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New fractional integral inequalities via Caputo-Fabrizio operator and an open problem concerning an integral inequality

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Abstract: In this paper, author introduces some new integral inequalities by using the Caputo-Fabrizio (CF) fractional integral and functions with the same sense of variation. Also an open problem concerning an integral inequality is discussed.

Keywords: Fractional integral inequalities, fractional integral operator, fractional Calculus, continuous function.

1 Introduction

A fractional derivative is just an operator which generalizes the ordinary derivative, such that if the fractional derivative is represented by the operator symbol D^{α} then, when $\alpha = n$, it coincides with the usual differential operator D^n [16]. Its origin dates back to 1695 when L'Hopital raised by a letter to Leibniz the question of how the expression

$$D^n u(t) = \frac{d^n}{dt^n} u(t),$$

should be understood if n was a real number [16]. Since then, the fractional derivative has become popular and useful due to its ability to describe some natural phenomena in numerous fields of engineering such as theory of viscoelasticity [3-5], study of the anomalous diffusion phenomenon [19-21], circuit theory [22-24], image processing [25, 26] and optimal control theory [12-15], among other applications. Various definition of fractional derivatives have been introduced [27-33]. In fact, the Grunwald-Letnikov fractional derivative, defined as a limit of a fractional order backward difference, is one of the first introduced fractional operators. Other definition which also plays a major role in Fractional Calculus is the Riemann-Liouville fractional derivative. The Caputo fractional derivative has also been defined via a modified Riemann-Liouville fractional derivative. This approach is useful for the formulation and solution of applied problems [29]. In 2015, Caputo and Fabrizio introduced a new fractional approach [31], which was born due to the necessity to describe a class of non-local systems which cannot be well described by classical local theories or by fractional models with singular kernel [31].

In recent years, many researchers have obtained integral inequalities using fractional integral operators [2, 5, 6, 10]. For example, in [5] appear fractional integral inequalities using Riemann-Liouville fractional integral:

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Theorem 1. *Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\big(g(\tau)-g(\rho)\big)\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\geq 0;\quad \tau,\rho\in[0,t],\quad t>0,$$

then we have

$$\frac{J^{\alpha}\big(f(t)\big)}{J^{\alpha}\big(h(t)\big)}\geq\frac{J^{\alpha}\big(gf(t)\big)}{J^{\alpha}\big(gh(t)\big)},$$

for all $\alpha>0,t>0$.

Theorem 2. *Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\big(g(\tau)-g(\rho)\big)\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\geq 0;\quad \tau,\rho\in[0,t],\quad t>0,$$

then for all $\alpha>0$, $w,t>0$, we have

$$\frac{J^{\alpha}\big(f(t)\big)\cdot J^{w}\big(gh(t)\big)+J^{w}\big(f(t)\big)\cdot J^{\alpha}\big(gh(t)\big)}{J^{\alpha}\big(h(t)\big)\cdot J^{w}\big(gf(t)\big)+J^{w}\big(h(t)\big)\cdot J^{\alpha}\big(gf(t)\big)}\geq 1.$$

Theorem 3. *Let f and h be two positive continuous functions and $f\leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p\geq 1,\alpha>0,t>0$, the inequality*

$$\frac{J^{\alpha}\big(f(t)\big)}{J^{\alpha}\big(h(t)\big)}\geq\frac{J^{\alpha}\big(f^{p}(t)\big)}{J^{\alpha}\big(h^{p}(t)\big)},$$

is valid.

