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An Accurate and Efficient Technique for Approximating Fuzzy Fredholm Integral Equations of the Second Kind Using Triangular Functions

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Abstract: In this work an accurate and efficient method is suggested to solve the Fredholm fuzzy integral equations of the second kind. The orthogonal triangular function (TF) based method is first applied to transform the fuzzy Fredholm integral equations to a coupled system of matrix algebraic equations. An iterative algorithm of finite nature is then applied to solve the coupled system to obtain the coefficients used to obtain the form of approximate solution of the unknown functions of the integral problems. Finally, an algorithm is presented to solve the fuzzy integral equation by using the trapezoidal rule. This algorithm is implemented on some numerical examples by using software MATLAB. The obtained numerical results are compared with other numerical method and the exact solutions.

The main purpose of this paper is to approximate the solution of linear dimensional fuzzy Fredholm integral equations of the second kind (1D-FFIE-2). We use fuzzy triangular functions (1D-TFs) to replace the Fredholm fuzzy integral with a coupled system of matrix algebraic equations. An iterative algorithm of finite nature is then applied to solve the coupled system to obtain the coefficients used to obtain the form of approximate solution of the unknown functions of the integral problems. Moreover, we prove the convergence of the method. Finally we illustrate this method with some numerical examples to demonstrate the validity and applicability of the technique.

In this work an accurate and efficient hybrid technique is suggested to solve the Fredholm fuzzy integral equations of the second kind. First, a two m- sets of orthogonal triangular basis functions (TF$_s$) method is first used to replace the Fredholm fuzzy integral with a coupled system of matrix algebraic equations. An iterative algorithm of finite nature is then applied to solve the coupled system to obtain the coefficients used to obtain the form of approximate solution of the unknown functions of the integral problems. To illustrate the accuracy and the efficiency of the proposed method, set of numerical examples are solved where obtained numerical results are compared with other numerical method and the exact solutions.

Keywords: Fuzzy number, Fredholm fuzzy integral equations, generalized Sylvester matrix equation, Finite iterative algorithm, orthogonal triangular functions.

1 Introduction

The importance of fuzzy integral equations appears in studying and solving a large proportion of the problems for different topics in applied mathematics, in particular in relation to biology, physics, medical and geographic. Usually in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them. Numerous techniques have been recently implemented for solving integral equations. Many different methods have been used to approximate the solution of integral equation systems [18-20]. Many basic and fundamentals functions are recently used to approximate the solution of integral equations like wavelet basis orthogonal bases, see Maleknejad et al. [13], Rationalized Haar functions are developed by Maleknejad and Mirzaee[4]

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to approximate the solutions of the linear Fredholm integral equations system. A general method by Jahantigh et al. [9] for solving fuzzy Fredholm integral equation of the second kind is introduced. Triangular functions direct method for solving Fredholm integral equations of second kind are proposed in [6], A Direct Method for Numerically Solving Integral Equations System Using Orthogonal Triangular Functions is introduced in [21]. [22], Introduce Application of Triangular Functions to Numerical Solution of Stochastic Volterra Integral Equations. Fredholm integral equations of second kind are solved by using triangular functions method hybrid with iterative algorithm [1]. Also, Fredholm fuzzy integral equations of the second kind is solved via direct method using triangular functions [2] and numerical solution of linear Fredholm fuzzy equation of the second kind by block-pulse functions is considered in [5]. A numerical method for solving the fuzzy Fredholm integral equation of second kind is presented Barkhordary et al. [8] where the trapezoidal rule is used to compute the integrals. Maleknejad et al. [3] proposed a numerical solution of integral equation system of the second kind by block pulse functions, and Babolian et al. [6] proposed a method for solving Fredholm integral equations. Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind is presented via direct method using triangular functions [14].H. Nouriani et al. [15] is proposed a quadrature iterative method for numerical solution of two-dimensional fuzzy Fredholm integral equations. R.Ezzati et al. [16] is given numerical solution of two-dimensional fuzzy Fredholm integral equations using bivariate bernstein polynomials. In [17], Modified homotopy perturbation method is solving two-dimensional fuzzy Fredholm integral equation. In this paper we are going to use a kind of these bases that is orthogonal triangular functions. In many applications some of the parameters in our problems are usually represented by fuzzy number rather than crisp state, and thus developing mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them is important. The paper is organized as follows. In Section 2, some definitions and properties of the orthogonal triangular functions (TFs) are presented. Also, expanding two variable functions by TFs and fuzzy numbers is given. In section 3, to solve coupled system of matrix equations a finite iterative algorithm is presented. In section 4, the suggested method is introduced. In section 5, we solve some numerical examples to illustrate the applicability and the accuracy of the proposed technique.

2 Brief review of triangular functions (TFs)

2.1 Triangular Functions (TFs)

Definition 1. *Two m-sets of triangular functions (TFs) are defined over the interval [0, T) as:*

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (1+i)h \ , \\ 0 \ , & o.w \end{cases} \tag{1}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t < (1+i)h \ , \\ 0 \ , & o.w \end{cases} \tag{2}$$

where $i = 0,\ 1, \ldots, m-1$ *and m has a positive integer value. Also, consider* $h = \frac{T}{m}$ *and* $\mathrm{T1_i}$ *as the ith left-handed triangular function and* $\mathrm{T2_i}$ *as the ith right-handed triangular function. In this paper, it is assumed that* $T = 1$*, so TFs are defined over* $[0\ ,\ 1\ )$ *and* $h = \frac{T}{m}$ *. From the definition of TFs, it is clear that triangular functions are disjoint, orthogonal and complete [4]. We can write*

$$\begin{gathered} \int_0^1 T1_i(t)\,T1_j(t)\,dt = \int_0^1 T2_i(t)\,T2_j(t)dt = \begin{cases} \frac{h}{3}\,, & i=j \\ 0\,, & i \neq j \end{cases} \ , \\ \int_0^1 T1_i(t)\,T2_j(t)\,dt = \int_0^1 T2_i(t)\,T1_j(t)dt = \begin{cases} \frac{h}{6}\,, & i=j \\ 0\,, & i \neq j \end{cases} \ , \end{gathered} \tag{3}$$

