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Hermite-Hadamard type inequalities for fourth-times differentiable arithmetic-harmonically functions

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Abstract: In this study, by using an integral identity together with both the Hölder integral inequality and the power-mean integral inequality we establish several new inequalities for fourth-times differentiable arithmetic-harmonically-convex function. Also, some applications are given for arithmetic-harmonically convex functions.

Keywords: Convex function, arithmetic-harmonically-convex function, Hermite-Hadamard's inequality, Hölder inequality, power-mean inequality.

1 Introduction

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

valid for all $x,y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [2,3,17,19,20]. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex functions see for instance the recent papers [5,6,7,8,10] and the references within these papers.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a<b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

holds.

This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [5, 9,11,12,13,14,18], for the results of the generalization, improvement and extention of the famous integral inequality (1). It was firstly discovered by Ch. Hermite in 1881 in the journal Mathesis (see [15]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result. E. F. Beckenbach, a leading expert on the

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history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinovic found Hermite's note in Mathesis [15]. Since (1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

Definition 2([4]). *A function $f : I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x,y \in I$ and $t \in [0,1]$ the equality*

$$f(tx+(1-t)y) \leq \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{2}$$

holds. If the inequality (2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

Theorem 2(Hölder Inequality for Integrals [16]). *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A\,|f(x)|^p = B\,|g(x)|^q$ almost everywhere, where A and B are constants.

Theorem 3(Power-mean Integral Inequality). *Let $q \geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

In this study, in order to establish some new inequalities of Hermite-Hadamard type inequalities for arithmetic harmonically convex functions, we will use the following lemma obtained in the specials case of identity given in [14].

Lemma 1.*Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a fourth-times differentiable mapping on I° and $f^{(4)} \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$, we have the identity*

$$\begin{aligned} &\frac{bf(b)-af(a)}{1!} - \frac{b^2f'(b)-a^2f'(a)}{2!} + \frac{b^3f''(b)-a^3f''(a)}{3!} \\ &-\frac{b^4f'''(b)-a^4f'''(a)}{4!} - \int_a^b f(x)dx = -\frac{1}{4!}\int_a^b x^4 f^{(4)}(x)dx. \end{aligned} \tag{3}$$

where an empty sum is understood to be nil.

By using above Lemma together with Hölder and power-mean integral inequalities, we derive a general integral identity for differentiable functions in order to provide inequality for functions whose fourth derivatives in absolute value at certain power are arithmetic-harmonically-convex functions.

Let $0<a<b$, throught this paper, we will use

$$A(a,b) = \frac{a+b}{2}$$

$$G(a,b) = \sqrt{ab}$$

$$L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & p \neq -1,0 \\ \frac{b-a}{\ln b - \ln a}, & p=-1 \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p=0. \end{cases}$$

for the arithmetic, the geometric and generalized logarithmic mean for $a,b>0$ respectively. In addition, we will use the following notation for shortness:

$$I_f(a,b)=\frac{bf(b)-af(a)}{1!}-\frac{b^2f'(b)-a^2f'(a)}{2!}+\frac{b^3f''(b)-a^3f''(a)}{3!}$$
$$-\frac{b^4f'''(b)-a^4f'''(a)}{4!}-\int_a^b f(x)dx=-\frac{1}{4!}\int_a^b x^4f^{(4)}(x)dx.$$

2 Main results

In this section, we will obtain our main results by using the Lemma 1.

Theorem 4. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be a fourth-times differentiable mapping on I°, and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(4)}\right|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f^{(4)}(a)\right|^q-\left|f^{(4)}(b)\right|^q\neq 0$, then

$$\left|I_f(a,b)\right|\leq\frac{b-a}{4!}\frac{L_{4p}^4(a,b)G^2\left(\left|f^{(4)}(a)\right|,\left|f^{(4)}(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f^{(4)}(a)\right|^q,\left|f^{(4)}(b)\right|^q\right)},\tag{4}$$

