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An analysis of discrete time retrial queuing system with starting failures, Bernoulli feedback with general retrial times and a vacation

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Abstract: This article is concerned with a discrete time Geo/G/1 retrial queue with general retrial times, Bernoulli feedback and the server subject to starting failures and a vacation. In this article we generalize the previous works in discrete time retrial queue with unreliable server due to starting failures in the sense that we consider general service with Bernoulli feed back and general retrial times with single vacation. In this model arrival time follows geometrical distribution and vacation times are generally distributed. In this model the PGF is derived by using generating function technique and also we obtain the analytical expression for mean queue length in performance measure. In numerical examples we analyzed the effects of mean queue length in several possible ways.

Keywords: Discrete time retrial queue, Bernoulli feedback, unreliable server, mean queue length, markov chain, single vacation.

1 Introduction

In Queuing models many researchers have found a lot of application in computer communications and manufacturing systems. Currently many researchers are interested in discrete queue, due to applications in a various slotted digital communicated systems and other related areas. The analysis of discrete queuing model has received considerable attention in the scientific literature over the past years because of its applications which are widely used in the real life.

In brief, in telecommunication and computer systems the role of retrial queues is very important and are characterized by the fact that a customer who leaves the service area and joins a retrial group (orbit) when the server is busy. In the customers in the orbit can not receive service immediately, when the server is idle but, it is not so standard queues this is the main difference between retrial and standard queues. Falm (1990), Falm and Templation (1997), Kulkarni and Liang (1997), yang and Templation (1987) have discussed on retrial queues and analyzed the fundamental methods on retrial queues.

In the most of literature the researchers analyze continuous queuing model, but only some of the authors concentrate on discrete queues since in practice it is applied many systems which shows an inherent genetic slotted time scale (time shared computing system). Initially, the discrete queues are discussed by Meisling (1958), Bindsall, Ristenbatt, and Weinstein (1962) and also by powell and Avi – Lizha (1967). In modelling computers and telecommunications the role of discrete queuing models are the most important when compared with continuous time models. The concept feedback is initiated by Takacs (1963) which has been widely investigated in continuous time [5,6,7,8,14,15,16] whereas it has been rarely analyzes in discrete time [1].Takacs thinks about that the number of services needed by a customer is geometrically distributed, that is, after receiving each service a customers quit the system with probability 1-α or rejoins the end of the queue for another service with probability α. This phenomenon of feed back has many practical applications. Also, Atencia, Fortes, and Sanchez (2009) have analyzed a discrete queue with Bernoulli feedback and starting failures.

In this article we have developed a new concept in discrete retrial queue deals with Bernoulli feedback, starting failures and a vacation. Since the role of vacation in discrete queue with feedback and starting failures has wide application in many real situations of our life, which is motivated me to develop this article.

The aim of this article is to discuss the problem like that arises in telecommunication systems where messages that produce errors at the destination are sent again in a call centre, where customers may call again (repeat their service)

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if their problems are completely solved after the service. Also, in the telecommunication system the starting failures occurred and at the time of starting failures no service is produced to customers and the server also takes a vacation of random time. This process is the most suitable for our model under consideration.

2 Model Description

In this model we deal with discrete time queuing system where the time axis is splitted into constant length interval of one unit called slots. In continuous queue the probability of an arrival and a departure occurring simultaneously is zero, whereas it is not so in discrete queues. Also, in this discrete retrial queue all the queuing activities occur around the slot boundary. We divide the time axis marked by 0,1,2,...m. Consider the epoch m and assume that departures and the end of repair times take place just before the slot boundary m ie in the interval (m^-, m) and arrivals, retrials, beginning of repairs and vacations just after the slot boundary ie in the interval (m, m^+). That is, we follow an Early arrival system or departure first policy.

Customers arrival times follow geometrical distribution with probability p and an arriving customer who finds the server down or busy joins a group of blocked customers called orbit with a FCFS discipline, that is the customer who is only at head of the orbit is allowed to access to the server. When the server is idle an arriving customer (external or repeated) must turn on the service station, if the server has started its service successfully (with probability α) the customer has served immediately, otherwise, if the server has started its service unsuccessfully (with probability $\overline{\alpha} - 1 - \alpha$) the repair time of the server begins immediately and the customer must join the orbit. Also, the server takes a vacation of random time if no customers present in the system and upon return from vacation if any customer present in the system the server starts its service otherwise it is idle.

Successive inter retrial times $\{a_i\}$ are generally distributed with generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$, service times$\{s_i\}$, repair times$\{r_i\}$ and vacation times$\{v_i\}$ are also generally distributed with generating function $S(x) = \sum_{i=1}^{\infty} s_i x^i$, $R(x) = \sum_{i=1}^{\infty} r_i x^i$ and $V(x) = \sum_{i=0}^{\infty} v_i x^i$and nthfactorial moments $\beta_{1,n}$, $\beta_{2.n}$ and $\beta_{3.n}$respectively.

After completion of service the customer decide either to join the retrial group again for another service with probability θ or leave the system with complementary probability $\overline{\theta}$. It is assumed that inter arrival times, repair times, service times and vacation times are mutually independent. In order to avoid trivial cases, it is also assumed that 0<p<1, 0<v<1 and 0<θ<1.

