

# Approximation by univariate, bivariate and GBS operators of Chlodowsky-Kantorovich operators

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**Abstract:** We define univariate, bivariate and GBS operators of Chlodowsky-Kantorovich operators and we give their approximations and rate of their approximations for B-continuous and B-differentiable functions. Finally we give some plots to compare operators and their GBS operators.

**Keywords:** GBS-operators of Chlodowsky-Kantorovich operators, Bögel continuous, Bögel differentiable, rate of approximation, bivariate functions.

## 1 Introduction and the construct of the one variable Chlodowsky-Kantorovich operators

Classical approximation theory, including polynomial approximation is a fundamental research area in applied mathematics. Development in approximation theory plays an important role in numerical solution of partial differential equations, image processing as well as in data sciences and many other disciplines. For example, radial basis functions and shift-invariant spaces are widely used for geometric modeling in aerospace and automobile industries. Due to the importance of polynomials, a variety of their generalizations and related topics have been studied in [1],[2], [3] and [4].

Bernstein polynomials are not useful for discontinuous measurable functions. More suitable modification of the Bernstein polynomials, for this kind functions, introduced by Kantorovich [5] : For any  $f \in L_p[0, 1]$ ,  $x \in [0, 1]$  and  $p \geq 1$

$$K_n(f;x) = (n+1) \sum_{k=0}^n \varphi_n^k(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt. \quad (1)$$

(1) If we take definite integral operator of  $f$  as

$$F(x) = \int_0^x f(t) dt$$

then it is possible to connect the Kantorovich polynomials with the Bernstein polynomials by means of the identity

$$K_n(f;x) = \frac{d}{dx} B_{n+1}(f;x)$$

(see [3], cf:30). Chlodowsky [6] modified Bernstein operators as the following:

$$C_n(f;x) = \sum_{k=0}^n f\left(\frac{k}{n}b_n\right)\varphi_n^k\left(\frac{x}{b_n}\right), \quad 0 \leq x \leq b_n \quad (2)$$

where  $(b_n)$  is a sequence with positive terms which increasing and satisfying the conditions

$$\lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0. \quad (3)$$

Bernstein-Chlodowsky polynomials are studied in (for instance [7], [8], [9], [10]).

Chlodowsky-Kantorovich operators were introduced by Habib and Wafi in [11]. We produced our study by making use of the "Degree of Approximation of Functions by Modified Bernstein Polynomials On Unbounded Interval" article studied by Habib and Wafi in [11].

In this article we intend to study the approximation features of the operator

$$H_n(f;x) = \frac{(n+1)}{b_n} \sum_{k=0}^n \varphi_n^k\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} f(t)dt. \quad (4)$$

which is Chlodowsky-Kantorovich operators. Here  $\varphi_n^k(x) = \binom{n}{k} x^k (1-x)^{n-k}$ ,  $x \in [0, b_n]$ ,  $f \in C[0, b_n]$  and  $b_n$  is an increasing and positive sequence with features  $\lim_{n \rightarrow \infty} b_n = \infty, \lim_{n \rightarrow \infty} \frac{b_n^2}{n} = 0$  instead of (3).

**Lemma 1.** Let  $e_i := e_i(x) = x^i$ ,  $E_i := E_i(u, x) = (u-x)^i$  and  $f \in C[0, b_n]$ , we have

$$H_n(e_0;x) = 1,$$

$$H_n(e_1;x) = x + \frac{b_n - 2x}{2(n+1)},$$

$$H_n(e_2;x) = x^2 + \frac{x[2nb_n - (3n+1)x]}{(n+1)^2} + \frac{b_n^2}{3(n+1)^2},$$

$$\begin{aligned} H_n(e_3;x) &= x^3 - \frac{6n^2 + n + 1}{(n+1)^3} x^3 + \frac{9n(n-1)b_n}{2(n+1)^3} x^2 + \frac{15nb_n^2}{2(n+1)^3} x + \frac{b_n^3}{4(n+1)^3} \\ &= x^3 + \frac{x}{2(n+1)^3} [15nb_n^2 + 9n(n-1)b_n x - (12n^2 + 2n + 2)x^2] + \frac{b_n^3}{4(n+1)^3}, \end{aligned}$$