Now, write the first m terms of the left-hand triangular functions and the first m terms of the right-hand triangular functions as m-vectors:

$$T1(t) = [T1_0\ (t), T1_1(t), \ldots,\ T1_{m-1}\ (t)]^T\ , \tag{4}$$

$$T2(t) = [T2_0\ (t), T2_1(t), \ldots,\ T2_{m-1}\ (t)]^T\ , \tag{5}$$

We call $T1(t)$ and $T2(t)$ as left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively. The product of two TFs vectors are presented by:

$$T1(t)T1^T(t) \cong \begin{pmatrix} T1_0\ (t) & 0 & \cdots & 0 \\ 0 & T1_1\ (t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T1_{m-1}\ (t) \end{pmatrix}, \tag{6}$$

$$T2(t)T2^T(t) \cong \begin{pmatrix} T2_0\ (t) & 0 & \cdots & 0 \\ 0 & T2_1\ (t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T2_{m-1}\ (t) \end{pmatrix}, \tag{7}$$

and

$$T1(t)T2^T(t) \cong 0 \tag{8}$$

$$T2(t)T1^T(t) \cong 0 \tag{9}$$

where 0 is the zero $m \times m$ matrix. Also,

$$\int_0^1 T1(t)T1^T(t)\,dt = \int_0^1 T2(t)T2^T(t)\,dt \cong \frac{h}{3}I \tag{10}$$

$$\int_0^1 T1(t)T2^T(t)\,dt = \int_0^1 T2(t)T1^T(t)\,dt \cong \frac{h}{6}I. \tag{11}$$

In which Iis an $m \times m$ identity matrix.

The expansion of function $f(t)$ over [0, 1) with respect to TFs, may be compactly written as

$$f(t) \cong \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = c^T T1(t) + d^T T2(t), \tag{12}$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for i=0, 1...... m-1.

2.2 Expanding two variables function by TFs [2]

Each function $f(t,\ s) \in L^2([0,1) \times [0\ ,1))$ can be expanded by two TFs vectors with m_1 and m_2 components, respectively. For convenience, take $m_1 = m_2 = m$. To get desired results, first fix the independent variables. Then, expand $f(t,\ s)$ by

TFs with respect to independent variable t as follows:

$$f(t,s) \cong T1^T(t)\begin{pmatrix} f(0,s) \\ f(h,s) \\ \vdots \\ f((m-1)h,s) \end{pmatrix} + T2^T(t)\begin{pmatrix} f(h,s) \\ f(2h,s) \\ \vdots \\ f(mh,s) \end{pmatrix} \tag{13}$$

Now, each of the functions $f(ih,\ s)$'s for $i = 0,1,\ldots\ldots,m-1$ is expanded by TFs with respect to independent variable s. Thus, the expansion of $f(t,s)$ takes the form:

$$T1^T(t)\begin{pmatrix} F11_1^T T1(s) + F12_1^T T2(s) \\ F11_2^T T1(s) + F12_2^T T2(s) \\ \vdots \\ F11_m^T T1(s) + F12_m^T T2(s) \end{pmatrix} + T2^T(t)\begin{pmatrix} F21_1^T T1(s) + F22_1^T T2(s) \\ F21_2^T T1(s) + F22_2^T T2(s) \\ \vdots \\ F21_m^T T1(s) + F22_m^T T2(s) \end{pmatrix}$$

$$= T1^T(t)\left(\begin{pmatrix} F11_1^T \\ F11_2^T \\ \vdots \\ F11_m^T \end{pmatrix} T1(s) + \begin{pmatrix} F12_1^T \\ F12_2^T \\ \vdots \\ F12_m^T \end{pmatrix} T2(s)\right) + T1^T(t)\left(\begin{pmatrix} F21_1^T \\ F21_2^T \\ \vdots \\ F21_m^T \end{pmatrix} T1(s) + \begin{pmatrix} F22_1^T \\ F22_2^T \\ \vdots \\ F22_m^T \end{pmatrix} T2(s)\right)$$

$$= T1^T(t)\,F11\,T1(s) + T1^T(t)\,F12\,T2(s) + T2^T(t)\,F21\,T1(s) + T2^T(t)\,F22\,T2(s)$$