3 Markov Chain

At time m$^+$ the system can be described by the markov process $\{y_m; m \geq 1\}$ with $y_m = \{c_m, \xi_{\mathrm{om}}, \xi_{1\mathrm{m}}, \xi_{2\mathrm{m}}, \xi_{3\mathrm{m}}, N_m\}$, where c_m denotes the state of the server 0,1,2, and 3 according to whether the server is on vacation, idle, busy and down respectively and N_m represents the number of customers in the retrial group. If c_m=0, then ξ_{om} represents remaining vacation time , if c_m=1, then $\xi_{1\mathrm{m}}$ represents remaining retrial time, if c_m=2, then $\xi_{2\mathrm{m}}$ represents remaining service time, if c_m=3, then $\xi_{3\mathrm{m}}$ represents remaining repair time. It is clear that $\{y_m; m \geq 1\}$ is the markov chain of our queuing model, whose state space is $\{(0,i,k), i \geq 1, k \geq 0, (1,i,k), i \geq 1, k \geq 1, (2,i,k), i \geq 1, k \geq o, (3,i,k) i \geq 1, k \geq 1\}$

The limiting probabilities are defined as;

$$\pi_{0,i,k} = \lim_{m\to\infty} \Pr\{C_m = 0, \xi_{0,m} = i,\, N_m = k\},$$

$$\pi_{1,i,k} = \lim_{m\to\infty} \Pr\{C_m = 1, \xi_{1,m} = i,\, N_m = k\},$$

$$\pi_{2,i,k} = \lim_{m\to\infty} \Pr\{C_m = 2, \xi_{2,m} = i,\, N_m = k\},$$

$$\pi_{3,i,k} = \lim_{m\to\infty} \Pr\{C_m = 3, \xi_{3,m} = i,\, N_m = k\}.$$

The Kolmogorov equations for the stationary distributions are;

$$\pi_{0,1,0} = p\pi_{0,1,0} + \overline{p}\pi_{0,0} i \geq 1. \tag{1}$$

$$\pi_{0,i,k} = \overline{p}\pi_{0,i+1,k} + pv_i\pi_{0,1,k} + \overline{p}v_i\pi_{0,1,k+1} i \geq 1. \tag{2}$$

$$\pi_{1,1,0} = \overline{p}\pi_{0,0} + \overline{\theta}\,\overline{p}\pi_{1,1,0} i \geq 1. \tag{3}$$

$$\pi_{1,i,k} = \theta \overline{p} a_i \pi_{0,1,k-1} + \overline{\theta} \overline{p} a_i \pi_{0,1,k} + \overline{p} \pi_{1,i+1,k} + \theta \overline{p} a_i \pi_{2,1,k+1} + \overline{\theta} \overline{p} a_i \pi_{2,1,k} + \overline{p} a_i \pi_{3,1,k}, i \geq 1, k \geq 1. \tag{4}$$

$$\begin{aligned}\pi_{2,i,k} = p s_i \delta_{0k} \alpha \pi_{00} + \alpha p s_i \pi_{0,1,k-1} + \alpha \overline{p} s_i \pi_{0,1,k} + \alpha \overline{p} s_i \pi_{1,1,k+1} + (1-\delta_{0k}) p \alpha s_i \sum_{j=1}^{\infty} \pi_{1,j,k} \\ + (1-\delta_{0k}) \theta p \alpha s_i \pi_{2,1,k-1} + \left(\overline{\theta} p \alpha s_i + \theta \overline{p} \alpha a_0 s_i\right) \pi_{2,1,k} + \overline{\theta} \overline{p} a_0 \alpha s_i \pi_{2,1,k-1} + (1-\delta_{0k}) \pi_{2,i+1,k-1} \\ + \overline{p} \pi_{2,ii+1,k} + (1-\delta_{0k}) p \alpha s_i \pi_{3,1,k} + \overline{p} \alpha s_i a_0 \pi_{3,1,k+1} i \geq 1, k \geq 0.\end{aligned} \tag{5}$$

$$\begin{aligned}\pi_{3,i,k} = p r_i \overline{\alpha} \delta_{ok} \pi_{00} + p r_i \overline{\alpha} \pi_{0,1,k-1} + \overline{p} r_i \overline{\alpha} \pi_{0,1,k} + (1-\delta_{1k}) p \overline{\alpha} r_i \sum_{j=1}^{\infty} \pi_{1,j,k-1} + \overline{p} \overline{\alpha} r_i \pi_{1,1,k} + (1-\delta_{0k}) \vartheta p r_i \overline{\alpha} \pi_{2,1,k-2} \\ + \left(\overline{\theta} \overline{\alpha} p r_i + \vartheta \overline{p} a_0 \alpha r_i\right) \pi_{2,1,k-1} + \overline{\theta} \overline{p} \overline{\alpha} a_0 r_i \pi_{2,1,k} + (1-\delta_{1k}) p \overline{\alpha} r_i \pi_{3,1,k-1} + \overline{p} \overline{\alpha} a_0 \pi_{3,i,k} \\ + (1-\delta_{1k}) p \pi_{3,i+1,k-1} + \overline{p} \pi_{3,i=1,k} i \geq 1 k \geq 1.\end{aligned} \tag{6}$$