ii) If $\left|f^{(4)}(a)\right|^q-\left|f^{(4)}(b)\right|^q=0$, then

$$\left|I_f(a,b)\right|\leq\frac{b-a}{n!}\left|f^{(4)}(b)\right|L_{4p}^4(a,b),$$

where

$$B_{q,f}=B_{q,f}(a,b)=\left|f^{(4)}(a)\right|^q-\left|f^{(4)}(b)\right|^q,$$

$$C_{q,f}=C_{q,f}(a,b)=\frac{b\left|f^{(4)}(b)\right|^q-a\left|f^{(4)}(a)\right|^q}{B_{q,f}},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $\left|f^{(4)}(a)\right|^q-\left|f^{(4)}(b)\right|^q\neq 0$. If $\left|f^{(4)}\right|^q$ for $q>1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then using Lemma 1, well known Hölder integral inequality and the following identity

$$\left|f^{(4)}(x)\right|^q=\left|f^{(4)}\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\right|^q\leq\frac{(b-a)\left|f^{(4)}(a)\right|^q\left|f^{(4)}(b)\right|^q}{(b-x)\left|f^{(4)}(b)\right|^q+(x-a)\left|f^{(4)}(a)\right|^q},$$

we obtain

$$\begin{aligned}\left|I_f(a,b)\right|&\leq\frac{1}{4!}\left(\int_a^b x^{4p}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(4)}(x)\right|^q dx\right)^{\frac{1}{q}}\\&\leq\frac{1}{4!}\left(\int_a^b x^{4p}dx\right)^{\frac{1}{p}}\left(\int_a^b\frac{(b-a)\left|f^{(4)}(a)\right|^q\left|f^{(4)}(b)\right|^q}{(b-x)\left|f^{(4)}(b)\right|^q+(x-a)\left|f^{(4)}(a)\right|^q}dx\right)^{\frac{1}{q}}.\end{aligned}\tag{5}$$

Since

$$0<(b-x)\left|f^{(4)}(b)\right|^{q}+(x-a)\left|f^{(4)}(a)\right|^{q}=B_{q,f}\left(x+C_{q,f}\right),$$

we can write the following inequality:

$$I_f(a,b)\leq\frac{(b-a)L_{4p}^{4}(a,b)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\left(\frac{1}{B_{q,f}}\int_a^b\frac{1}{x+C_{q,f}}dx\right)^{\frac{1}{q}}.$$

It is easily seen that

$$\frac{1}{B_{q,f}}\int_a^b\frac{1}{x+C_{q,f}}dx=\frac{1}{B_{q,f}}\ln\frac{\left|f^{(4)}(a)\right|^{q}}{\left|f^{(4)}(b)\right|^{q}}=\frac{1}{L\left(\left|f^{(4)}(a)\right|^{q},\left|f^{(4)}(b)\right|^{q}\right)} \tag{6}$$

Therefeore, we have

$$\left|I_f(a,b)\right|\leq\frac{b-a}{4!}\frac{L_{4p}^{4}(a,b)G^{2}\left(\left|f^{(4)}(a)\right|,\left|f^{(4)}(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f^{(4)}(a)\right|^{q},\left|f^{(4)}(b)\right|^{q}\right)},$$

where

$$\int_a^b x^{4p}dx=(b-a)L_{4p}^{4p}(a,b).$$

ii) Let $\left|f^{(4)}(a)\right|^{q}-\left|f^{(4)}(b)\right|^{q}=0$. In this case, by substituting $\left|f^{(4)}(a)\right|^{q}=\left|f^{(4)}(b)\right|^{q}$ in (5) we obtain the following inequality:

$$\left|I_f(a,b)\right|\leq\frac{b-a}{4!}\left|f^{(4)}(b)\right|L_{4p}^{4}(a,b). \tag{7}$$

This completes the proof of the Theorem.

Theorem 5. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be a fourth-times differentiable mapping on I°, and $a,b\in I^{\circ}$ with $a<b$. If $\left|f^{(4)}\right|^{q}, q\geq 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f^{(4)}(a)\right|^{q}-\left|f^{(4)}(b)\right|^{q}\neq 0$, then

$$\begin{aligned}\left|I_f(a,b)\right|&\leq\frac{(b-a)L_4^{4\left(1-\frac{1}{q}\right)}(a,b)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\\&\times\left[\frac{1}{B_{q,f}}\sum_{k=0}^{3}(-1)^{k}C_{q,f}^{k}\left(\frac{b^{4-k}-a^{4-k}}{4-k}\right)+\frac{C_{q,f}^{4}}{L\left(\left|f^{(4)}(a)\right|^{q},\left|f^{(4)}(b)\right|^{q}\right)}\right]^{\frac{1}{q}}.\end{aligned} \tag{8}$$