In which,

$$F11 = \begin{pmatrix} f(0,0) & f(0,h) & \cdots & f(0,(m-1)h) \\ f(h,0) & f(h.h) & \cdots & f(h,(m-1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f((m-1)h,0) & f((m-1)h,h) & \cdots & f((m-1)h,(m-1)h) \end{pmatrix}, \tag{14}$$

$$F12 = \begin{pmatrix} f(0,h) & f(0,2h) & \cdots & f(0,mh) \\ f(h,h) & f(h.2h) & \cdots & f(h,mh) \\ \vdots & \vdots & \ddots & \vdots \\ f((m-1)h,h) & f((m-1)h,2h) & \cdots & f((m-1)h,mh) \end{pmatrix}, \tag{15}$$

$$F21 = \begin{pmatrix} f(h,0) & f(h,h) & \cdots & f(h,(m-1)h) \\ f(2h,0) & f(2h.h) & \cdots & f(2h,(m-1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f(mh,0) & f(mh,h) & \cdots & f(mh,(m-1)h) \end{pmatrix}, \tag{16}$$

$$F22 = \begin{pmatrix} f(h,h) & f(h,2h) & \cdots & f(h,mh) \\ f(2h,h) & f(2h.2h) & \cdots & f(2h,mh) \\ \vdots & \vdots & \ddots & \vdots \\ f(mh,h) & f(mh,2h) & \cdots & f(mh,mh) \end{pmatrix}, \tag{17}$$

Let $T(t)$ be a $2m$-vector defined as:

$$T(t) = \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix} \quad ;0 = t < 1\ .$$

The two vector functions $T1(t)$ and $T2(t)$ defined in Eqs. (4) and (5). Now, suppose that $f(t,s)$ is a function of two variables. Thus, we can expand it with respect to TFs as follows:

$$f(s,t) \cong T^T(s) F T(t) \tag{18}$$

where $T(s)$ and T(t) are $2m_1$ and $2m_2$ dimensional TFs and F a $2m_1 \times 2m_2$ is TFs coefficient matrix. For convenience, we put $m_1 = m_2 = m$, so matrix F can be written as

$$F = \begin{pmatrix} (F11)_{m\times m} & (F12)_{m\times m} \\ (F21)_{m\times m} & (F22)_{m\times m} \end{pmatrix},$$

where $F11, F12, F21$ *and* $F22$ in above-stated Equation, are previously defined in Eqs. (14): (17).

2.3 Fuzzy functions

In this subsection, two definitions that are needed in this work are stated.

Definition 2. *A fuzzy number is a fuzzy set u: $R^1 \rightarrow$ [0, 1] where the following conditions are to be hold:*

(a) *u is upper semi continuous.*
(b) $u(x) = 0$ *outside some interval [c, d].*
(c) *there are real numbers a and b, $c \le a \le b \le d$, for which*

(1) $u(x)$ *is a monotonicly increasing on [c, a],*
(2) $u(x)$ *is monotonicly decreasing on [b, d],*
(3) $u(x) = 1$ *for $a \le x \le b$.*

Definition 3. *A fuzzy number u is a pair$(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, satisfying the following requirement:*

(a) $\underline{u}(r)$*is bounded and monotonic increasing as well as left continuous function,*
(b) $\overline{u}(r)$ *is bounded ,monotonic decreasing and left continuous function,*
(c) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ for $k > 0$, addition $(u+v)$ and multiplication by k are defined as:

$$\begin{array}{r} (\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r) \\ (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r) \end{array}, \tag{19}$$

$$\begin{array}{r} (\underline{ku})(r) = k\underline{u}(r) \\ (\overline{ku})(r) = k\overline{u}(r). \end{array} \tag{20}$$

3 Solving coupled system of matrix equations using finite iterative algorithm [1]

There are many variant forms of finite iterative algorithms for solving matrix equations, see for example [1, 10-12]. We are concerned with iterative solutions to coupled system of similar forms of the Sylvester matrix equations [1].

$$AV + BW = C \tag{21}$$

and second algorithm for solving coupled system of Sylvester matrix equations

$$A_1V + B_1W = C_1$$

$$A_2V+B_2W=C_2 \tag{22}$$

Algorithm 1. To solve the matrix equation (21) a finite iterative algorithm is constructed and used as follows,

1- input A,B,C

2- choose arbitrary matrices $V_1 \in \Re^{n\times p} and \ \ W_1 \in \Re^{r\times p}$

3- set

$$R_1 = C - AV_1 - BW_1$$
$$P_1 = A^T R_1$$
$$Q_1 = B^T R_1$$
$$K = 1$$

4- if $R_K = 0$ then stop and V_K , W_K is the solution else let $K = K+1$ go to step 5,

5- compute

$$V_{K+1} = V_K + \frac{\| R_K \|^2}{\| P_K \|^2 + \| Q_K \|^2} P_K, \quad W_{K+1} = W_K + \frac{\| R_K \|^2}{\| P_K \|^2 + \| Q_K \|^2} Q_K$$

$$R_{K+1} = C - AV_{K+1} - BW_{K+1} = R_K - \frac{\| R_K \|^2}{\| P_K \|^2 + \| Q_K \|^2} |AP_K + BQ_K|$$

$$P_{K+1} = A^T R_{K+1} + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} P_K, Q_{K+1} = B^T R_{K+1} + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} Q_K$$

Algorithm 2.

A finite iterative algorithm is constructed to coupled system of Sylvester matrix equations (22)

1- Input A_1,B_1,A_2,B_2,C_1,C_2

2- Choose arbitrary matrices $Y_{1_1} \in C^{n\times p} and \ \ Y_{2_1} \in C^{r\times p}$

3- Set

$$R_1 = diag(C_1 - f(Y_{1_1},\ Y_{2_1}), C_2 - g(Y_{1_1}, Y_{2_1}))$$
$$S_1 = A_1{}^T(C_1 - f(Y_{1_1},\ Y_{2_1})) + A_2{}^T(C_2 - g(Y_{1_1},\ Y_{2_1}))$$
$$T_1 = B_1{}^T(C_1 - f(Y_{1_1},\ Y_{2_1})) + B_2{}^T(C_2 - g(Y_{1_1},\ Y_{2_1}))$$

4- if $R_K = 0$ then stop and Y_{1_K}, Y_{2_K} is the solution else let $K = K+1$ go to step 5.