Theorem 4. *Let f and h be two positive continuous functions and $f\leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p\geq 1,\alpha>0,w>0,t>0$, we have*

$$\frac{J^{\alpha}\big(f(t)\big)\cdot J^{w}\big(h^{p}(t)\big)+J^{w}\big(f(t)\big)\cdot J^{\alpha}\big(h^{p}(t)\big)}{J^{\alpha}\big(h(t)\big)\cdot J^{w}\big(f^{p}(t)\big)+J^{w}\big(h(t)\big)\cdot J^{\alpha}\big(f^{p}(t)\big)}\geq 1.$$

In [7], it is established the following integral inequalities:

Theorem 5. *Let $f(x)\geq 0$ be a continuous function on $[a,b]$ and satisfies $[(y-a)^{\alpha}\cdot f^{\alpha}(x)-(x-a)^{\alpha}\cdot f^{\alpha}(y)]\cdot[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)]\geq 0,\forall x,y\in[a,b]$ and $f^{\beta}(x)\leq f^{\gamma}(x),\forall x\in[a,b]$. Then for every positive real number $\alpha>0$ and $\beta\geq\gamma>0$ the inequality*

$$\frac{\int_a^b f^{\alpha+\beta}(x)dx}{\int_a^b f^{\alpha+\gamma}(x)dx}\geq\frac{\left(\int_a^b(x-a)^{\alpha}\cdot f^{\beta}(x)dx\right)^{\delta}}{\left(\int_a^b(x-a)^{\alpha}\cdot f^{\gamma}(x)dx\right)^{\lambda}},$$

holds under each of the following conditions

(1) $\lambda=\delta=0$ *and* $\beta=\gamma$, $\forall x\in[a,b]$,
(2) $\lambda=\delta\in[1,+\infty),\forall x\in[a,b]$,
(3) *If* $\int_x^b f^{\beta}(t)dt\geq\frac{1}{(b-a)^{\alpha}}$ *and* $1\leq\delta<1,\forall x\in[a,b]$,
(4) *If* $\int_x^b f^{\beta}(t)dt\leq\frac{1}{(b-a)^{\alpha}}$ *and* $1\leq\lambda<\delta,\forall x\in[a,b]$.

Theorem 6. *Let $f(x),g(x)>0$ continuous function on $[a,b]$ and satisfies*

$$[g^{\alpha}(y)f^{\alpha}(x)-g^{\alpha}(x)f^{\alpha}(y)]\cdot[f^{\beta-\gamma}(x)-f^{\beta-\gamma}(y)]\geq 0,\quad\forall x,y\in[a,b],$$

and $f^{\beta}(x) \le f^{\gamma}(x), \forall x \in [a,b]$. *Then for every positive real number* $\alpha > 0$ *and* $\beta \ge \gamma > 0$*, the inequality*

$$\frac{\int_a^b f^{\alpha+\beta}(x)dx}{\int_a^b f^{\alpha+\gamma}(x)dx} \ge \frac{\left(\int_a^b g^{\alpha}(x)\cdot f^{\beta}(x)dx\right)^{\delta}}{\left(\int_a^b g^{\alpha}(x)\cdot f^{\gamma}(x)dx\right)^{\lambda}},$$

holds under each of the following conditions

(1) $\lambda = \delta = 0$ *and* $\beta = \gamma$, $\forall x \in [a,b]$,
(2) $\lambda = \delta \in [1,+\infty), \forall x \in [a,b]$,
(3) *If* $g^{\alpha}(a) \ge \frac{1}{(b-a)f^{\gamma}(a)}$ *and* $1 \le \delta < \lambda$,
(4) *If* $g^{\alpha}(b) \le \frac{1}{(b-a)f^{\beta}(b)}$ *and* $1 \le \lambda < \delta$.

Also we can find some interesting integral inequalities in [11]:

Theorem 7. *Let* $\alpha > 0, \mu > 0, \delta > 0, \beta > 0, \gamma > 0$ *and let* $f(t)$ *be a positive and continuous function on* $(0,\infty)$ *such that*

$$(\tau^{\mu} - \rho^{\mu})\left(f^{\delta+\beta-\gamma}(\rho) - f^{\delta+\beta-\gamma}(\tau)\right) \ge 0,$$
$$(\tau^{\mu} - \rho^{\mu})\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \ge 0,$$
$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \ge 0,$$

for all $\tau, \rho \in (0,t]$. *Then we have*

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \ge \frac{J^{\alpha}\left(t^{\mu} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\gamma}(t)\right)}.$$