$$\begin{aligned} H_n(e_4;x) &= x^4 - \frac{10n^3 - 5n^2 + 10n + 1}{(n+1)^4} x^4 + \frac{8n(n-1)(n-2)b_n}{(n+1)^4} x^3 + \frac{15n(n-1)b_n^2}{(n+1)^4} x^2 + \frac{38nb_n^3}{(n+1)^4} x + \frac{b_n^4}{5(n+1)^4} \\ &= x^4 + \frac{x}{(n+1)^4} [38nb_n^3 + 15n(n-1)b_n^2 x + 8n(n-1)(n-2)b_n x^2 - (10n^3 - 5n^2 + 10n + 1)x^3] + \frac{b_n^4}{5(n+1)^4} \end{aligned}$$

From Lemma 1 we get the following equalities and inequalities:

$$H_n(E_2;x) = \frac{(n-1)}{(n+1)^2} x(b_n - x) + \frac{b_n^2}{3(n+1)^2}, \quad (5)$$

$$H_n(E_4;x) = \frac{3n^2 - 20n + 1}{(n + 1)^4}x^4 - \frac{6n^2 - 40n + 2}{(n + 1)^4}b_n x^3 + \frac{13n^2 + 41n + 2}{(n + 1)^4}b_n^2 x^2 + \frac{37n - 1}{(n + 1)^4}b_n^3 x + \frac{b_n^4}{5(n + 1)^4} \tag{6}$$

It is easy to see that on  $[0, b_n]$

$$H_n(E_2;x) \leq \frac{3n + 1}{12(n + 1)^2}b_n^2 < \frac{b_n^2}{4(n + 1)} \tag{7}$$

and

$$\begin{aligned} H_n(E_4;x) &\leq \frac{(3n^2 - 20n + 1) + (13n^2 + 41n + 2) + (37n - 1) + \frac{1}{5}}{(n + 1)^4}b_n^4 \\ &< \frac{16n^2 + 58n + 3}{(n + 1)^4}b_n^4 \\ &< \frac{29}{(n + 1)^2}b_n^4. \end{aligned} \tag{8}$$

And also it is seen that on  $[0, A] \subset [0, b_n]$

$$\begin{aligned} H_n(E_2;x) &< \frac{(n - 1)}{(n + 1)^2}A + \frac{b_n^2}{3(n + 1)^2} = \frac{3(n - 1)b_n A + b_n^2}{(n + 1)^2} \\ &< \frac{(3nA + b_n)b_n}{3(n + 1)^2} < \frac{b_n}{(n + 1)}, \end{aligned} \tag{9}$$

and

$$H_n(E_4;x) < \frac{29}{(n + 1)^2}b_n^4. \tag{10}$$

**Theorem 1.** Let  $x \in [0, b_n]$ ,  $f \in C[0, b_n]$  and  $H_n(f;x)$  be linear positive operators. If

- (i)  $\lim_{n \rightarrow \infty} H_n(e_0;x) = 1$
- (ii)  $\lim_{n \rightarrow \infty} H_n(e_1;x) = x$
- (iii)  $\lim_{n \rightarrow \infty} H_n(e_2;x) = x^2$

$H_n$  converges to  $f$ , for  $f \in C[0, b_n]$ .

From Lemma 1, (3) and defined feature  $b_n$ , the result is easily visible. Due to this theorem,  $H_n(f;x)$  operators fulfills the Korovkin condition.

## 2 The construct of the bivariate operators of Chlodowsky-Kantorovich type

In this section first we construct the bivariate operators of Chlodowsky-Kantorovich type on rectangle. Bivariate operators of Kantorovich type on simplex was introduced in ([26]).

The purpose of this paper is to give a representation for the bivariate Chlodowsky-Kantorovich operators.