5 – Compute

$$Y_{1_{K+1}} = Y_{1_K} + \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} S_K$$

$$Y_{2_{K+1}} = Y_{2_K} + \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} T_K$$

$$R_{K+1} = \ diag(C_1 - f\left(\ Y_{1_{K+1}},\ Y_{2_{K+1}}\right), C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right))$$
$$= R_K - \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} diag(f(S_K, T_K), g(S_K, T_K))$$

$$S_{K+1} = A_1{}^T\left(C_1 - f\left(Y_{1_{K+1}},\, Y_{2_{K+1}}\right)\right) + A_2{}^T\left(C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} S_K$$

$$T_{K+1} = B_1{}^T\left(C_1 - f\left(Y_{1_{K+1}},\, Y_{2_{K+1}}\right)\right) + B_2{}^T\left(C_2 - g\left(Y_{1_{K+1}}, Y_{2_{K+1}}\right)\right) + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} T_K$$

4 Proposed hybrid iterative technique for solving linear fuzzy Fredholm integral equation

4.1 Converting Fredholm integral equations of second kind to two crisp coupled systems

In this subsection, a TFs method is presented to transform the fuzzy Fredholm integral equation of second kind linear (FFIE-2) to two crisp coupled systems. First consider the following equation:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)\, u(t) dt, \tag{23}$$

where $k(x,t)$ is an arbitrary kernel function over the square $0 \leq$ x, t ≤ 1 and $u(x)$ is a fuzzy real valued function.

The main task is to determine TFs coefficients of $u(x)$ in the interval [0, 1) from the know functions $f(x)$ and kernel $k(x,t)$.

So, we present the parametric form of FFIE-2 with respect to definition 3. Let $(\underline{f}(x,\ r), \overline{f}(x,\ r))$ and $(\underline{u}(x,\ r), \overline{u}(x,\ r))$, 0 $\leq$ r $\leq$ 1 and $x \in [0,1)$ be parametric forms of $f(x)$ and $u(x)$, respectively.

Therefore, we rewrite system (23) in the following form

$$\underline{u}(x,r) = \underline{f}(x,\ r) + \lambda \int_0^1 k(x,t)\, \underline{u}(x,r) dt, \tag{24}$$

$$\overline{u}(x,\ r) = \overline{f}(x,\ r) + \lambda \int_0^1 k(x,t)\, \overline{u}(x,\ r) dt, \tag{25}$$

Let us expand $\underline{u}(x,r)$, $\underline{f}(x,\ r)$ and $k(s,t)$ by TFs (LHTF and RHTF) as follows:

$$\underline{u}(x,r) \simeq T1^T(x)U11T1(r) + T1^T(x)U12T2(r) + T2^T(x)U21T1(r) + T2^T(x)U22T2(r) = T^T(x)UT(r)$$

$$\underline{f}(x,r) \simeq T1^T(x)F11T1(r) + T1^T(x)F12T2(r) + T2^T(x)F21T1(r) + T2^T(x)F22T2(r) = T^T(x)FT(r)$$

$$k(x,t) \simeq T1^T(x)K11T1(t) + T1^T(x)K12T2(t) + T2^T(x)K21T1(t) + T2^T(x)K22T2(t) = T^T(x)KT(t)$$

with $U = \begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix}$, $F = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}$ and $K = \begin{pmatrix} K11 & K12 \\ K21 & K22 \end{pmatrix}$ Substituting in Eq. (24)

$$T^T(x)UT(r) \simeq T^T(x)FT(r) + \lambda \int_0^1 T^T(x)KT(t)T^T(t)UT(r)\, dt$$

$$T^T(x)UT(r) \simeq T^T(x)FT(r) + \lambda T^T(x)K(\int_0^1 T(t)T^T(t)\, dt)UT(r)$$

with the equation

$$\int_0^1 T(t)T^T(t)\, dt = \int_0^1 \begin{pmatrix} T1(t) & T2(t) \end{pmatrix} \begin{pmatrix} T1^T(t) & T2^T(t) \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} T1(t)\,T1^T(t) & T1(t)\,T2^T(t) \\ T2(t)\,T1^T(t) & T2(t)\,T2^T(t) \end{pmatrix} dt$$

$$\simeq \begin{pmatrix} \frac{h}{3} l_{m\times m} & \frac{h}{6} l_{m\times m} \\ \frac{h}{6} l_{m\times m} & \frac{h}{3} l_{m\times m} \end{pmatrix} = D_{2m\times 2m}$$

We have

$$T^T(x)\,UT(r) \simeq T^T(x)\,FT(r) + \lambda T^T(x)\,KDUT(r),$$

then

$$U = F + \lambda KDU \Rightarrow U = (l - \lambda KD)^{-1} F$$

$$(l - \lambda KD)U = F$$

$$\left[\begin{pmatrix} l_m & 0 \\ 0 & l_m \end{pmatrix} - \begin{pmatrix} K11 & K12 \\ K21 & K22 \end{pmatrix} \begin{pmatrix} \frac{h}{3} l_{m\times m} & \frac{h}{6} l_{m\times m} \\ \frac{h}{6} l_{m\times m} & \frac{h}{3} l_{m\times m} \end{pmatrix}\right] \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}$$