Theorem 8. *Let* $\alpha > 0, \mu > 0, \gamma > 0, \delta > 0, \beta > 0, t > 0$ *and let* $f(t)$ *be a positive and continuous function on* $(0,\infty)$ *such that*

$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \ge 0,$$

for all $\tau, \rho \in (0,t]$. *Then we have*

$$\frac{J^{\alpha}\left(t^{\mu} f^{\gamma}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\delta+\gamma}(t)\right)} \ge \frac{J^{\alpha}\left(t^{\mu} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\delta+\beta}(t)\right)}.$$

Theorem 9. *Let* $\alpha > 0, \mu > 0, \delta > 0, \beta > 0, \gamma > 0$ *and let* $f(t)$ *be a positive and continuous function on* $(0,\infty)$ *such that*

$$(\tau^{\mu} - \rho^{\mu})\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \ge 0,$$
$$\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \ge 0,$$

for all $\tau, \rho \in (0,t]$. *Then we have*

$$\frac{J^{\alpha}\left(f^{\gamma}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \ge \frac{J^{\alpha}\left(t^{\mu} f^{\beta}(t)\right)}{J^{\alpha}\left(t^{\mu} f^{\delta+\beta}(t)\right)}.$$

Theorem 10. *Let* $\alpha > 0, \delta > 0, \beta > 0$ *and* $f(t)$ *be a continuous function on* $(0,\infty)$ *such that* $f(t) \ge t$ *on* $(0,\infty)$.

(a) *If* $0 < \gamma < 1$ *and* $J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right) \ge 1$, *then we have*

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \ge \left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{\gamma}.$$

(b) *If* $\gamma \geq 1$ *and* $0 < J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right) < 1$, *then we have*

$$J^{\alpha}\left(f^{\delta+\beta}(t)\right) \geq \left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{\gamma}.$$

Theorem 11. *Let* $\alpha > 0, \delta > 0, \beta > 0, \gamma > 0, 0 < r < 1, s \geq 1$ *and let* $f(t)$ *be a positive and continuous function on* $(0,\infty)$ *such that*

$$\begin{aligned} J^{\alpha}(t^{\delta} \cdot f^{\gamma}(t)) \geq 1, \qquad J^{\alpha}(t^{\delta} \cdot f^{\beta}(t)) \geq 1, \\ (\tau^{\mu} - \rho^{\mu})\left(f^{\delta+\beta-\gamma}(\rho) - f^{\delta+\beta-\gamma}(\tau)\right) \geq 0, \\ (\tau^{\mu} - \rho^{\mu})\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0, \\ \left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right)\left(\frac{1}{f^{\delta}(\rho)} - \frac{1}{f^{\delta}(\tau)}\right) \geq 0, \end{aligned}$$

for all $\tau, \rho \in (0,t]$. *Then we have*

$$\frac{J^{\alpha}\left(f^{\delta+\beta}(t)\right)}{J^{\alpha}\left(f^{\delta+\gamma}(t)\right)} \geq \frac{\left(J^{\alpha}\left(t^{\delta} f^{\beta}(t)\right)\right)^{r}}{\left(J^{\alpha}\left(t^{\delta} f^{\gamma}(t)\right)\right)^{s}}.$$

And in [8], authors proved the following results:

Theorem 12. *Let* $f(x), g(x) \geq 0$ *be continuous functions on* $[a,b]$ *and satisfy*

$$\left[g^{\alpha}(y) \cdot f^{\alpha}(x) - g^{\alpha}(x) \cdot f^{\alpha}(y)\right]\left[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right] \geq 0, \forall x, y \in [a,b].$$

Then the inequality

$$\frac{\int_a^b f^{\alpha+\beta}(x)dx}{\int_a^b f^{\alpha+\gamma}(x)dx} \geq \frac{\int_a^b g^{\alpha}(x) \cdot f^{\beta}(x)dx}{\int_a^b g^{\alpha}(x) f^{\gamma}(x)dx},$$

holds for every positive real number $\alpha > 0$ *and* $\beta \geq \gamma > 0$.