For  $n, m \in \mathbb{N}$  we define the operator  $H_{n,m}: C([0, b_n] \times [0, b_m]) \rightarrow C([0, b_n] \times [0, b_m])$  be defined for any function  $f \in C([0, b_n] \times [0, b_m])$  by

$$H_{n,m}(f;x,y) = \frac{(n + 1)(m + 1)}{b_n b_m} \sum_{k=0}^n \sum_{j=0}^m \varphi_n^k\left(\frac{x}{b_n}\right) \varphi_m^j\left(\frac{y}{b_m}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} \int_{\frac{j b_m}{m+1}}^{\frac{(j+1)b_m}{m+1}} f(t,u) dt du \tag{11}$$

for any  $(x, y) \in [0, b_n] \times [0, b_m]$ .

The operators defined above are called the bivariate operators of Chlodowsky-Kantorovich type. Clearly, the bivariate operators of Chlodowsky-Kantorovich type are linear and positive. And also we can see that easily

$$H_{n,m}(t^r u^s; x, y) = H_n(t^r; x)H_m(u^s; y) \text{ and } H_{n,m}(t^r + u^s; x, y) = H_n(t^r; x) + H_m(u^s; y)$$

From Lemma 1 we can give the following lemma:

**Lemma 2.** Let  $e_{ij} := e_{ij}(x, y) = x^i y^j$ ,  $E_{ij} := E_{ij}(u, v; x, y) = (u - x)^i (v - y)^j$  and  $f \in C([0, b_n] \times [0, b_m])$ , we have

$$H_{n,m}(e_{00}; x, y) = 1 \quad (12)$$

$$H_{n,m}(e_{10}; x, y) = x + \frac{b_n - 2x}{2(n+1)}, H_{n,m}(e_{01}; x, y) = y + \frac{b_m - 2y}{2(m+1)} \quad (13)$$

$$H_{n,m}(e_{20}; x, y) = x^2 + \frac{x[2nb_n - (3n+1)x]}{(n+1)^2} + \frac{b_n^2}{3(n+1)^2}$$

$$H_{n,m}(e_{02}; x, y) = y^2 + \frac{y[2mb_m - (3m+1)y]}{(m+1)^2} + \frac{b_m^2}{3(m+1)^2}$$

$$H_{n,m}(e_{30}; x, y) = x^3 - \frac{6n^2 + n + 1}{(n+1)^3} x^3 + \frac{9n(n-1)b_n}{2(n+1)^3} x^2 + \frac{15nb_n^2}{2(n+1)^3} x + \frac{b_n^3}{4(n+1)^3}$$

$$H_{n,m}(e_{03}; x, y) = y^3 - \frac{6m^2 + m + 1}{(m+1)^3} y^3 + \frac{9m(m-1)b_m}{2(m+1)^3} y^2 + \frac{15mb_m^2}{2(m+1)^3} y + \frac{b_m^3}{4(m+1)^3}$$

$$\begin{aligned} H_{n,m}(e_{40}; x, y) &= x^4 - \frac{10n^3 - 5n^2 + 10n + 1}{(n+1)^4} x^4 + \frac{8n(n-1)(n-2)b_n}{(n+1)^4} x^3 + \frac{15n(n-1)b_n^2}{(n+1)^4} x^2 + \frac{38nb_n^3}{(n+1)^4} x + \frac{b_n^4}{5(n+1)^4} \\ &= x^4 + \frac{x}{(n+1)^4} [38nb_n^3 + 15n(n-1)b_n^2 x + 8n(n-1)(n-2)b_n x^2 \\ &\quad - (10n^3 - 5n^2 + 10n + 1)x^3] + \frac{b_n^4}{5(n+1)^4} \end{aligned}$$

$$\begin{aligned} H_{n,m}(e_{04}; x, y) &= y^4 - \frac{10m^3 - 5m^2 + 10m + 1}{(m+1)^4} y^4 + \frac{8m(m-1)(m-2)b_m}{(m+1)^4} y^3 + \frac{15m(m-1)b_m^2}{(m+1)^4} y^2 + \frac{38mb_m^3}{(m+1)^4} y + \frac{b_m^4}{5(m+1)^4} \\ &= y^4 + \frac{y}{(m+1)^4} [38mb_m^3 + 15m(m-1)b_m^2 y + 8m(m-1)(m-2)b_m y^2 \\ &\quad - (10m^3 - 5m^2 + 10m + 1)y^3] + \frac{b_m^4}{5(m+1)^4}. \end{aligned}$$