The normalization condition is

$$\pi_{00} + \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \pi_{0,i,k} + \sum_{j=1}^{3} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \pi_{j,i,k} = 1.$$

To solve above Kolmogorov equations we define generating function and auxiliary generating function as follows:

$$\phi_0(x,z) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \pi_{0,i,k} x^i z^k,$$

$$\phi_1(x,z) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \pi_{1,i,k} x^i z^k,$$

$$\phi_2(x,z) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \pi_{2,i,k} x^i z^k,$$

$$\phi_3(x,z) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \pi_{3,i,k} x^i z^k,$$

$$\phi_{0,i}(z) = \sum_{k=0}^{\infty} \pi_{0,i,k} z^k,$$

$$\phi_{1,i}(z) = \sum_{k=1}^{\infty} \pi_{1,i,k} z^k,$$

$$\phi_{2,i}(z) = \sum_{k=0}^{\infty} \pi_{2,i,k} z^k,$$

$$\phi_{3,i}(z) = \sum_{k=1}^{\infty} \pi_{3,i,k} z^k.$$

In equation (2) multiply both sides by $z^k$and taking summation over k and using (1), we get;

$$\phi_{oi}(z) = \overline{p} \phi_{oi+1}(z) + v_i \phi_{01}(z) \left[\frac{\overline{p} + pz}{z}\right] - \frac{\overline{p} v_i}{z} \pi_{00}. \tag{7}$$

In the above equations multiply both sides by $x^i$and taking summation over i and after some algebraic simplifications, we get;

$$\phi_o(x,z) \left(\frac{x - \overline{p}}{x}\right) = \phi_{01}(z) \left[\frac{(\overline{p} + pz)}{z} V(x) - \overline{p}\right] - \frac{\overline{p} V(x)}{z} \pi_{00}. \tag{8}$$

In equation (4) multiply both sides by $z^k$and taking summation over k and using (3), we get;

$$\phi_{1i}(z) = \phi_{01}(z)a_i\overline{p}\left(\overline{\theta}+\theta z\right)+\overline{p}\phi_{i+1}(z)+\phi_{21}(z)a_i\overline{p}\left(\theta z+\overline{\theta}\right)+\overline{p}a_i\phi_{31}(z)-pa_i\pi_{00}. \tag{9}$$

In the above equations multiply both sides by $x^i$and taking summation over i and after some algebraic simplifications, we get

$$\phi_1(x,z)\left(\frac{x-\overline{p}}{x}\right) = \phi_{01}(z)\overline{p}\left(\overline{\theta}+\theta z\right)(A(x)-a_0)-\overline{p}\phi_{11}(z)+\phi_{21}(z)\overline{p}\left(\overline{\theta}+\theta z\right)(A(x)-a_0)+ \phi_{31}(z)\overline{p}(A(x)-a_0)-p\pi_{00}(A(x)-a_0). \tag{10}$$

In equation (5) multiply both sides by $z^k$and taking summation over k and using (3), we get;

$$\phi_{2i}(z) = \alpha s_i\phi_{01}(z)(\overline{p}+pz)+\frac{\overline{p}\alpha s_i}{z}\phi_{11}(z)+(\overline{p}+pz)\phi_{2i+1}(z)+p\alpha s_i\phi_1(1,z)+ \alpha s_i\frac{\left(\overline{\theta}+\theta z\right)(\overline{p}+pz)}{z}\phi_{21}(z)+p\alpha s_i\left(\frac{z-a_0}{z}\right)\pi_{00}. \tag{11}$$

In the above equations multiply both sides by $x^i$and taking summation over i and after some algebraic simplifications, we get;

$$\phi_2(x,z)\left(\frac{x-(\overline{p}+pz)}{x}\right) = (\overline{p}+pz)\alpha S(x)\phi_{01}(z)+\frac{\overline{p}\alpha S(x)}{z}\phi_{11}(z)+\alpha S(x)\left(\frac{a_0\overline{p}+pz}{z}\right)\phi_{31}(z)+p\alpha S(x)\phi_1(1,z) +\left[\frac{\left(\overline{\theta}+\theta z\right)(a_0\overline{p}+pz)}{z}\alpha S(x)-(\overline{p}+pz)\right]\phi_{21}(z)+\frac{\overline{p}(z-a_0)}{z}\alpha S(x)\pi_{00}. \tag{12}$$

In equation (6) multiply both sides by $z^k$and taking summation over k and using (3), we get;

$$\phi_{3i}(z) = \overline{\alpha}r_i(\overline{p}+pz)\phi_{01}(z)+(\overline{p}+pz)\phi_{3i+1}(z)+pz\overline{\alpha}r_i\phi_1(1,z)+\overline{\alpha}r_i(\overline{p}a_0+pz)\phi_{31}(z) +(\overline{p}a_0+pz)\left(\overline{\theta}+\theta z\right)\overline{\alpha}r_i\phi_{21}(z)\overline{p}\overline{\alpha}r_i\phi_{11}(z)+p(z-a_0)\overline{\alpha}r_i\pi_{00}. \tag{13}$$