$$\left[\begin{pmatrix} l_m & 0 \\ 0 & l_m \end{pmatrix} - \begin{pmatrix} \frac{h}{3}K11 + \frac{h}{6}K12 & \frac{h}{6}K11 + \frac{h}{3}K12 \\ \frac{h}{3}K21 + \frac{h}{6}K22 & \frac{h}{6}K21 + \frac{h}{3}K22 \end{pmatrix}\right] \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}$$

$$\begin{bmatrix} l_m - (\frac{h}{3}K11 + \frac{h}{6}K12) & -(\frac{h}{6}K11 + \frac{h}{3}K12) \\ -(\frac{h}{3}K21 + \frac{h}{6}K22) & l_m - (\frac{h}{6}K21 + \frac{h}{3}K22) \end{bmatrix} \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}$$

$$\begin{bmatrix} (l_m - (\frac{h}{3}K11 + \frac{h}{6}K12))U11 & (-(\frac{h}{6}K11 + \frac{h}{3}K12) \\ -(\frac{h}{3}K21 + \frac{h}{6}K22) & l_m - (\frac{h}{6}K21 + \frac{h}{3}K22) \end{bmatrix} \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}$$

$$(l_m - (\frac{h}{3}K11 + \frac{h}{6}K12))U11 - (\frac{h}{6}K11 + \frac{h}{3}K12)U21 = F11$$

$$-\left(\frac{h}{3}K21 + \frac{h}{6}K22\right)U11 + \left(l_m - \left(\frac{h}{6}K21 + \frac{h}{3}K22\right)\right)U21 = F21$$

$$(l_m - (\frac{h}{3}K11 + \frac{h}{6}K12))U12 - (\frac{h}{6}K11 + \frac{h}{3}K12)U22 = F12$$

$$-\left(\frac{h}{3}K21 + \frac{h}{6}K22\right)U12 + \left(l_m - \left(\frac{h}{6}K21 + \frac{h}{3}K22\right)\right)U22 = F22.$$

Set,

$$A_1 = (l_m - (\frac{h}{3}K11 + \frac{h}{6}K12)), B_1 = -(\frac{h}{6}K11 + \frac{h}{3}K12),$$

$$A_2 = -\left(\frac{h}{3}K21 + \frac{h}{6}K22\right), B_2 = l_m - \left(\frac{h}{6}K21 + \frac{h}{3}K22\right),$$

which lead to the following two coupled crisp linear systems

$$\begin{aligned} A_1U11 + B_1U21 &= F11, \\ A_2U11 + B_2U21 &= F21, \end{aligned} \tag{26}$$

and

$$\begin{aligned} A_1U1\,2 + B_1U22 &= F12\,, \\ A_2U12 + B_2U22 &= F22. \end{aligned} \tag{27}$$

Similarly, we expand $\overline{u}(x,\ r)$ and $\overline{f}(x,\ r)$ by TFs (LHTF and RHTF) and substituting in Eq. (25) two coupled crisp linear systems, similar to (26) and (27) are obtained. It is clear that all matrices in the two coupled crisp linear systems (26) and (27) are square matrices of dimensions$m \times m$. Thus, to obtain the coefficient matrices $U11$, $U21$, $U12$ and $U22$ in order to get the approximate numerical solution of the form:

$$\underline{u}(x,r)_{approx.} \;=\; T^T(x)\,U\,T(r)\;.$$

The following efficient finite iterative algorithm is proposed which is a generalization of algorithm 2.

4.2 Proposed iterative algorithm for solving the two coupled systems

A proposed iterative algorithm is presented in this subsection to solve the two coupled systems (26) and (27) as a modification to algorithm 2.

Algorithm 3.
In this algorithm we modified and generalized algorithm 2 to work out for systems (26) and (27) as follows. First for the coupled system (26)

1- Input A_1 , A_1 , B_1 , B_2, $F11$, $F21$.
2- Choose arbitrary matrices$U11$, $U21$.
3- For $k=1$, set

$$R_k = diag(F11 - f(U11,\ U21), F21 - g(U11,U21))$$
$$S_k = {A_1}^T(F11 - f(U11,U21)) + {A_2}^T(F21 - g(U11,,\ U21))$$
$$T_k = {B_1}^T(F11 - f(U11,U21)) + {B_2}^T(F21 - g(U11,\ U21))$$

4- if $R_K = 0$ then stop and $U11,\ U21$ is the solution else let $K = K+1$ go to step 5.
5 – Compute

$$U11 = U11 + \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} S_K\ ,$$
$$U21 = U21 + \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} T_K,$$
$$R_{k+1} = diag(F11 - f(U11,\ U21), F21 - g(U11,U21)),$$
$$= R_K - \frac{\| R_K \|^2}{\| S_K \|^2 + \| T_K \|^2} diag(f(S_K,T_K), g(S_K,T_K)),$$
$$S_{K+1} = {A_1}^T(F11 - f(U11,U21)) + {A_2}^T(F21 - g(U11,,\ U21)) + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} S_K,$$
$$T_{K+1} = {B_1}^T(F11 - f(U11,U21)) + {B_2}^T(F21 - g(U11,\ U21)) + \frac{\| R_{K+1} \|^2}{\| R_K \|^2} T_K.$$

For the coupled system (26), the algorithm is repeated with replacing $U11,\ U21$ by $U12,\ U22$ and $F11$, $F21$ by $F12,\ F22$ where the $2m \times 2m$ block Umatrix is computed. The approximate crisp numerical solution for equation (24) of the form $\underline{u}(x,r)_{approx.} \;=\; T^T(x)\,U\,T(r)$ is then obtained. In a similar manner, the crisp numerical solution for equation (25) of the form

$$\bar{u}(x,r)_{approx.} \;=\; T^T(x)\,U\,T(r)$$

can be obtained by carrying out the above proposed algorithm for the other two coupled crisp systems similar to (26) and (27). Finally, the solution for (23) is then given as $u(x) = \left(\underline{u}_{approx.}(x,\ r), \overline{u}approx.(x,\ r)\right)$, $0 \leq \mathrm{r} \leq 1$ and $x \in [0,1)$.