From Lemma 2, we have

$$H_{n,m}(E_{20}; x, y) = \frac{(n-1)}{(n+1)^2} x(b_n - x) + \frac{b_n^2}{3(n+1)^2}, \quad (14)$$

$$H_{n,m}(E_{02}; x, y) = \frac{(m-1)}{(m+1)^2} y(b_m - y) + \frac{b_m^2}{3(m+1)^2} \quad (15)$$

and

$$H_{n,m}(E_{40};x,y) = \frac{3n^2 - 20n + 1}{(n + 1)^4}x^4 - \frac{6n^2 - 40n + 2}{(n + 1)^4}b_n x^3 + \frac{13n^2 + 41n + 2}{(n + 1)^4}b_n^2 x^2 + \frac{37n - 1}{(n + 1)^4}b_n^3 x + \frac{b_n^4}{5(n + 1)^4}, \tag{16}$$

$$H_{n,m}(E_{04};x,y) = \frac{3m^2 - 20m + 1}{(m + 1)^4}y^4 - \frac{6m^2 - 40m + 2}{(m + 1)^4}b_m y^3 + \frac{13m^2 + 41m + 2}{(m + 1)^4}b_m^2 y^2 + \frac{37m - 1}{(m + 1)^4}b_m^3 y + \frac{b_m^4}{5(m + 1)^4}. \tag{17}$$

**Theorem 2.** Let  $(x, y) \in [0, b_n] \times [0, b_m]$ . and  $f \in C([0, b_n] \times [0, b_m])$ ;  $H_{n,m}(f;x,y)$  be linear positive operators. If

- (i)  $\lim_{n,m \rightarrow \infty} H_n(e_{00};x,y) = 1$
- (ii)  $\lim_{n,m \rightarrow \infty} H_n(e_{10};x,y) = x$
- (iii)  $\lim_{n,m \rightarrow \infty} H_n(e_{01};x,y) = y$
- (iv)  $\lim_{n,m \rightarrow \infty} H_n((e_{20} + e_{02};u,v) = x^2 + y^2)$   $H_{n,m}$  converges to  $f$ , for  $f \in C([0, b_n] \times [0, b_m])$ .

From Lemma 2, (3) and defined feature  $b_n$ , the result is easily visible. Due to this theorem,  $H_{n,m}(f;x,y)$  operators fulfills the Korovkin condition.

### 3 Main results

GBS operators of Chlodowsky-Kantorovich type, to establish a convergence theorem for these operators. We also give an approximation theorem for these operators in terms of the first modulus of smoothness and of the mixed modulus of smoothness for bivariate functions.

The terms of "B-Continuity" and "B-Differentiability" (Bögel-Continuity and Bögel-Differentiability) were introduced by K.Bögel ([13], [14], [15]) as the following. Let  $X$  and  $Y$  be compact real intervals,  $f : X \times Y \rightarrow \mathbb{R}$  a function, and  $\Delta f [(x,y), (x_0,y_0)] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ .  $\Delta f [(x,y), (x_0,y_0)]$  is called mixed difference of  $f$ .

**Definition 1.** (a) A function  $f$  is called a B-Continuous function in  $(x_0,y_0) \in X \times Y$  if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \Delta f [(x,y), (x_0,y_0)] = 0. \tag{18}$$

(b) A function  $f$  is called a B-Differentiable function in  $(x_0,y_0) \in X \times Y$  if the following limit is exist and finite,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{\Delta f [(x,y), (x_0,y_0)]}{(x - y)(x_0 - y_0)}. \tag{19}$$

This B-Differentiable of  $f$  in  $(x_0,y_0)$  is denoted by  $D_B f(x_0,y_0)$ .