In the above equations multiply both sides by $x^i$and taking summation over i and after some algebraic simplifications, we get;

$$\phi_3(x,z)\left(\frac{x-(\overline{p}+pz)}{x}\right) = (\overline{p}+pz)\overline{\alpha}R(x)\phi_{01}(z)+pz\overline{\alpha}\phi_1(1,z)R(x)+\left[\overline{\alpha}(\overline{p}a_0+pz)R(x)-(\overline{p}+pz)\right]\phi_{31}(z) +(\overline{p}a_0+pz)\left(\overline{\theta}+\theta z\right)\overline{\alpha}R(x)\phi_{21}(z)+\overline{p}\overline{\alpha}R(x)\phi_{11}(z)+p(z-a_0)\overline{\alpha}R(x)\pi_{00}. \tag{14}$$

Put x = 1 in (10) and after some algebraic simplifications, we get;

$$\phi_1(1,z)p = \overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\phi_{01}(z)-\overline{p}\phi_{11}(z)+\phi_{21}(z)\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)+\overline{p}(1-a_0)\phi_{31}(z)-p(1-a_0)\pi_{00}. \tag{15}$$

Using (15) in (12) and after some algebraic simplifications, we get;

$$\phi_2(x,z)\left(\frac{x-(\overline{p}+pz)}{x}\right) = \alpha S(x)\left[(\overline{p}+pz)+\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\right]\phi_{01}(z)+\overline{p}S(x)\left(\frac{1-z}{z}\right)\phi_{11} (z)+\left[\frac{z+\overline{p}a_0(1-z)}{z}\alpha S(x)\right]\phi_{31}(z)+\left[\frac{z+\overline{p}a_0(1-z)}{z}\alpha S(x)\right]\phi_{31}(z) +\left[\frac{(z+\overline{p}a_0(1-z))\left(\overline{\theta}+\theta z\right)}{z}\alpha S(x)-(\overline{p}+pz)\right]\phi_{21}(z)+\frac{pa_0(z-1)}{z}\alpha S(x)\pi_{00}. \tag{16}$$

Using (15) in (14) and after some algebraic simplifications, we get;

$$\phi_3(x,z)\left[\frac{x-(\overline{p}+pz)}{x}\right]=\left[(\overline{p}+pz)+z\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\right]\overline{\alpha}R(x)\,\phi_{01}(z)+\left[\left[z+\overline{p}a_0(1-z)\right]\overline{\alpha}R(x)-(\overline{p}+pz)\right]\phi_{31}(z)$$
$$+\left[z+\overline{p}a_0(1-z)\right]\left(\overline{\theta}+\theta z\right)\overline{\alpha}R(x)\,\phi_{21}(z)+\overline{p\overline{\alpha}}R(x)(1-z)\,\phi_{11}(z)+\overline{\alpha}R(x)\,a_0(z-1)\,\pi_{00}. \tag{17}$$

Put x= $\overline{p}$ in (8) and solving for $\phi_{01}(z)$, we get;

$$\phi_{01}(z)=\frac{\overline{p}V(\overline{p})}{\left[(\overline{p}+pz)V(\overline{p})-\overline{p}z\right]}\pi_{00}. \tag{18}$$

Put x= $\overline{p}$ in (8) and after simplification , we get;

$$\overline{p}\phi_{11}(z)=\overline{p}\left(\overline{\theta}+\theta z\right)\left(A(\overline{p})-a_0\right)\phi_{01}(z)+\overline{p}\left(\overline{\theta}+\theta z\right)\left(A(\overline{p})-a_0\right)\phi_{21}(z)+\overline{p}\left(A(\overline{p})-a_0\right)\phi_{31}(z)-p\left(A(\overline{p})-a_0\right)\pi_{00}. \tag{19}$$

In equation (16) put x =$(\overline{p}+pz$) and solving for $\Phi_{2,1}(z)$, we get;

$$\phi_{21}(z)=\frac{\begin{array}{l}\alpha zS(\overline{p}+pz)\left[(\overline{p}+pz)+\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\right]\phi_{01}(z)+\alpha S(\overline{p}+pz)(1-z)\,\overline{p}\phi_{11}(z)\\+\left[z+\overline{p}a_0z-1\right]\alpha S(\overline{p}+pz)\,\phi_{31}(z)+\alpha S(\overline{p}+pz)\,pa_0(z-1)\,\pi_{00}\end{array}}{\left\{S(\overline{p}+pz)-\left[(z+\overline{p}a_0(1-z))\,\alpha S(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)\right]\right\}}. \tag{20}$$

In equation (17) put x =$(\overline{p}+pz)$ and solving for $\Phi_{3,1}(z)$, we get;

$$\phi_{31}(z)=\frac{\begin{array}{l}\left[(\overline{p}+pz)+z\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\right]\overline{\alpha}R(\overline{p}+pz)\,\phi_{01}(z)+\overline{\alpha}R(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)\left[z+\overline{p}a_0(1-z)\right]\phi_{21}(z)\\+\overline{p}\overline{\alpha}R(\overline{p}+pz)(1-z)\,\phi_{11}(z)+\overline{\alpha}R(\overline{p}+pz)(z-1)\,a_0\pi_{00}\end{array}}{\left\{(\overline{p}+pz)-\left[(z+\overline{p}a_0(1-z))\,\overline{\alpha}R(\overline{p}+pz)\right]\right\}}. \tag{21}$$