ii) If $\left|f^{(4)}(a)\right|^{q}-\left|f^{(4)}(b)\right|^{q}=0$, then

$$\left|I_f(a,b)\right|\leq\frac{(b-a)L_4^{4}(a,b)\left|f^{(4)}(b)\right|}{4!},$$

where

$$B_{q,f}=B_{q,f}(a,b)=\left|f^{(4)}(a)\right|^{q}-\left|f^{(4)}(b)\right|^{q},$$

$$C_{q,f}=C_{q,f}(a,b)=\frac{b\left|f^{(4)}(b)\right|^{q}-a\left|f^{(4)}(a)\right|^{q}}{B_{q,f}}.$$

*Proof.*i) Let $\left|f^{(4)}(a)\right|^q - \left|f^{(4)}(b)\right|^q \neq 0$. If $\left|f^{(4)}\right|^q$ for $q \geq 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then using Lemma 1, well known power-mean integral inequality and

$$\left|f^{(4)}(x)\right|^q \leq \left|f^{(4)}\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\right|^q = \frac{(b-a)\left|f^{(4)}(a)\right|^q\left|f^{(4)}(b)\right|^q}{(b-x)\left|f^{(4)}(b)\right|^q+(x-a)\left|f^{(4)}(a)\right|^q}$$

we write the following ineqaulity:

$$\begin{aligned}
&\left|I_f(a,b)\right| \qquad (9)\\
&\leq \frac{1}{4!}\left(\int_a^b x^4dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^4\left|f^{(4)}(x)\right|^q dx\right)^{\frac{1}{q}}\\
&\leq \frac{1}{4!}\left(\int_a^b x^4dx\right)^{1-\frac{1}{q}}\left(\int_a^b \frac{x^4(b-a)\left|f^{(4)}(a)\right|^q\left|f^{(4)}(b)\right|^q}{(b-x)\left|f^{(4)}(b)\right|^q+(x-a)\left|f^{(4)}(a)\right|^q}dx\right)^{\frac{1}{q}}\\
&= \frac{b-a}{4!}L_4^{4\left(1-\frac{1}{q}\right)}(a,b)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|\left(\int_a^b \frac{x^4}{(b-x)\left|f^{(4)}(b)\right|^q+(x-a)\left|f^{(4)}(a)\right|^q}dx\right)^{\frac{1}{q}}\\
&= \frac{(b-a)L_4^{4\left(1-\frac{1}{q}\right)}(a,b)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\left(\frac{1}{B_{q,f}}\int_a^b \frac{x^4}{x+C_{q,f}}dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_a^b \frac{x^4}{x+C_{q,f}}dx &= \int_a^b \sum_{k=0}^{3}(-1)^k C_{q,f}^k x^{3-k}dx + \int_a^b \frac{C_{q,f}^4}{x+C_{q,f}}dx\\
&= \sum_{k=0}^{3}(-1)^k C_{q,f}^k \int_a^b x^{3-k}dx + C_{q,f}^4\int_a^b \frac{1}{x+C_{q,f}}dx\\
&= \sum_{k=0}^{3}(-1)^k C_{q,f}^k \left(\frac{b^{4-k}-a^{4-k}}{4-k}\right) + C_{q,f}^4\int_a^b \frac{1}{x+C_{q,f}}dx,
\end{aligned}$$

we have the following inequality:

$$\begin{aligned}
\left|I_f(a,b)\right| \leq{}& \frac{(b-a)L_4^{4\left(1-\frac{1}{q}\right)}(a,b)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\\
&\times\left[\frac{1}{B_{q,f}}\sum_{k=0}^{3}(-1)^k C_{q,f}^k\left(\frac{b^{4-k}-a^{4-k}}{4-k}\right)+\frac{C_{q,f}^4}{L\left(\left|f^{(4)}(a)\right|^q,\left|f^{(4)}(b)\right|^q\right)}\right]^{\frac{1}{q}}.
\end{aligned}$$

ii) Let $\left|f^{(4)}(a)\right|^q - \left|f^{(4)}(b)\right|^q = 0$. By using the inequality (9), we obtain

$$\left|I_f(a,b)\right| \leq \frac{(b-a)L_4^4(a,b)\left|f^{(4)}(b)\right|}{4!} \qquad (10)$$

This completes the proof of the Theorem.