5 Numerical results and discussions

To demonstrate the accuracy and effectiveness of our proposed hybrid method, $TFs$and an iterative algorithm, some examples are considered. The solution of each example is obtained for different values of r , x and m and is compared with the exact solution and the direct method presented by Mirzaee et al. [2] and with Ghanbari et al. [5].

Example 51 *Consider the following FFIE-2 with*

$$\underline{f}(x,r) = \frac{1}{6}rx^2, \overline{f}(x,r) = (2-r)x^2,$$

and $k(x,t) = x^2(1+2t)$, $0 < x,\ t < 1$ *and* $\lambda = 1$. *The exact solution in this case is given by*

$$\underline{u}(x,r) = rx^2,\ \ \overline{u}(x,r) = (2-r)x^2.$$

From the obtained numerical results of the first test example, we can see that our proposed hybrid iterative method gives

Table 1: The numerical results for Example 1 with TFs method when the tolerance criteria is residual $> \mathrm{e}^{-4}$.

r	Exact solution $\underline{u}(x,r)$	Direct method for $x = 0.25$ $m = 4$	Absolute error	Presented method for $x = 0.25$ $m = 4$	Absolute error
0	0.00000	0.00000	0.00000e+000	0.00003	2.78283e-005
0.1	0.00625	0.00716	9.09951e-004	0.00719	9.35707e-004
0.2	0.01250	0.01432	1.81990e-003	0.01434	1.84359e-003
0.3	0.01875	0.02148	2.72985e-003	0.02150	2.75147e-003
0.4	0.02500	0.02864	3.63981e-003	0.02866	3.65935e-003
0.5	0.03125	0.03580	4.54976e-003	0.03582	4.56722e-003
0.6	0.03750	0.04296	5.45971e-003	0.04298	5.47510e-003
0.7	0.04375	0.05012	6.36966e-003	0.05013	6.38298e-003
0.8	0.05000	0.05728	7.27961e-003	0.05729	7.29086e-003
0.9	0.05625	0.06444	8.18956e-003	0.06445	8.19874e-003

the same accuracy compared with the direct method. Also, It is worth noting that the number of iterations to execute the algorithm taking tolerance criteria is residual $> e^{-4}$ *was* $k = 5$ *which means that the technique is quite efficient. The accuracy can be further improved by increasing the stopping tolerance.*

Example 52 *Consider the following FFIE-2 with*

$$\underline{f}(x,r) = -\frac{1}{3}x^2 + rx^2 + \frac{1}{3}x + \frac{1}{4}r - \frac{1}{12}, \overline{f}(x,r) = \frac{1}{3}x - x^2r - \frac{1}{4}r + \frac{5}{3}x^2 + \frac{5}{12},$$

and $k(x,t) = (2t-1)^2(1-2x)$, $0 < x,\ t < 1$ *and* $\lambda = 1$. *The exact solution in this case is given by*

$$\underline{u}(x,r) = rx, \overline{u}(x,r) = (2-r)x.$$

The problem in example 2 is solved by proposed method and the results are given in Table 5 and the numerical results are which compared with the obtained results using the direct method [2]. The last column in this table shows the absolute

Table 2: The numerical results for Example 1 with TFs method when the tolerance criteria is residual $> \mathrm{e}^{-4}$.

r	Exact solution $\overline{u}(x,r)$	Direct method for $x = 0.25$ $m = 4$	Absolute error	Presented method for $x = 0.25$ $m = 4$	Absolute error
0	0.12500	0.14320	1.81990e-002	0.14320	1.82034e-002
0.1	0.11875	0.13604	1.72891e-002	0.13605	1.72960e-002
0.2	0.11250	0.12888	1.63791e-002	0.12889	1.63886e-002
0.3	0.10625	0.12172	1.54692e-002	0.12173	1.54812e-002
0.4	0.10000	0.11456	1.45592e-002	0.11457	1.45739e-002
0.5	0.09375	0.10740	1.36493e-002	0.10742	1.36665e-002
0.6	0.08750	0.10024	1.27393e-002	0.10026	1.27591e-002
0.7	0.08125	0.09308	1.18294e-002	0.09310	1.18518e-002
0.8	0.07500	0.08592	1.09194e-002	0.08594	1.09444e-002
0.9	0.06875	0.07876	1.00095e-002	0.07879	1.00370e-002

Table 3: The numerical results for Example 1 with TFs method when the tolerance criteria is residual $> \mathrm{e}^{-4}$.

r	Exact solution $\underline{u}(x,r)$	Direct method for $x = 0.25$ $m = 8$	Absolute error	presented method for $x = 0.25$ $m = 8$	Absolute error
0	0.00000	0.00000	0.00000e+000	0.00001	5.54746e-006
0.1	0.00625	0.00646	2.12714e-004	0.00647	2.17706e-004
0.2	0.01250	0.01293	4.25428e-004	0.01293	4.29865e-004
0.3	0.01875	0.01939	6.38142e-004	0.01939	6.42024e-004
0.4	0.02500	0.02585	8.50857e-004	0.02585	8.54183e-004
0.5	0.03125	0.03231	1.06357e-003	0.03232	1.06634e-003
0.6	0.03750	0.03878	1.27628e-003	0.03878	1.27850e-003
0.7	0.04375	0.04524	1.48900e-003	0.04524	1.49066e-003
0.8	0.05000	0.05170	1.70171e-003	0.05170	1.70282e-003
0.9	0.05625	0.05816	1.91443e-003	0.05816	1.91498e-003