Using (21) in (20) and after some algebraic calculations , we get;

$$\phi_{21}(z)=\frac{\begin{array}{l}\phi_{01}(z)\,\overline{\alpha}R(\overline{p}+pz)\left\{\begin{array}{l}\alpha S(\overline{p}+pz)\left[(\overline{p}+pz)-\overline{\alpha}R(\overline{p}+pz)\right]\left[(\overline{p}+pz)-(1-z)\,\overline{p}\left(A(\overline{p})-a_0\right)\right]\\\left[\begin{array}{l}\overline{p}\left(\overline{\theta}+\theta z\right)(1-z)\left(A(\overline{p})-a_0\right)\left(1+z\overline{p}a_0(1-z)\right)+\left[\begin{array}{l}(\overline{p}+pz)\\+\overline{p}z\left(\overline{\theta}+\theta z\right)(1-a_0)\end{array}\right]\\\left[2z+\overline{p}a_0(1-z)\right]\end{array}\right]\\+\left[(\overline{p}+pz)+z\overline{p}a_0(1-z))\left(\overline{\theta}+\theta z\right)\right]+\left[\begin{array}{l}((\overline{p}+pz)-(z+\overline{p}a_0(1-z)))\overline{p}(1-z))\left(\overline{\theta}+\theta z\right)\\\left(A(\overline{p})-a_0\right)\end{array}\right]\\\overline{p}\left(A(\overline{p})-a_0\right)(1-z)\,\alpha S(\overline{p}+pz)\left(1+z+\overline{p}a_0(1-z)\right)\overline{\alpha}R(\overline{p}+pz)\end{array}\right\}\\+\pi_{00}\left\{\begin{array}{l}(z-1)\left[p+z+\overline{p}a_0(1-z))\right]\alpha S(\overline{p}+pz)\,\overline{\alpha}R(\overline{p}+pz)\left[a_0+\overline{p}\left(A(\overline{p})-a_0\right)\right]\left(\left[(\overline{p}+pz)-\overline{\alpha}R(\overline{p}+pz)\right]\right.\\\left[z+\overline{p}a_0(1-z)\right]-(1-z)\,\overline{p}\left(A(\overline{p})-a_0\right))+\overline{\alpha}R(\overline{p}+pz)(z-1)\,\overline{p}\left(A(\overline{p})-a_0\right)+\overline{p}\left(A(\overline{p})-a_0\right)(z-1)\\\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]\alpha S(\overline{p}+pz)\,\overline{\alpha}R(\overline{p}+pz)\left(1+z+\overline{p}a_0(1-z)\right)\overline{\alpha}R(\overline{p}+pz)\end{array}\right\}\end{array}}{\begin{array}{l}z(\overline{p}+pz)-\alpha S(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)\left[\begin{array}{l}(z+\overline{p}a_0(1-z))\left(1+\overline{\alpha}R(\overline{p}+pz)\right)(z+\overline{p}a_0(1-z))+\overline{p}\left(A(\overline{p})-a_0\right)\\(1-z)\left(1+z+\overline{p}a_0(1-z)\right)\overline{\alpha}R(\overline{p}+pz)\end{array}\right]\\\left[(\overline{p}+pz)-\overline{\alpha}R(\overline{p}+pz)\left[(z+\overline{p}a_0(1-z))-(1-z)\,\overline{p}\left(A(\overline{p})-a_0\right)\right]\right]-\left[\begin{array}{l}\overline{\alpha}R(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)(1-z)\\\overline{p}\left(A(\overline{p})-a_0\right)\end{array}\right]\\+\left[\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]+\left[\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]\overline{\alpha}R(\overline{p}+pz)\left(z+\overline{p}a_0(1-z)\right)\right]\right]\\\overline{p}\left(A(\overline{p})-a_0\right)(1-z)\,\alpha S(\overline{p}+pz)\left(1+z+\overline{p}a_0(1-z)\right)\overline{\alpha}R(\overline{p}+pz)\end{array}} \tag{22}$$

Using (19) and (22) in (21) and after some algebraic simplification, we get;