Theorem 13. *Let* $f(x), h(x) > 0$ *be continuous functions on* $[a,b]$ *with* $f(x) \leq h(x)$ *for all* x *and such that* $\frac{f(x)}{h(x)}$ *is decreasing and* $f(x)$ *increasing. Assume that* $\varphi(x)$ *is a convex function with* $\varphi(0) = 0$. *Then the inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\int_a^b \varphi(f(x))dx}{\int_a^b \varphi(h(x))dx},$$

holds.

Theorem 14. *Let* $f(x), g(x), h(x) > 0$ *be continuous functions on* $[a,b]$ *with* $f(x) \leq h(x)$ *for all* x *and such that* $\frac{f(x)}{h(x)}$ *is decreasing and* $f(x), g(x)$ *are increasing. Assume that* $\varphi(x)$ *is a convex function with* $\varphi(0) = 0$. *Then the inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx},$$

holds.

Next, they proposed the following open problems:

Open problem 1. Under what conditions does the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \left(\int_a^b (x-a)^\alpha \cdot f^\beta(x)dx\right)^\lambda,$$

hold for α, β and λ?.

Open problem 2. Under what conditions does the inequality

$$\frac{\int_a^b f^{\alpha+\beta}(x)dx}{\int_a^b f^{\alpha+\gamma}(x)dx} \geq \frac{\left(\int_a^b (x-a)^\alpha f^\beta(x)dx\right)^\delta}{\left(\int_a^b (x-a)^\alpha f^\gamma(x)dx\right)^\lambda},$$

hold for $\alpha, \beta, \gamma, \delta$ and λ?.

Open problem 3. Assume that $\phi(x)$ is a convex function with $\phi(0) = 0$. Under what conditions does the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\left(\int_a^b \phi(f(x))g(x)dx\right)^\delta}{\left(\int_a^b \phi(h(x))g(x)dx\right)^\lambda},$$

hold for δ and λ?.

In literature few results have been obtained on properties of the Caputo-Fabrizio fractional integral [10, 18]. Motivated from [1], the main purpose of this paper is to establish some new inequalities using Caputo-Fabrizio fractional integral. Also a solution to the **open problem 1** is established. The paper has been organized as follows, in Section 2, we define basic concepts and definitions. In Section 3, we give the main results. The paper finalize with the conclusion in the section 4.

2 Basic Concepts and definitions

Firstly, we give some necessary definitions and preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $\alpha > 0$. The Riemann-Liouville fractional integral of order α of a function f is defined by [29]*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

Definition 2. *Let $0 < \alpha < 1$. The Caputo-Fabrizio fractional integral of order α of a function f is defined by [9, 31]*

$$I_{0t}^\alpha f(t) = (1-\alpha)f(t) + \alpha \int_0^t f(\tau)d\tau.$$

Definition 3. *Let $0 < \alpha < 1$ The Caputo-Fabrizio fractional derivative of order α of a function f is defined by [9, 32]*

$$D_{at}^\alpha f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau)d\tau.$$

Definition 4. *We say that two functions f and g have the same sense of variation on $[0,\infty)$ if*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0, \quad \tau, \rho \in (0,t), t > 0.$$

Note 1. Let $M>0, p\geq 1$ and f,g be two positive functions on $[0,\infty)$. The inequality $\frac{f}{g}\geq M$ is equivalent to

$$M^p(f+g)^p \leq (M+1)^p \cdot f^p, \tag{1}$$

because

$$M^p(f+g)^p \geq (M+1)^p f^p \Leftrightarrow M(f+g)\geq (M+1)f \Leftrightarrow Mg \geq f \Leftrightarrow M \geq \frac{f}{p}.$$