Table 4: The numerical results for Example 1 with TFs method when the tolerance criteria is residual $> \mathrm{e}^{-4}$.

r	Exact solution $\overline{u}(x,r)$	Direct method for $x = 0.25$ $m = 8$	Absolute error	Presented method for $x = 0.25$ $m = 8$	Absolute error
0	0.12500	0.12925	4.25428e-003	0.12924	4.24263e-003
0.1	0.11875	0.12279	4.04157e-003	0.12278	4.02983e-003
0.2	0.11250	0.11250	3.82885e-003	0.11632	3.81704e-003
0.3	0.10625	0.10625	3.61614e-003	0.10985	3.60425e-003
0.4	0.10000	0.10340	3.40343e-003	0.10339	3.39146e-003
0.5	0.09375	0.09694	3.19071e-003	0.09693	3.17867e-003
0.6	0.08750	0.09048	2.97800e-003	0.09047	2.96588e-003
0.7	0.08125	0.08402	2.76528e-003	0.08400	2.75308e-003
0.8	0.07500	0.07755	2.55257e-003	0.07754	2.54029e-003
0.9	0.06875	0.07109	2.33986e-003	0.07108	2.32750e-003

Table 5: The numerical results for Example 2 with TFs method when the tolerance criteria is residual $> e^{-4}$.

r	Exact solution $\left(\underline{u}(x,r), \overline{u}(x,r)\right)$.	Direct method of [2] $x = 0.5$, $m = 2$	Presented method for $x = 0.5$, $m = 2$	Absolute error between Exact and presented method
0	(0.000000, 1.000000)	(0.000000, 1.000000)	(0.000005, 1.000002)	(5.277283e-006, 2.084783e-006)
0.1	(0.050000, 0.950000)	(0.050000, 0.950000)	(0.050005, 0.950001)	(4.727861e-006, 1.219422e-006)
0.2	(0.100000, 0.900000)	(0.100000, 0.900000)	(0.100004, 0.900000)	(4.178440e-006, 3.540609e-007)
0.3	(0.150000, 0.850000)	(0.150000, 0.850000)	(0.150004, 0.849999)	(3.629019e-006, 5.113003e-007)
0.4	(0.200000, 0.800000)	(0.200000, 0.800000)	(0.200003, 0.799999)	(3.079598e-006, 1.376662e-006)
0.5	(0.250000, 0.750000)	(0.250000, 0.750000)	(0.250003, 0.749998)	(2.530177e-006, 2.242023e-006)
0.6	(0.300000, 0.700000)	(0.300000, 0.700000)	(0.300002, 0.699997)	(1.980756e-006, 3.107384e-006)
0.7	(0.350000, 0.650000)	(0.350000, 0.650000)	(0.350001, 0.649996)	(1.431335e-006, 3.972745e-006)
0.8	(0.400000, 0.600000)	(0.400000, 0.600000)	(0.400001, 0.599995) (	(8.819137e-007, 4.838107e-006)
0.9	(0.450000, 0.550000)	(0.450000, 0.550000)	0.450000, 0.549994)	(3.324926e-007, 5.703468e-006)

Table 6: The numerical results for Example 2 with TFs method when the tolerance criteria is residual $> e^{-4}$.

r	Exact solution $\underline{u}(x,r)$	presented method for $x = 0.5$ $m = 32$	Absolute error	Method [5] for $x = 0.5$ $m = 32$	Absolute error
0	0.00000000	-0.00000001	1.45012526e-008	0.007956	7.956000e-03
0.1	0.05000000	0.04999999	1.27389860e-008	0.056347	6.347000e-03
0.2	0.10000000	0.09999999	1.09767186e-008	0.104737	4.737000e-03
0.3	0.15000000	0.14999999	9.21445154e-009	0.153128	3.128000e-03
0.4	0.20000000	0.19999999	7.45218404e-009	0.201519	1.519000e-03
0.5	0.25000000	0.24999999	5.68991698e-009	0.266040	1.604000e-02
0.6	0.30000000	0.30000000	3.92765037e-009	0.314430	1.443000e-02
0.7	0.35000000	0.35000000	2.16538287e-009	0.362820	1.282000e-02
0.8	0.40000000	0.40000000	4.03115374e-010	0.411210	1.121000e-02
0.9	0.45000000	0.45000000	1.35915212e-009	0.359603	9.039700e-02

Table 7: The numerical results for example 2 with TFs method when the tolerance criteria is residual $> e^{-4}$.