Corollary 1. *If we take $q=1$ in (8), we get the following inequality:*

$$\left|I_f(a,b)\right| \le \frac{(b-a)G^2\left(\left|f^{(4)}(a)\right|,\left|f^{(4)}(b)\right|\right)}{4!} \times \left[\frac{1}{B_f}\sum_{k=0}^{3}(-1)^k C_f^k\left(\frac{b^{4-k}-a^{4-k}}{4-k}\right)+\frac{C_f^4}{L\left(\left|f^{(4)}(a)\right|,\left|f^{(4)}(b)\right|\right)}\right].$$

Theorem 6. *Let $f: I\subset\mathbb{R}\to(0,\infty)$ be a fourth-times differentiable mapping on I°, and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(4)}\right|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:*

i) If $\left|f^{(4)}(a)\right|-\left|f^{(4)}(b)\right|\neq 0$, then

$$\left|I_f(a,b)\right| \le \frac{(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\left[\frac{1}{B_f}\sum_{k=0}^{3}(-1)^k C_f^k\left(\frac{b^{4-k}-a^{4-k}}{n-k}\right)+\frac{C_f^4}{L\left(\left|f^{(4)}(a)\right|^q,\left|f^{(4)}(b)\right|^q\right)}\right], \tag{11}$$

ii) If $\left|f^{(4)}(a)\right|-\left|f^{(4)}(b)\right|=0$, then

$$\left|I_f(a,b)\right| \le \frac{(b-a)\left|f^{(4)}(b)\right|}{4!}L_4^4(a,b),$$

where

$$B_f=B_f(a,b)=\left|f^{(4)}(a)\right|-\left|f^{(4)}(b)\right|,$$

$$C_f=C_f(a,b)=\frac{b\left|f^{(4)}(b)\right|-a\left|f^{(4)}(a)\right|}{B_f}.$$

Proof. i) Let $\left|f^{(4)}(a)\right|-\left|f^{(4)}(b)\right|\neq 0$. If $\left|f^{(4)}\right|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, using Lemma 1 and

$$\left|f^{(4)}(x)\right|=\left|f^{(4)}\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\right|\le\frac{(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{(b-x)\left|f^{(4)}(b)\right|+(x-a)\left|f^{(4)}(a)\right|}$$

we get

$$\begin{aligned}
&\left|I_f(a,b)\right|\\
&\le \frac{1}{4!}\int_a^b x^4\left|f^{(4)}(x)\right|dx\\
&\le \frac{1}{4!}\int_a^b \frac{x^4(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{(b-x)\left|f^{(4)}(b)\right|+(x-a)\left|f^{(4)}(a)\right|}dx\\
&= \frac{(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\int_a^b \frac{x^4}{(b-x)\left|f^{(4)}(b)\right|+(x-a)\left|f^{(4)}(a)\right|}dx.
\end{aligned} \tag{12}$$

Since

$$0<(b-x)\left|f^{(4)}(b)\right|+(x-a)\left|f^{(4)}(a)\right|=B_f\left(x+C_f\right),$$

we can write the following inequality:

$$I_f(a,b)\le\frac{(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!B_f}\int_a^b\frac{x^4}{x+C_f}dx. \tag{13}$$

Sample calculation give us that

$$\int_a^b \frac{x^4}{x+C_f}dx = \int_a^b \sum_{k=0}^{3}(-1)^k C_f^k x^{3-k}dx + \int_a^b \frac{C_f^4}{x+C_f}dx \tag{14}$$
$$= \sum_{k=0}^{3}(-1)^k C_f^k \int_a^b x^{3-k}dx + C_f^4 \int_a^b \frac{1}{x+C_f}dx$$
$$= \sum_{k=0}^{3}(-1)^k C_f^k \left(\frac{b^{4-k}-a^{4-k}}{4-k}\right) + C_f^4 \int_a^b \frac{1}{x+C_f}dx.$$

From the inequalities (12), (13) and (14), we get the desired inequality.

ii) Let $\left|f^{(4)}(a)\right| - \left|f^{(4)}(b)\right| = 0$. Then, by substituting $\left|f^{(4)}(a)\right| = \left|f^{(4)}(b)\right|$ in (12), we obtain

$$\left|I_f(a,b;n)\right| \le \frac{(b-a)\left|f^{(4)}(a)\right|\left|f^{(4)}(b)\right|}{4!}\int_a^b \frac{x^4}{(b-x)\left|f^{(n)}(b)\right| + (x-a)\left|f^{(n)}(a)\right|}dx$$
$$= \frac{(b-a)\left|f^{(4)}(b)\right|}{4!}L_4^4(a,b). \tag{15}$$

This completes the proof of theorem.

