of gains to the expected values of losses.

Table 2: Optimal solutions for Model 3, Example 6

θ	$Fe_{\omega}(Z)$	(a,b,c,d)	$E[Z]$	x^{*}
0	2390.000	(540,730,1590,6700)	2390	$(30,30,10,0,0,70)^T$
0.5	2768.164	(540,730,1590,6700)	2390	$(30,30,10,0,0,70)^T$
1	3043.949	(540,730,1590,6700)	2390	$(30,30,10,0,0,70)^T$
2	3433.825	(540,730,1590,6700)	2390	$(30,30,10,0,0,70)^T$
3	3693.215	(540,880,1980,6400)	2450	$(0,30,40,30,0,40)^T$
8	4350.555	(540,880,1980,6400)	2450	$(0,30,40,30,0,40)^T$
18	4861.094	(540,880,1980,6400)	2450	$(0,30,40,30,0,40)^T$

Table 3: Simulation Results for Example 5 and 6

θ	Theoretic gain-loss ratio $(=\frac{\omega}{1-\omega})$	Simulated gain-loss ratio for Example 5	Simulated gain-loss ratio for Example 6
0	1	3.949	1.575
0.5	1.5	2.253	2.414
1	2	3.338	3.321
2	3	6.436	5.808
3	4	7.016	7.031
8	9	119.923	15.146
18	19	Inf	Inf

5 Conclusions

This paper offers a way of risk quantification which is based on the credibility theory. In the proposed models, FE-VaR values are assumed to be objective functions subjected to the original constraints of the problems. They are concerned with the identification of the optimal values of the decision variables for achieving the optimal objective value at the minimum risk. For future studies, optimization models with generalized-quantiles criteria and nonlinear L-R type fuzzy parameters are going to be considered. Besides, hybrid chance-constrained programming can also be applied when both costs and demands are fuzzy.

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