r	Exact solution $\overline{u}(x,r)$	presented method for $x = 0.5$ $m = 32$	Absolute error	Method [5] for $x = 0.5$ $m = 32$	Absolute error
0	1.00000000	1.00000000	3.13259463e-009	1.024160	2.416000e-02
0.1	0.95000000	0.95000000	1.26375377e-009	0.975770	2.577000e-02
0.2	0.90000000	0.90000000	6.05086314e-010	0.927379	2.737900e-02
0.3	0.85000000	0.85000000	2.47392629e-009	0.878988	2.898800e-02
0.4	0.80000000	0.80000000	4.34276726e-009	0.830598	3.059800e-02
0.5	0.75000000	0.74999999	6.21160723e-009	0.766077	1.607700e-02
0.6	0.70000000	0.69999999	8.08044720e-009	0.717986	1.798600e-02
0.7	0.65000000	0.64999999	9.94928728e-009	0.669290	1.929000e-02
0.8	0.60000000	0.59999999	1.18181291e-008	0.630905	3.090500e-02
0.9	0.55000000	0.54999999	1.36869673e-008	0.572514	2.251400e-02

error between the proposed method and exact solution. We can see that the method is of good accuracy and can be further improved by increasing the stooping criteria (the tolerance of the residual).

$$\|R_{1k}\| = \|diag(F11 - f1 \quad 00; \; 00 \quad F21 - g1)\| = 1.0000e - 002,$$

and

$$\|R_{2k}\| = \|diag(F12 - f1 \quad 00;\ 00 \quad F22 - g1)\| = 1.0000e - 002,$$

where

$$f1 = A_1\,U11 \ + B_1\,U21\,,\ \ f1 = A_1\,U12 \ + B_1\,U22\,,$$
$$g1 = A_2\,U21 \ + \ B_2\,U21\,,\, g1 = A_2\,U12 \ + \ B_2\,U22\,,$$

for the system (4.4) and system (4.5) respectively in our suggested iterative algorithm to obtain the coefficient matrices in (4.4) and (4.5). The number of iterations is $k = 5$. Moreover, the presented method is compared with the block-pulse function method proposed by Ghanbari et al. [5]. As we can see from the numerical results in Tables 6 and 7, the proposed method is of high accuracy as it is also highly efficient.

Example 53 *Consider the following FFIE-2 with*

$$\underline{f}(x,r) = rx - x^2[\frac{2}{3}rx^3 - \frac{4}{3}x^3 - \frac{1}{2}rx^2 + x^2 + \frac{1}{12}r - \frac{1}{12}],$$

$$\overline{f}(x,r) = (2-r)x + x^2[\frac{2}{3}rx^3 - \frac{1}{2}rx^2 + \frac{1}{12}r - \frac{1}{12}],$$

and

$$k(x,t) = x^2(1-2t) \quad ,0 \le x,t \le 1 \text{ and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,\ \ \overline{u}(x,r) = (2-r)x.$$

The results are shown in Tables 8 and 9. TheThe problem in example 3 is also solved by the proposed method and the

Table 8: The numerical results for Example 3 with TFs method when the tolerance criteria is residual $> e^{-4}$.

r	Exact solution $\underline{u}(x,r)$	Direct method [2] for $x = 0.1$, $m = 10$	Absolute error	presented method for $x = 0.1$, $m = 10$	Absolute error
0	0.0000	0.0003	3.9582e-04	0.0029	2.9089e-003
0.1	0.0100	0.0102	1.9745e-04	0.0128	2.8161e-003
0.2	0.0200	0.0200	9.0743e-07	0.0227	2.7232e-003
0.3	0.0300	0.0298	1.9927e-04	0.0326	2.6304e-003
0.4	0.0400	0.0396	3.9763e-04	0.0425	2.5376e-003
0.5	0.0500	0.0494	5.9599e-04	0.0524	2.4447e-003
0.6	0.0600	0.0592	7.9436e-04	0.0624	2.3519e-003
0.7	0.0700	0.0690	9.9272e-04	0.0723	2.2590e-003
0.8	0.0800	0.0788	1.2000e-03	0.0822	2.1662e-003
0.9	0.0900	0.0886	1.4000e-03	0.0921	2.0734e-003

results are given in tables 8, 9 which are compared with the one obtained using the direct method [2]. From the forth and last column, we can see that both have almost the same accuracy.

Table 9: The numerical results for Example 3 with TFs method when the tolerance criteria is residual $> e^{-4}$.

r	Exact solution $\overline{u}(x,r)$	Direct method [2] for $x=0.1$ $m=10$	Absolute error	Presented method for $x=0.1$ $m=10$	Absolute error
0	0.2000	0.1964	3.6000e-03	0.1929	7.1479e-003
0.1	0.1900	0.1866	3.4000-03	0.1831	6.8967e-003
0.2	0.1800	0.1768	3.2000-03	0.1734	6.6455e-003
0.3	0.1700	0.1670	3.0000-03	0.1636	6.3943e-003
0.4	0.1600	0.1572	2.8000e-03	0.1539	6.1431e-003
0.5	0.1500	0.1474	2.6000e-03	0.1441	5.8919e-003
0.6	0.1400	0.1376	2.4000e-03	0.1344	5.6407e-003
0.7	0.1300	0.1278	2.0000e-03	0.1246	5.3896e-003
0.8	0.1200	0.1180	5.1384e-03	0.1149	5.1384e-003
0.9	0.1100	0.1082	1.8000e-03	0.1051	4.8872e-003

6 Conclusion

In this paper, an approximate numerical solution for linear FFIE-2 is considered. The original Fredholm integral equations of second kind are first transformed to two crisp coupled systems. Then, we use the two m- sets of TFs to approximate of the unique solution of FFIE-2. Here, a hybrid method of a triangular functions and an iterative algorithm are considered. By examining this hybrid method, the numerical results obtained for three examples show that: the proposed method produces results almost similar to that obtained using the direct method and other numerical method based on block-pulse function method with acceptable percentage error and the absolute error is reduced with the reduction of the solution tolerance. The advances of this method are the number of calculations is very low as well as a good accuracy is mentioned.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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