Let  $F$  be the class of all functions  $f : X \times Y \rightarrow \mathbb{R}$ . Then we use subsets of  $F$  which are given in the following:

$$B(X \times Y) = \{f \in F : f \text{ bounded on } X \times Y\}$$

with usual sup-norm  $\| \cdot \|_\infty$ .

$$B_b(X \times Y) = \{f \in F : |\Delta f [(x,y), (x_0,y_0)]| \leq K, \text{ on } X \times Y, K > 0\}$$

is called  $B$ -bounded functions class with the norm

$$\| \cdot \|_b = \sup_{(x,y),(x_0,y_0) \in X \times Y} |\Delta f[(x,y),(x_0,y_0)]|.$$

$$C_b(X \times Y) = \{f \in F : f \text{ is } B\text{-Continuous on } X \times Y\},$$

$$D_b(X \times Y) = \{f \in F : f \text{ is } B\text{-Differentiable on } X \times Y\}$$

The term GBS(Generalized Boolean Sum) operators were introduced by Badea and Kottin as the following [16].

An important theorem (the analogous of the well-known Korovkin theorem), for approximation of  $B$ -Continuous functions using GBS-operators is due to C. Badea and his collaborates [17].

In order to estimate the order approximation of  $B$ -Continuous functions using GBS operators, we use "the mixed modulus of smoothness" which was introduced by I. Badea [18]. It is analogous of first modulus of smoothness for univariate functions.

**Definition 2.** Let  $f \in B_b(X \times Y)$ . For any  $(\delta_1, \delta_2) \in \mathbb{R}_{0,+}^2$ , the mixed modulus of smoothness is the function  $\omega_{mixed}(f; \delta_1, \delta_2) : \mathbb{R}_{0,+}^2 \rightarrow \mathbb{R}$  defined by

$$\omega_{mixed}(f; \delta_1, \delta_2) = \sup \{ |\Delta f[(x,y),(u,v)]| : |u-x| \leq \delta_1, |v-y| \leq \delta_2 \} \quad (20)$$

where  $\mathbb{R}_{0,+} := [0, \infty)$ .

The features of mixed moduli of continuity ;

$$\omega_{mixed}(f; \lambda_1 \delta_1, \lambda_1 \delta_2) \leq (1 + \lambda_1)(1 + \lambda_2) \omega_{mixed}(f; \delta_1, \delta_2) \quad (21)$$

we can write,

$$\begin{aligned} |\nabla h[(u,v);(x,y)]| &\leq \omega_{mixed}(f; |u-x|, |v-y|) \\ &\leq \left(1 + \frac{|u-x|}{\delta_1}\right) \left(1 + \frac{|v-y|}{\delta_2}\right) \omega_{mixed}(f; \delta_1, \delta_2) \end{aligned} \quad (22)$$

There are many studies about approximation by associated GBS operators of some well-known operators and their generalizations. For instance, ([18]-[19]), ([20]-[22]) and [23].

The approximation theorems for bivariate functions were first given by Volkov in [24] and approximation of the GBS operators of associate with operators of two variables were established by [17]. More detail about GBS it can be seen in ([16], [25], [13], [14], [15], [21]).

Now from (11) we construct GBS operator associated to the operator  $H_{n,m}$  :

$$H_{n,m}^*(f; x, y) = H_{n,m}((f(t, y) + f(x, u) - f(t, u)); x, y) \quad (23)$$

Define  $D_{AB} = [0, A] \times [0, B]$  where  $[0, A] \times [0, B] \subset [0, b_n] \times [0, b_m]$ .

In the following theorems we examine approximation of  $(P_{n,m}^*)_{n,m \in \mathbb{N}}$  to  $f$  (here  $f \in C_b(D_{AB})$  or  $f \in D_b(D_{AB})$ ) and rate of approximation of  $(P_{n,m}^*)_{n,m \in \mathbb{N}}$  to  $f$ . Taking into account the relations theorems defined by Bögel we obtain the following theorem.