$$\phi_{31}(z)=\frac{\begin{array}{l}\phi_{01}(z)\left\{\begin{array}{l}\overline{\alpha}R(\overline{p}+pz)\left[\begin{array}{l}(\overline{p}+pz)+z\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]\overline{\alpha}R(\overline{p}+pz)(1-z)\\ \overline{p}\left(\overline{\theta}+\theta z\right)(A(\overline{p})-a_0)\left[z(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\alpha S(\overline{p}+pz)-\left(\overline{\theta}+\theta z\right)\right]\end{array}\right]\\ +\overline{p}\left(\overline{\theta}+\theta z\right)(A(\overline{p})-a_0)z(1-z)+\left[\begin{array}{l}\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right](z+\overline{p}a_0(1-z))\alpha S(\overline{p}+pz)\\ \left[(\overline{p}+pz)+\overline{p}\left(\overline{\theta}+\theta z\right)(1-a_0)\right]\end{array}\right]\end{array}\right\}\\ +\phi_{21}(z)\left\{\begin{array}{l}\overline{p}\alpha S(\overline{p}+pz)(1-z)\overline{\alpha}R(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)\{(1-z)\overline{p}(A(\overline{p})-a_0)+\overline{p}(A(\overline{p})-a_0)\\ \overline{\alpha}R(\overline{p}+pz)(z+\overline{p}a_0(1-z))\left(\overline{\theta}+\theta z\right)(A(\overline{p})-a_0)\end{array}\right\}\\ +\pi_{00}\left\{\begin{array}{l}\alpha S(\overline{p}+pz)\overline{\alpha}R(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)(1-z)\{(1-z)\overline{p}(A(\overline{p})-a_0)+\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]\}\\ \overline{\alpha}R(\overline{p}+pz)(z+\overline{p}a_0(1-z))\left[a_0-\overline{p}(A(\overline{p})-a_0)\right]\end{array}\right\}\end{array}}{\left\{\begin{array}{l}z\left[\begin{array}{l}\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]((z+\overline{p}a_0(1-z))-(1-z)\overline{p}(A(\overline{p})-a_0))\\ z(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\alpha S(\overline{p}+pz)-\left(\overline{\theta}+\theta z\right)\end{array}\right]\\ -\left[\begin{array}{l}\alpha S(\overline{p}+pz)\overline{\alpha}R(\overline{p}+pz)\{(1-z)\overline{p}(A(\overline{p})-a_0)+\left[(\overline{p}+pz)-(z+\overline{p}a_0(1-z))\right]\}\\ \overline{\alpha}R(\overline{p}+pz)(z+\overline{p}a_0(1-z))z\left(\overline{\theta}+\theta z\right)\overline{p}(A(\overline{p})-a_0)\end{array}\right]\end{array}\right\}}. \tag{23}$$

The PGF of the model under consideration is obtained from

$$\phi(z)-\phi_{0,1}(1,z)+\phi_{1,1}(1,z)+\phi_{2,1}(1,z)+\phi_{3,1}(1,z),$$

$$\phi(z)=\phi_{01}(z)\left[\frac{\overline{p}(1-z)^2+z\left[1+\overline{p}\left(\overline{\theta}+\theta z\right)((1-a_0)(2-z)+z)\right]}{pz(1-z)}\right]+\phi_{11}(z)\left[\frac{\overline{p}(1-z\alpha)}{p}\right]$$

$$+\phi_{21}(z)\left[\frac{\overline{p}(1-a_0)}{p}+\frac{\left[z+\overline{p}a_0(1-z)\right]\left(\overline{\theta}+\theta z\right)(\overline{\alpha}+\alpha z)}{pz(1-z)}-\frac{(\overline{p}+pz)}{p(1-z)}\right]$$

$$+\phi_{31}(z)\left[\frac{\overline{p}(1-a_0)}{p}+\frac{(z+\overline{p}a_0(1-z))(\overline{\alpha}+\alpha z)}{pz(1-z)}-\frac{(\overline{p}+pz)}{p(1-z)}\right]-\pi_{00}\left[\frac{a_0(\alpha p+\overline{\alpha}z)-\overline{p}}{pz}-(1-a_0)\right].$$

Where $\phi_{01}(z)$, $\phi_{11}(z)$, $\phi_{21}(z)$, $\phi_{31}(z)$ are respectively given by the equations (18) , (19) , (22) and (23).

3.1 Steady State Condition

The steady state condition for the model under consideration is given below;

$$\pi_{00}=\frac{2p\left[A(\overline{p})-a_0\right]\left[\overline{p}V(\overline{p})-\overline{p}\right]\left[\overline{p}-V(\overline{p})\right]\left[2\overline{p}(1+\alpha)+(\alpha\overline{\alpha})^2\left[A(\overline{p})-a_0\right]+\overline{p}(1-\overline{p}a_0-\overline{p})\right]}{\begin{array}{l}2p\left[A(\overline{p})-a_0\right]\left[\overline{p}V(\overline{p})-\overline{p}\right]\left[\overline{p}-V(\overline{p})\right]\left[2\overline{p}(1+\alpha)+(\alpha\overline{\alpha})^2\left[A(\overline{p})-a_0\right]+\overline{p}(1-\overline{p}a_0-\overline{p})\right]\\ \{\left[pV(\overline{p})(2-a_0)(1-\overline{p}a_0)+a_0-p\overline{p}(1-a)\overline{p}\left[A(\overline{p})-a_0\right]\right]+\left[\overline{p}-V(\overline{p})\right]\}\\ +2\left[\overline{p}-V(\overline{p})\right]\left\{\begin{array}{l}2(\overline{p})^2\left[A(\overline{p})-a_0\right](\overline{\alpha})^2(2\overline{p}+2+2\overline{p}\alpha)+2(\overline{p})^2V(\overline{p})(1-a_0)^2\left[A(\overline{p})-a_0\right]\\ +(\overline{p})^2(1-a_0)\alpha\overline{\alpha}(2-\overline{p}a_0-a_0)\end{array}\right\}\end{array}},$$

which is obtained from PGF by substituting z =1 and equating to one and it is clearly less than one.

