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Rough Convergence For Difference Sequences

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Abstract: As known, difference sequences have their own characteristics. In this paper, we study the concept of rough convergence for difference sequences in a finite dimensional normed space. At the same time, we examine some properties of the set

$$LIM^r_{\Delta x_i} = \left\{ x_* \in X : \Delta x_i \xrightarrow{r} x_* \right\}$$

which is called as r-limit set of the difference sequence $\Delta x = (\Delta x_i)$.

Keywords: Convergence, difference sequences, rough convergence, limit points.

1 Introduction and Background

It is indisputable that the concept of convergence of a sequence is one of the most important concepts in Summability Theory. Also, determining the place of sequences in that does not satisfy the convergence condition is as important as convergent ones. Although not convergent, the existence of this kind of sequences that show similar characteristics to the concept of convergent sequence under certain conditions, has led to the emergence of different types of convergence. One of these is the concept of rough convergence defined by Phu ([12]) in finite dimensional normed spaces. According to this idea, rough convergence of a sequence can be obtained by extending the range of convergence by a number $r > 0$. Here, it should be noted that rough convergence has quite interesting applications in numerical analysis**.** After Phu's work, Aytar ([3]) studied about rough limit set and the core of a real sequence. Then, Phu ([13]) examined these results in infinite dimensional normed spaces and obtained more general results.

Accordingly, the definition of rough convergence in a finite dimensional normed space can be given as follows:
Let $(X, \|.\|)$ be a normed linear space and r be a nonnegative real number. Then, the sequence $x = (x_i)$ in X is said to be rough convergent or $r-$convergent to x_*; if for any $\varepsilon > 0$ there exists an $i_\varepsilon \in \mathbb{N}$ such that

$$\|x_i - x_*\| < r + \varepsilon$$

for all $i \geq i_\varepsilon$. This expression means that

$$\limsup \|x_i - x_*\| < r$$

and r is called by roughness degree. In this definition, we say that x_* is an $r-$limit point of (x_i) and it is denoted by $x_i \xrightarrow{r} x_*$.

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Let (x_i) be a rough convergent sequence in a finite dimensional normed space $(X, \|.\|)$ and r be a non-negative real number. For each $r > 0$, we obtain a different x_* point. So, this point which is called by the $r-$limit point of the sequence is unique. Therefore, a set of these points can be mentioned. This set is called by the set of $r-$limit points and is indicated by $LIM^r_{x_i}$. As seen, the topological and analytical features of the set are very important. The $r-$limit point set of the sequence (x_i) is defined by

$$LIM^r_{x_i} = \left\{ x_* \in X : x_i \xrightarrow{r} x_* \right\}.$$

Phu investigated boundedness and convexity of this set in ([12]). At the same time, he proved that this set is closed.

Following the definition of Phu, the concept of rough convergence was studied by Arslan and Dündar ([1]), Dündar and Çakan ([6]), Dündar([7]) and Kişi and Dündar([10]) for ideal convergence.

Now, lets briefly talk about difference sequences and their main properties. Difference sequences are defined by Kizmaz ([11]) for a real sequence $x = (x_i)$ by $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$ for all $i \in \mathbb{N}$. He examined the basic properties of $c_0(\Delta)$, $c(\Delta)$ and $l_\infty(\Delta)$ sequence spaces defined as

$$\begin{aligned} c_0(\Delta) &= \{x = (x_i) : \Delta x \in c_0\} \\ c(\Delta) &= \{x = (x_i) : \Delta x \in c\} \\ l_\infty(\Delta) &= \{x = (x_i) : \Delta x \in l_\infty\}. \end{aligned}$$

In these definitions, c_0, c and l_∞ are null, convergent and bounded linear sequence spaces, respectively Kizmaz proved that these spaces are Banach spaces by the norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$ and he also investigated α, β and $\gamma-$duals of these spaces but we do not interested in the duals in our study. Later on, Aydın and Başar ([2]), Başarır ([4]), Et ([8]), Et and Çolak ([9]) and many others interested in some properties of difference sequences.

2 Main Results

After specifying our purpose, let's start by giving the definition of rough convergence for difference sequences in a finite dimensional normed space.

Definition 1. *Let $(X, \|.\|)$ be a normed space, r be a non-negative real number and (Δx_i) be a difference sequence in X. For every $\varepsilon > 0$ and $i \geq i_\varepsilon$; if there is an i_ε such that*

$$\|\Delta x_i - x_*\| < r + \varepsilon$$

or equivalently

$$\limsup_{i\to\infty} \|\Delta x_i - x_*\| \leq r,$$

then, the sequence (Δx_i) is rough convergent to x_*, where $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$. We denote $r-$limit set of (Δx_i) by

$$LIM^r_{\Delta x_i} = \left\{ x_* \in X : \Delta x_i \xrightarrow{r} x_* \right\}.$$

If we obtain a new type of convergence, it would be interesting to compare this type of convergence with the known types of convergence. We can explain this comparison with some examples. The first example is an example of a difference sequences that are not convergent but $r-$convergent.