**Theorem 3.** *The sequence  $(H_{n,m}^*)_{n,m \in \mathbb{N}}$  converges to any  $f \in C(D_{AB})$  uniformly. It easy to see the following relation:*

$$H_{n,m}^*(t^i u^j; x, y) = x^i y^j \text{ for all } i, j = 0, 1, 2, \dots$$

That means there is no approximation for any usual continuous function  $f$  on  $[0, A]$ . Mean  $H_{n,m}^*(f; x, y) = f$  for all  $f \in C(D_{AB})$ .

**Theorem 4.** *If  $f \in C_b(D_{b_n b_m})$ , then for any  $(x, y) \in R_{AB}$ , the following relation holds for all :*

$$|H_{n,m}^*(f(u, v); x, y) - f(x, y)| \leq 3 \cdot \omega_{mixed}(f; \sqrt{\frac{b_n}{(n+1)}}, \sqrt{\frac{b_m}{(m+1)}}).$$

*Proof.* Applying Theorem given in [16] for rate of GBS operators, Lemma 2 ,( by taking in account (4),(11), (19) and (20) we have

$$\begin{aligned} |H_{n,m}^*(f(u, v); x, y) - f(x, y)| &\leq \left( \frac{1}{\delta_1} \sqrt{H_{n,m}(E_{20}; x, y)} + \frac{1}{\delta_2} \sqrt{H_{n,m}(E_{02}; x, y)} \right. \\ &\quad \left. + \frac{1}{\delta_1 \delta_2} \sqrt{H_{n,m}(E_{20}; x, y) H_{n,m}(E_{02}; x, y)} \right) \omega_{mixed}(f; \delta_1, \delta_2) \\ &\leq \left( \frac{1}{\delta_1} \sqrt{\frac{b_n}{(n+1)}} + \frac{1}{\delta_2} \sqrt{\frac{b_m}{(m+1)}} \right. \\ &\quad \left. + \frac{1}{\delta_1 \delta_2} \sqrt{\frac{b_n}{(n+1)}} \sqrt{\frac{b_m}{(m+1)}} \right) \omega_{mixed}(f; \delta_1, \delta_2) \end{aligned}$$

If we choose  $\delta_1 = \sqrt{\frac{b_n}{(n+1)}}$  and  $\delta_2 = \sqrt{\frac{b_m}{(m+1)}}$ , we get

$$|H_{n,m}^*(f(u, v); x, y) - f(x, y)| \leq 3 \cdot \omega_{mixed}(f; \sqrt{\frac{b_n}{(n+1)}}, \sqrt{\frac{b_m}{(m+1)}}),$$

then the proof is being completed.

**Theorem 5.** *If  $f \in D_B(R_{b_n b_m})$  with  $D_B f \in B(R_{b_n b_m})$ , then for any  $(x, y) \in R_{AB}$  and sufficiently large  $n$  and  $m$*

$$|H_{n,m}^*(f(u, v); x, y) - f(x, y)| \leq 3 \|D_B f\|_\infty + \frac{64}{5} \cdot \omega_{mixed}(D_B f; \frac{b_n^2}{(n+1)}, \frac{b_m^2}{(m+1)}).$$

*Proof.* Applying Theorem which given [21] for B-Differentiable function , Lemma 2 ,( by taking in account (4) and (9)), (18), theorems defined by Bögel, we have