3 Applications for special means

We know that if $p \in (-1,0)$ then the function $f(x) = x^p, x > 0$ is an arithmetic harmonically-convex function [4]. By using this function we obtain following propositions related to means:

Proposition 1. *Let $a,b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-1,0)$. Then, we have the following inequality:*

$$L_{\frac{m}{q}+4}^{\frac{m}{q}+4}(a,b) \le \frac{L_{4p}^4(a,b)G^{\frac{2m}{q}}(a,b)}{\left[L(a,b)L_{m-1}^{m-1}(a,b)\right]^{\frac{1}{q}}}.$$

Proof. The assertion follows from the inequality (4) in the Theorem 4. Let

$$f(x) = \frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)\left(\frac{m}{q}+3\right)\left(\frac{m}{q}+4\right)}x^{\frac{m}{q}+4}, \quad x \in (0,\infty).$$

Then $\left|f^{(4)}(x)\right|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 4.

Proposition 2. *Let $a,b \in (0,\infty)$ with $a<b$, $q>1$ and $m \in (-1,0)$. Then, we have the following inequality:*

$$L_{\frac{m}{q}+4}^{\frac{m}{q}+4}(a,b) \le L_4^{4\left(1-\frac{1}{q}\right)}(a,b) G^{\frac{2m}{q}}(a,b) \tag{16}$$
$$\times \left\{ -\frac{1}{mL_{m-1}^{m-1}(a,b)} \left[A(a,b)A(a^2,b^2) + \frac{m+1}{m}\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)} \left(\frac{2A(a^2,b^2)+G^2(a,b)}{3}\right) \right.\right.$$
$$\left. +A(a,b)\left(\frac{m+1}{m}\right)^2 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^2 + \left(\frac{m+1}{m}\right)^3 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^3 \right]$$
$$\left. + \left(\frac{m+1}{m}\right)^4 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^4 \frac{1}{L(a,b)L_{m-1}^{m-1}(a,b)} \right\}^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (8) in the Theorem 5. Let

$$f(x) = \frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)\left(\frac{m}{q}+3\right)\left(\frac{m}{q}+4\right)} x^{\frac{m}{q}+4}, \quad x \in (0,\infty).$$

Then $\left|f^{(4)}(x)\right|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 5.

Proposition 3. *Let $0<a<b$ and $p \in (-1,0)$. Then we have the following inequalities:*

$$L_{p+4}^{p+4}(a,b) \le \frac{G^{2p}(a,b)}{a^p - b^p} \left\{ -\frac{1}{mL_{m-1}^{m-1}(a,b)} \left[A(a,b)A(a^2,b^2) + \frac{m+1}{m}\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)} \left(\frac{2A(a^2,b^2)+G^2(a,b)}{3}\right) \right.\right.$$
$$\left. +A(a,b)\left(\frac{m+1}{m}\right)^2 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^2 + \left(\frac{m+1}{m}\right)^3 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^3 \right]$$
$$\left. + \left(\frac{m+1}{m}\right)^4 \left(\frac{L_m^m(a,b)}{L_{m-1}^{m-1}(a,b)}\right)^4 \frac{1}{L(a,b)L_{m-1}^{m-1}(a,b)} \right\}^{\frac{1}{q}}.$$

Proof. Let be $p \in (-1,0)$. Then we consider the function

$$f(x) = \frac{x^{p+4}}{(p+1)(p+2)(p+3)(p+4)}, \quad x>0.$$

Under the assumption of the Proposition

$$\left|f^{(4)}(x)\right| = x^p$$

is an AH-convex on $(0,\infty)$. Therefore, the assertion follows from the inequality (11) in the Theorem 6, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = \frac{x^{p+4}}{(p+1)(p+2)(p+3)(p+4)}$.

4 Conclusion

In this work, we established several new inequalities for fourth-times differentiable arithmetic-harmonically-convex function and obtained some new Hermite-Hadamard type inequalities connected with means. Similar method can be applied to the different type of convex functions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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