Example 1. Take the sequence $x_i = \begin{cases} 2, \text{ if } i \text{ is odd} \\ 1, \text{ if } i \text{ is even} \end{cases}$. Then $(\Delta x_i) = (-1)^i$ and we can easily say that (Δx_i) is not convergent but $r-$convergent. Because, from the definition 2.1, if $(\Delta x_{2i}) = (1,1,1,1,...)$, then for every $\varepsilon > 0$

$$-r-\varepsilon < 1-x_* < r+\varepsilon \Longrightarrow 1-r-\varepsilon < x_* < 1+r+\varepsilon$$

$$\Longrightarrow x_* \in [1-r, 1+r]$$

and if $(\Delta x_{2i-1}) = (-1,-1,-1,-1,...)$, then for every $\varepsilon > 0$

$$-r-\varepsilon < -1-x_* < r+\varepsilon \Longrightarrow -1-r-\varepsilon < x_* < -1+r+\varepsilon$$

$$\Longrightarrow x_* \in [-1-r, -1+r]$$

and so

$$LIM^r_{\Delta x_i} = \begin{cases} \emptyset, & \text{if } r < 1 \\ [1-r, -1+r], & \text{if } r \geq 1 \end{cases}.$$

This result gives us the $r-$convergence of (Δx_i).

The sequence given in the second example is both convergent and $r-$convergent.

Conclusion Let $(\Delta x_i) = \left(1+\frac{1}{i}\right)$. Then (Δx_i) is both convergent and $r-$convergent. Because

$$\begin{aligned} -r-\varepsilon+\Delta x_i < x_* < r+\varepsilon+\Delta x_i &\Longrightarrow -r-\varepsilon+1+\tfrac{1}{i} < x_* < r+\varepsilon+1+\tfrac{1}{i} \\ &\Longrightarrow \text{for every } \varepsilon > 0,\ \tfrac{1}{i} \to 0,\ x_* \in [1-r, 1+r] \end{aligned}$$

and so

$$LIM^r_{\Delta x_i} = [1-r, 1+r].$$

If $LIM^r_{\Delta x_i} \neq \emptyset$, then $LIM^r_{\Delta x_i} = [\limsup \Delta x_i - r, \liminf \Delta x_i + r]$.

As is known, in the classical sense, a convergent sequence has a single limit and each subsequence of the sequence converges to the same point. The following theorems will explain how these states will find a response for difference sequences and rough convergence.

Theorem 1. *For any difference sequence $\Delta x = (\Delta x_i)$, diameter of $LIM^r_{\Delta x_i}$ is not greater than $2r$. Generally, there is no smaller bound.*

Proof. If we show that

$$diam\left(LIM^r_{\Delta x_i}\right) = \sup\left\{\|y-z\| : y,z \in LIM^r_{\Delta x_i}\right\} \leq 2r,$$

then we will have the proof.

Suppose that $diam\left(LIM^r_{\Delta x_i}\right) > 2r$. Then there exists $y,z \in LIM^r_{\Delta x_i}$ such that

$$d := \|y-z\| > 2r$$

and for an arbitrary $\varepsilon \in \left(0, \frac{d}{2-r}\right)$ there exist an $i_0 \in \mathbb{N}$ such that

$$\|\Delta x_i - y\| < r+\varepsilon$$

and

$$\|\Delta x_i - z\| < r + \varepsilon,$$

for $i \geq i_0$. In this case, we obtain

$$\|y - z\| \leq \|\Delta x_i - y\| + \|\Delta x_i - z\| < 2(r + \varepsilon) < 2r + 2\left(\frac{d}{2-r}\right) = d$$

but this result contradicts with $d := \|y - z\|$. So, $diam\left(LIM^r_{\Delta x_i}\right) \leq 2r$ is true.

Now, let's show that there is generally no smaller bound. For this, we show that $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$. We know that $diam\bar{B}_r(x_*) = 2r$ for

$$\bar{B}_r(x_*) := \{y \in X : \|y - x_*\| \leq r\}.$$

Choose a convergent difference sequence (Δx_i) with $\lim \Delta x_i = x_*$. For each $\varepsilon > 0$ and for all $i \geq i_0$, there is an $\exists i_0 \in \mathbb{N}$ such that $\|\Delta x_i - x_*\| < \varepsilon$.

$$\|\Delta x_i - y\| \leq \|\Delta x_i - x_*\| + \|x_* - y\| \leq \|\Delta x_i - x_*\| + r, (\text{ for } y \in \bar{B}_r(x_*)\ )$$

and from the definition of rough limit point set we have $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$.