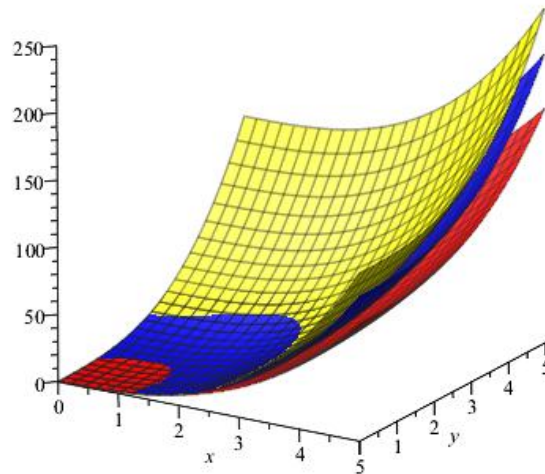
$$\begin{aligned} |H_{n,m}^*(f(u, v); x, y) - f(x, y)| &\leq 3 \|D_B f\|_\infty \sqrt{\frac{b_n}{(n+1)} \cdot \frac{b_m}{(m+1)}} + \left[ \sqrt{\frac{b_n}{(n+1)} \cdot \frac{b_m}{(m+1)}} + \frac{1}{\delta_1} \sqrt{\frac{29b_n^4}{(n+1)^2} \cdot \frac{b_m}{(m+1)}} \right. \\ &\quad \left. + \frac{1}{\delta_2} \sqrt{\frac{b_n}{(n+1)} \cdot \frac{29b_m^4}{(m+1)^2}} + \frac{1}{\delta_1 \delta_2} \cdot \frac{b_n}{(n+1)} \cdot \frac{b_m}{(m+1)} \right] \omega_{mixed}(D_B f, \delta_1, \delta_2). \end{aligned}$$

If we choose  $\delta_1 = \frac{b_n^2}{(n+1)}$  and  $\delta_2 = \frac{b_m^2}{(m+1)}$  and take into account that  $\frac{b_n}{(n+1)} < 1$  for sufficiently large  $n$  since  $\lim_{n \rightarrow \infty} \frac{b_n^2}{n} = 0$  then we get desired result.

#### 4 Some plots

The following calculations are made by Maple17.

- (1) We apply (11) for function  $f(x,y) = x^3 + y^3$  in the region  $[0,5] \times [0,5]$ , then we get the following graphics. Here  $(n,m) = (10,10)$  (red),  $(n,m) = (20,20)$  (blue).



**Fig. 1:** The convergence of the  $H_{n,m}(f(u,v);x,y)$  operator to the function  $f(x,y)$ .

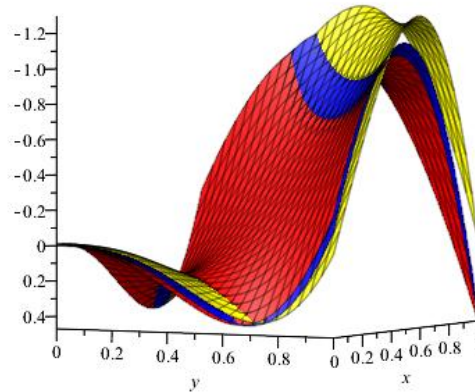
- (2) The convergence of  $H_{n,m}^*(f(u,v);x,y)$  GBS operator for  $(n,m) = (10,10)$  (red),  $(n,m) = (20,20)$  (blue) to the function  $f(x,y) = (x^2 + y^2) \sin(\pi(x+y))$  (yellow) in the region  $[0,1] \times [0,1]$  is pictorial as shown in Figure 2.
- (3) For  $(n,m) = (20,20)$ , in the region  $[0,5] \times [0,5]$ , The convergence of  $H_{n,m}(f(u,v);x,y)$  operator (red) and  $H_{n,m}^*(f(u,v);x,y)$  GBS operator (blue) to the function  $f(x,y) = (1+x+y) \sin(x+y)$  (yellow) is pictorial as shown in Figure 3.