4 Expected Queue Length

Performance degree of the queue length distribution that is predictable queue length is attained below. The mean of the queue length is attained by differentiating the PGF with respect to z and then putting z=1,

$$E(Q)=\frac{\begin{array}{l}[\overline{p}-V(\overline{p})]^2\left\{\begin{array}{l}2(\overline{p})^2[A(\overline{p})-a_0](\overline{\alpha})^2(2\overline{p}+2+2\overline{\alpha})+(\overline{p})^2v(\overline{p})(1-a_0)^2(A(\overline{p})-a_0)\\+(\overline{p})^2(1-a_0)\alpha\overline{\alpha}(2-\overline{p}a_0-a_0)+2[A(\overline{p})-a_0]\left[2\overline{p}(1+\alpha)+(\alpha\overline{\alpha})^2[A(\overline{p})-a_0]\right]\\+[\overline{p}[1-\overline{p}a_0-\overline{p}][\overline{p}V(\overline{p})-\overline{p}][\alpha p\overline{p}[A(\overline{p})-a_0]+a_0p(\alpha p+\overline{\alpha})-(1-a_0)]]\end{array}\right\}\pi_{00}\\+[\overline{p}V(\overline{p})-\overline{p}]\left\{\begin{array}{l}\overline{p}V(\overline{p})[A(\overline{p})-a_0]^2[[V(\overline{p})-\overline{p}]\overline{p}(\overline{\alpha}\theta-\alpha)-\overline{\alpha}p[[\overline{p}V(\overline{p})-\overline{p}]+[V(\overline{p})-\overline{p}]]]\\2p[\overline{p}V(\overline{p})-\overline{p}]\left[2p(1+\alpha)+(\alpha\overline{\alpha})^2\right.\\ \left.[[A(\overline{p})-a_0]+\overline{p}(1-\overline{p}a_0-p)]\right]\\+2\overline{p}V(\overline{p})[1+\overline{p}(1+\theta)][\overline{p}-V(\overline{p})][A(\overline{p})-a_0][\overline{p}V(\overline{p})-\overline{p}]\left[2\overline{p}(1+\alpha)+(\alpha\overline{\alpha})^2\right.\\ \left.\left[\begin{array}{l}[A(\overline{p})-a_0]\\+\overline{p}(1-\overline{p}a_0)\end{array}\right]\right]\\+[\overline{p}-V(\overline{p})]^2\left[\begin{array}{l}2(\overline{p})^2[A(\overline{p})-a_0](\overline{\alpha})^2(2\overline{p}+2+2\overline{\alpha})+(\overline{p})^2\alpha(\overline{p})(1-a_0)^2(A(\overline{p})-a_0)\\+(\overline{p})^2(1-a_0)\alpha\overline{\alpha}(2-\overline{p}a_0-a_0)\end{array}\right]\end{array}\right\}\pi_{00}\end{array}}{[\overline{p}-V(\overline{p})][\overline{p}V(\overline{p})-\overline{p}][2p[A(\overline{p})-a][2\overline{p}(1+\alpha)+(\alpha\overline{\alpha})^2[A(\overline{p})-a]+\overline{p}(1-\overline{p}a_0-\overline{p})]]}.$$

4.1 Particular Case

When the vacation is zero and there is no threshold policy the PGF is reduced into

$$\phi(z)=\frac{A(\overline{p})v(\overline{p}+pz)(1-z)[1-S(\overline{p}+pz)\theta]}{\left[vS(\overline{p}+pz)\left(\overline{\theta}+\theta z\right)+\overline{v}zR(\overline{p}+pz)\right][z+(1-z)\overline{p}A(\overline{p})]-z(\overline{p}+pz)}\pi_{00},$$

which is the PGF of discrete time retrial queue with starting failures, Bernoulli feedback and general retrial times.

5 Numerical Examples

This section has numerical examples are briefly analyzed in two different cases. In both of these two cases mean queue length is investigated in following manner,

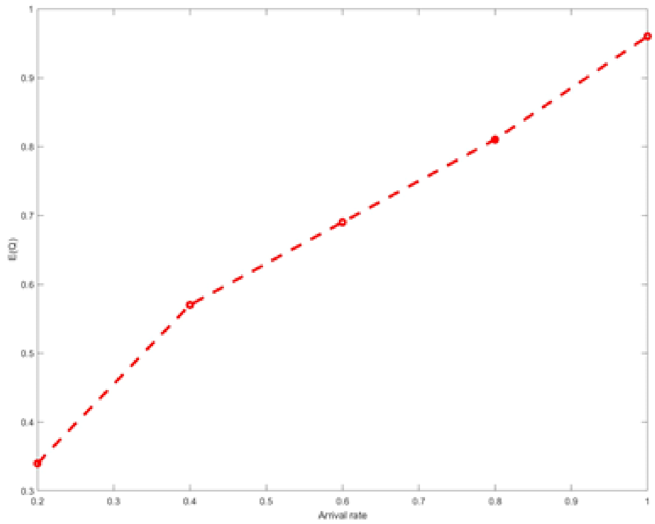
1.The result on mean queue length when the arrival rate increases.
2.The result on mean queue length when the service rate increases.

Case (I): In case (I), customer arrival times. Service times, vacation times are all geometrically distributed.

1.When the customer arrival rate rises the result on mean queue length is investigated below with the following index with graph.

Arrival Rate	π_{00}	E(Q)
.2	.16	.231
.4	.33	.456
.6	.48	.837
.8	.63	1.558
1.0	.86	2.130

Index & Graph 1.1 Arrival Rate vs Mean Queue Length.