Note 2. Let $m>0, p\geq 1$ and f,g be two positive functions on $[0,\infty)$. The inequality $\frac{f}{g}\geq m$ is equivalent to

$$(1/m)^p(f+g)^p \geq (1/m+1)^p \cdot g^p, \tag{2}$$

because

$$(1/m)^p(f+g)^p \geq (1/m+1)^p g^p \Leftrightarrow (1/m)(f+g)\geq (1/m+1)g \Leftrightarrow$$
$$\Leftrightarrow (1/m)f \geq g \Leftrightarrow \frac{f}{p}\geq m.$$

3 Main Results

In literature few results have been obtained on some fractional integral inequalities using Caputo-Fabrizio fractional integral [10]. The purpose of this section is to establish some new inequalities using the Caputo-Fabrizio fractional integral.

Theorem 15. *Let $p\geq 1$ and let f,g be two positive and continuous functions on $[0,\infty)$. If $0<m\leq \frac{f(\tau)}{g(\tau)}\leq M, \tau\in(0,t)$, then we have*

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{\frac{1}{p}} + \left[I_{0t}^{\alpha} g^p(t)\right]^{\frac{1}{p}} \leq \frac{M(m+2)+1}{(M+1)(m+1)}\left[I_{0t}^{\alpha}\left[(f+g)^p(t)\right]\right]^{\frac{1}{p}}. \tag{3}$$

Proof. Using the condition $\frac{f(\tau)}{g(\tau)}\leq M, \tau\in(0,t), t>0$, we can write

$$(M+1)^p \cdot f(\tau) \leq M^p \cdot (f+g)^p(\tau). \tag{4}$$

Multiplying both sides of (4) by α, then integrating resulting identity with respect to τ from 0 to t, we get

$$(M+1)^p \cdot \left[(1-\alpha)f^p(t) + \alpha\int_0^t f^p(s)ds - (1-\alpha)f^p(t)\right]$$
$$\leq M^p \cdot \left[(1-\alpha)(f(t)+g(t))^p + \alpha\int_0^t (f(\tau)+g(\tau))^p d\tau - (1-\alpha)(f(t)+g(t))^p\right],$$

which is equivalent to

$$(M+1)^p I_{0t}^{\alpha} f^p(t) + (1-\alpha)M^p[f+g]^p \leq M^p I_{0t}^{\alpha}[f(t)+g(t)]^p + (M+1)^p(1-\alpha)f^p(t). \tag{5}$$

By using (1) in (5), follows

$$(M+1)^p \cdot I_{0t}^{\alpha} f^p(t) \leq M^p I_{0t}^{\alpha}[(f+g)^p(t)].$$

Hence, we can write

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{\frac{1}{p}} \leq \frac{M}{M+1}\left[I_{0t}^{\alpha}[(f+g)^p(t)]\right]^{\frac{1}{p}}. \tag{6}$$

On the other hand, from the condition $m \le \frac{f(\tau)}{g(\tau)}$, we obtain

$$\big(1+\frac{1}{m}\big)g(\tau) \le \frac{1}{m}\big(f(\tau)+g(\tau)\big).$$

Therefore

$$\big(1+\frac{1}{m}\big)^p g^p(\tau) \le \big(\frac{1}{m}\big)^p \big(f(\tau)+g(\tau)\big)^p. \tag{7}$$

Now, multiplying both sides of (7) by α, then integrating resulting identity with respect to τ from 0 to t, we have the inequality

$$\begin{aligned}&\big(1+\frac{1}{m}\big)^p \Big[(1-\alpha)g^p(t)+\alpha\int_0^t g^p(\tau)d\tau-(1-\alpha)g^p(t)\Big] \le \\ &\big(\frac{1}{m}\big)^p \Big[(1-\alpha)(f(t)+g(t))^p+\alpha\int_0^t \big(f(\tau)+g(\tau)\big)^p d\tau-(1-\alpha)(f(t)+g(t))^p\Big],\end{aligned}$$

which implies

$$\big(1+\frac{1}{m}\big)^p I_{0t}^{\alpha} g^p+(1-\alpha)\big(\frac{1}{m}\big)^p(f+g)^p \le \big(\frac{1}{m}\big)^p I_{0t}^{\alpha}[(f+g)^p(t)]+(1-\alpha)\big(1+\frac{1}{m}\big)^p g^p. \tag{8}$$