Theorem 2. *A difference sequence (Δx_i) is bounded if and only if there exists an $r \geq 0$ such that $LIM^r_{\Delta x_i} \neq \emptyset$.*

Proof. Assume that $LIM^r_{\Delta x_i} \neq \emptyset$ and $s := \sup\{\|\Delta x_i\| : i \in \mathbb{N}\} < \infty$ for some $r \geq 0$. Then $LIM^s_{\Delta x_i}$ contain the origin of X. On the other hand; if $LIM^r_{\Delta x_i} \neq \emptyset$ for some $r \geq 0$, then all Δx_i except finite elements are contained in some ball with any radius greater then r. So, the sequence (Δx_i) is bounded.

Now, suppose that (Δx_i) is bounded. In this case it is clear that it has a convergent subsequence $\left(\Delta x_{i_j}\right)$. Let x_* be the limit point of this subsequence. Then, $LIM^r_{\Delta x_{i_j}} = \bar{B}_r(x_*)$ and for $r > 0$,

$$LIM^{\left(\Delta x_{i_j}\right),r}\Delta x_{i_j} = \left\{\Delta x_{i_j} : \left\|x_* - \Delta x_{i_j}\right\| \leq r\right\} \neq 0.$$

As is known, each subsequence of a convergent sequence converges to the same limit point. Similarly, we have the following theorem for rough convergent difference sequences.

Theorem 3. *If $\left(\Delta x_{i_j}\right)$ is a subsequence of the difference sequence (Δx_i) then, $LIM^r_{\Delta x_i} \subseteq LIM^r_{\Delta x_{i_j}}$.*

Proof. Suppose that $\left(\Delta x_{i_j}\right)$ is a subsequence of the difference sequence (Δx_i) and $x_* \in LIM^r_{\Delta x_i}$. In this instance,

$$\|\Delta x_i - x_*\| < r + \varepsilon$$

and

$$\left\|\Delta x_{i_j} - x_*\right\| < r + \varepsilon$$

for $i \in \mathbb{N}$ which means $x_* \in LIM^r_{\Delta x_{i_j}}$. Then $LIM^r_{\Delta x_i} \subseteq LIM^r_{\Delta x_{i_j}}$.

It is also important to know the geometric and topological properties of the set of limit points. These properties will be explained in the theorems given below.

Theorem 4. *For an arbitrary difference sequence* (Δx_i) *and for all* $r \geq 0$ *the set* $LIM^r_{\Delta x_i}$ *is closed.*

Proof. We will use a theorem which is well known in functional analysis for this proof "Let $y = (y_j) \in c(\Delta)$ is a $\Delta-$convergent sequence and $\Delta y_j \to y_*$. When $y \in LIM^r_{\Delta x_i}$ is also $y_* \in LIM^r_{\Delta x_i}$, then the set $LIM^r_{\Delta x_i}$ is closed."

Now, assume that the sequence $y = (y_j) \in c(\Delta)$, $\Delta y_j \to y_*$ and $y \in LIM^r_{\Delta x_i}$. For every $\varepsilon > 0$ and for $i \geq i_{\varepsilon/2}$ there are a $j_{\varepsilon/2}$ and an $i_{\varepsilon/2}$ such that

$$\left\|\Delta y_{j_{\varepsilon/2}} - y_*\right\| < \frac{\varepsilon}{2}$$

and

$$\left\|\Delta x_i - \Delta y_{j_{\varepsilon/2}}\right\| < r + \frac{\varepsilon}{2}.$$

For every $i \geq i_{\varepsilon/2}$,

$$\|\Delta x_i - y_*\| \leq \left\|\Delta x_i - \Delta y_{j_{\varepsilon/2}}\right\| + \left\|\Delta y_{j_{\varepsilon/2}} - y_*\right\| < r + \varepsilon$$

and so $y_* \in LIM^r_{\Delta x_i}$.

Theorem 5. (a) *If* $y_0 \in LIM^{r_0}_{\Delta x_i}$ *and* $y_1 \in LIM^{r_1}_{\Delta x_i}$ *then* $y_\lambda := (1-\lambda)y_0 + \lambda y_1 \in LIM^{(1-\lambda)r_0+\lambda r_1}_{\Delta x_i}$ *for* $\lambda \in [0,1]$.
(b) *The set* $LIM^r_{\Delta x_i}$ *is convex.*

Proof.