Let  $x = (x_n)$  and  $y = (y_n)$  be sequences with limits  $x_0$  and  $y_0$  respectively. If

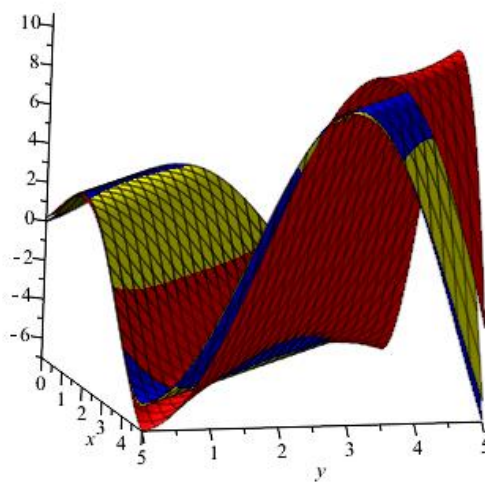
$$\lim_{n \rightarrow \infty} \frac{|y_n - y|}{|x_n - x|} = 0,$$

then it is said that  $y$  converges faster than  $x$  (see [27]). (Or equivalent  $\lim_{n \rightarrow \infty} \frac{|x_n - x|}{|y_n - y|} = \infty$ ). And if  $\lim_{n \rightarrow \infty} \frac{|y_n - y|}{|x_n - x|} = M < \infty$  then it is said that  $x$  and  $y$  have equivalent convergence. That means that there is any  $n_0$  such that for all  $n \geq n_0$  the inequality  $|y_n - y| \leq M|x_n - x|$  holds. If  $0 < M < 1$  then it is said that convergence of  $y$  is better than  $x$ , and if  $1 < M < \infty$  then it is said that convergence of  $x$  is better than  $y$ . (see [28]). Let  $(X, |||)$  and  $(Y, |||)$  be two function





**Fig. 2:** The convergence of the  $H_{n,m}^*(f(u,v);x,y)$  operator to the function  $f(x,y)$ .



**Fig. 3:** The convergence of the  $H_{n,m}(f(u,v);x,y)$  operator and  $H_{n,m}^*(f(u,v);x,y)$  Gbs operator to the function  $f(x,y)$ .

spaces  $(L_n)$  and  $(M_n)$  be two linear positive operators sequences acting from  $X$  into  $Y$ . And let these operators sequences approximate uniformly. If

$$\lim_{n \rightarrow \infty} \frac{\|L_n(f;x) - f(x)\|}{\|M_n(f;x) - f(x)\|} = 0,$$

(Or equivalent  $\lim_{n \rightarrow \infty} \frac{\|M_n(f;x) - f(x)\|}{\|L_n(f;x) - f(x)\|} = \infty$ ) for any  $f \in X$ , then it is said that  $(L_n)$  approximates faster than  $(M_n)$ .

And if  $\lim_{n \rightarrow \infty} \frac{\|L_n(f;x) - f(x)\|}{\|M_n(f;x) - f(x)\|} = K < \infty$ , then it is said that  $(L_n)$  and  $(M_n)$  have equivalent approximate. If

$0 < K < 1$  then it is said that approximate of  $(L_n)$  is better than  $(M_n)$ , and if  $1 < M < \infty$  then it is said that approximate of  $(M_n)$  is better than  $(L_n)$ .

We are ready to give the following example to compare approximation of  $H_{n,m}(f;x,y)$  and  $H_{n,m}^*(f;x,y)$  for some  $(n,m)$  and  $(x,y)$ .

- (4) We compare approximation of (11) and (23) to the function  $f(x,y) = (1+x+y)\sin(x+y)$  in the region  $[0,5] \times [0,5]$  for some  $(n,m)$  and  $(x,y)$  as following rate,

$$A_{n,m}(x,y) = \frac{|H_{n,m}^*(f(u,v);x,y) - f(x,y)|}{|H_{n,m}(f(u,v);x,y) - f(x,y)|}$$

and then we get the following numerical table.

**Table 1:** Error bounds at different  $n$  values for  $H_{n,m}(f;x,y)$  and  $H_{n,m}^*(f;x,y)$  GBS operator.

$(n,m)-(x,y)$	(1,1)	(2,2)	(3,3)	(4,4)
(10,10)	0.2748334033	0.0327836043	0.0096505536	0.4496459206
(20,20)	0.01368673105	0.03275994979	0.04873530553	0.1256181241
(50,50)	0.003614794524	0.003165557953	0.03180114071	0.006245528269

From the table we see that, approximation of (23) is better than (11) for the function  $f(x,y)$ .

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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