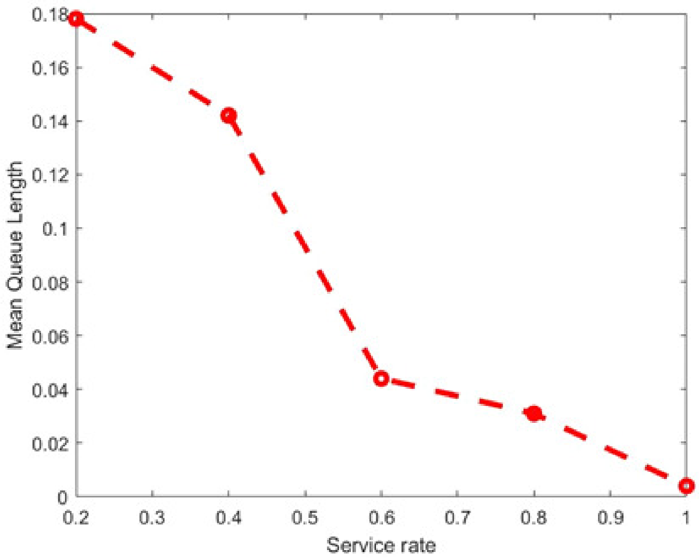


We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph.

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Service Rate	π_{00}	Mean Queue Length
0.2	.74	.178
0.4	.61	.142
0.6	.42	.044
0.8	.32	.031
1.0	.21	.004

Index & Graph 1.2 Service Rate vs Mean Queue Length.



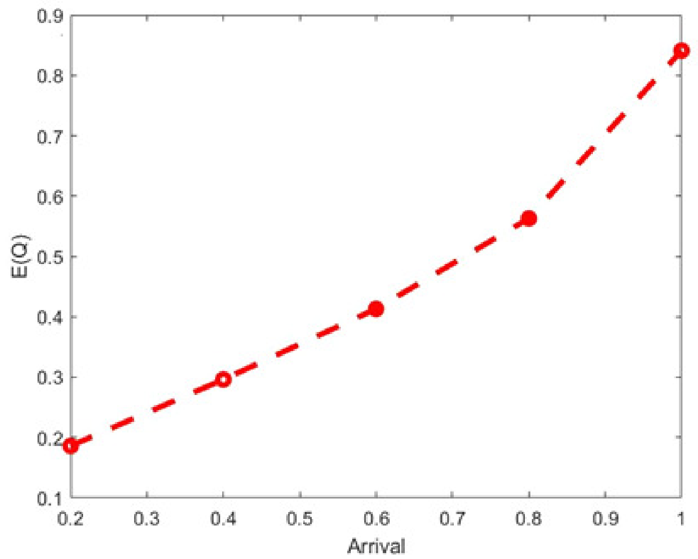
We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph.

Case (II): In case (II) customer arrival times follow geometrical distribution. Service times and vacation times are follow Poisson distribution.

1.When the customer arrival rate rises the result on mean queue length is investigated below with the following index with graph.

Arrival Rate	π_{00}	E(Q)
0	.32	0.086
.2	.48	0 .294
.4	.608	0 .792
.6	.73	1.47
.8	.88	2.26

Index & Graph 2.1 Arrival Rate vs Mean Queue Length.

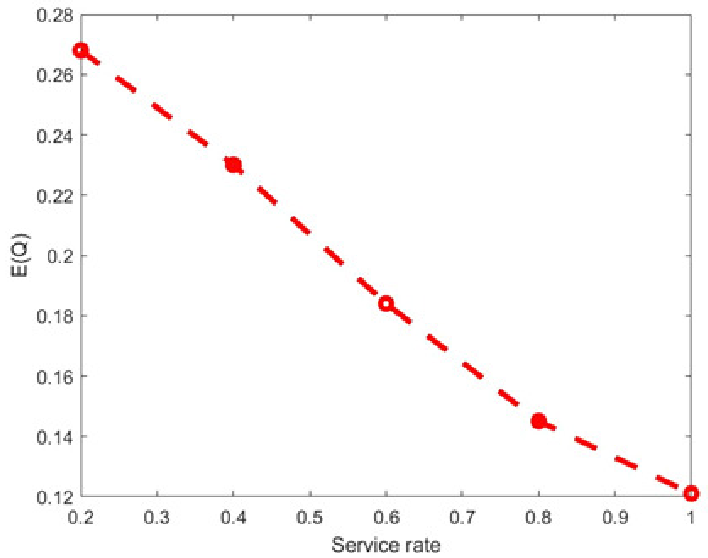


We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph.

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Service Rate	π_{00}	Mean Queue Length
0	.92	0.298
.2	.75	0.293
.4	.613	0.290
.6	.45	0.267
.8	.27	0.158

Index & Graph 2.2 Service Rate vs Mean Queue Length.



We notice that mean queue length decreases when service rate rises which is inferred by above index and table.

6 Conclusion

In this article a discrete time retrial queuing system with starting failures, Bernoulli feedback, general retrial times and a vacation has been analyzed briefly. In this model an analytical expression for PGF is derived by using generating function technique. In performance measure an expected queue length is derived in analytical expression form and by using this expression we investigate the length of the queue in several ways. In many real life situation this model is the most applicable.

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