(a) Assume that $y_0 \in LIM^{r_0}_{\Delta x_i}$ and $y_1 \in LIM^{r_1}_{\Delta x_i}$. In this case, for every $\varepsilon > 0$ there exists an i_ε such that $i \geq i_\varepsilon$ implies $\|\Delta x_i - y_0\| < r_0 + \varepsilon$ and $\|\Delta x_i - y_1\| < r_1 + \varepsilon$ which yields also

$$\|\Delta x_i - y_\lambda\| \leq (1-\lambda)\|\Delta x_i - y_0\| + \lambda\|\Delta x_i - y_1\|$$

$$< (1-\lambda)(r_0+\varepsilon) + \lambda(r_1+\varepsilon)$$

$$= (1-\lambda)r_0 + \lambda r_1 + \varepsilon.$$

Then, we have $y_\lambda \in LIM^{(1-\lambda)r_0+\lambda r_1}_{\Delta x_i}$.

(b) If we choose $r = r_0 = r_1$in (a) it is easily seen that $LIM^r_{\Delta x_i}$ is convex.

The following theorem formulates an additive property of rough convergence with difference sequences.

Theorem 6. *Let* $r_1 \geq 0$ *and* $r_2 \geq 0$. (Δx_i) *is* $(r_1+r_2)-$*convergent to* x_* *if and only if there exists a difference sequence* (Δy_i) *such that*

$$\Delta y_i \xrightarrow{r_1} x_* \text{ and } \|\Delta x_i - \Delta y_i\| \leq r_2 \ \ (i \in \mathbb{N}).$$

Proof. Suppose that $\Delta y_i \xrightarrow{r_1} x_*$ and $\|\Delta x_i - \Delta y_i\| \leq r_2$. Then, for every $\varepsilon > 0$ and $i \geq i_\varepsilon$ there exists an i_ε such that

$$\|\Delta y_i - x_*\| \leq r_1 + \varepsilon.$$

From $\|\Delta x_i - \Delta y_i\| \leq r_2$, we have

$$\|\Delta x_i - x_*\| \leq \|\Delta x_i - \Delta y_i\| + \|\Delta y_i - x_*\| < r_1 + r_2 + \varepsilon$$

if $i \geq i_\varepsilon$. So, (Δx_i) is $(r_1+r_2)-$convergent to x_*.

Now, assume that $\Delta x_i \overset{r_1+r_2}{\to} x_*$ and let's try to show that $\|\Delta y_i - x_*\| \leq r_1$ and $\|\Delta x_i - \Delta y_i\| \leq r_2$ for $i \geq i_\varepsilon$. With

$$\Delta y_i := \begin{cases} x_* & , \text{if } \|\Delta x_i - x_*\| \leq r_2 \\ \Delta x_i + r_2 \frac{x_* - \Delta x_i}{\|x_* - \Delta x_i\|} & , \text{if } \|\Delta x_i - x_*\| > r_2 \end{cases},$$

we have

$$\|\Delta y_i - x_*\| \leq \begin{cases} 0 & , \text{ if } \|\Delta x_i - x_*\| \leq r_2 \\ \|x_i - x_*\| & , \text{if } \|\Delta x_i - x_*\| > r_2 \end{cases}$$

and

$$\|\Delta x_i - \Delta y_i\| \leq r_2$$

for $i \in \mathbb{N}$. At the same time, we know that $\Delta x_i \overset{r_1+r_2}{\to} x_*$ implies

$$\limsup \|\Delta x_i - x_*\| \leq r_1 + r_2.$$

So,

$$\limsup \|\Delta y_i - x_*\| \leq r_1$$

and we have the proof.

Theorem 7. *A sequence $(\Delta x_i) \in \mathbb{R}^n$ convergent to x_* if and only if $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$ where* $\bar{B}_r(x_*) := \{y \in X : \|y - x_*\| \leq r\}$.

Proof. If $\Delta x_i \to x_*$, then we have $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$.

Now, assume that $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$ and (Δx_i) has a cluster point y_* different from x_*. Then the point

$$\bar{x}_* := x_* + \frac{r}{\|x_* - y_*\|}(x_* - y_*)$$

satisfies

$$\|\bar{x}_* - y_*\| = r + \|x_* - y_*\| > r.$$

From the fact that y_* is a cluster point, the last inequality implies that $\bar{x}_* \notin LIM^r_{\Delta x_i}$ and this contradicts with $\|\bar{x}_* - y_*\| = r$ and $LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$. So, our assumption is wrong and x_* is the only cluster point of the sequence. Then, $\Delta x_i \to x_*$.

3 Conclusions

As we will see from many studies, difference sequences have their own characteristics. For example, it is easy to see that $c \subseteq c(\Delta)$. Therefore, in this article, it was interesting to see the results obtained when the concept of rough convergence is studied for difference sequences.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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