By using (2) into (8), yields

$$\big(1+\frac{1}{m}\big)^p I_{0t}^{\alpha} g^p(t) \le \big(\frac{1}{m}\big)^p I_{0t}^{\alpha}[(f+g)^p(t)].$$

Hence, we can write

$$\Big[I_{0t}^{\alpha} g^p(t)\Big]^{\frac{1}{p}} \le \frac{1}{m+1}\Big[I_{0t}^{\alpha}[(f+g)^p(t)]\Big]^{\frac{1}{p}}. \tag{9}$$

The inequality (3) follows on adding the inequalities (6) and (9).

Remark. Let $m>0, p>1, \frac{1}{p}+\frac{1}{q}=1$ and f,g be two positive functions on $[0,\infty)$. The inequality $\frac{f}{g} \ge m$ is equivalent to

$$m^{1/p} g(t) \le g^{1/q} \cdot f^{1/p}, \tag{10}$$

as

$$\begin{aligned}&\frac{f}{g} \ge m \Leftrightarrow m \le \frac{f}{g} \Leftrightarrow m^{1/p} \le \Big(\frac{f}{g}\Big)^{1/p} \Leftrightarrow m^{1/p} \le g^{-\frac{1}{p}} f^{\frac{1}{p}}, \\ &\Leftrightarrow m^{1/p} \le g^{\frac{1}{q}-1} f^{\frac{1}{p}} \Leftrightarrow m^{1/p} g(t) \le g^{1/q} f^{1/p}.\end{aligned}$$

Remark. In the same way, inequality $\frac{f}{g} \le M$ is equivalent to

$$M^{-1/q} f(t) \le \Big[f(t)\Big]^{1/p} \Big[g(t)\Big]^{1/q}. \tag{11}$$

Lemma 1. *Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$ *and let* f *and* g *be two positive and continuous functions on* $[0,\infty)$*. If*

$$0<m \le \frac{f(\tau)}{g(\tau)} \le M<\infty, \quad \tau \in [0,t],$$

then the inequality

$$\big[I_{0t}^{\alpha} f(t)\big]^{1/p} \big[I_{0t}^{\alpha} g(t)\big]^{1/q} \le \big(\frac{M}{m}\big)^{\frac{1}{pq}} I_{0t}^{\alpha}\big[(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\big], \tag{12}$$

holds.

Proof. Since $\frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0,t], t > 0$, therefore

$$[g(\tau)]^{1/q} \geq M^{-1/q}[f(\tau)]^{1/q},$$

and so

$$[f(\tau)]^{1/p}[g(\tau)]^{1/q} \geq M^{-1/q}[f(\tau)]^{1/q}[f(\tau)]^{1/p} = M^{-1/q}f(\tau). \tag{13}$$

Integrating (13) with respect to τ from 0 to t, we have

$$\begin{aligned}&(1-\alpha)[f(t)]^{1/p}[g(t)]^{1/q} + \alpha\int_0^t [f(\tau)]^{1/p}[g(\tau)]^{1/q}d\tau - (1-\alpha)[f(\tau)]^{1/p}[g(\tau)]^{1/q}\\ &\geq M^{-1/q}[(1-\alpha)f(t) + \alpha\int_0^t f(\tau)d\tau - (1-\alpha)f(t)].\end{aligned}$$

That is

$$\begin{aligned}&I_{0t}^{\alpha}\left[[f(t)]^{1/p}[g(t)]^{1/q}\right] + (1-\alpha)M^{-1/q}f(t)\\ &\geq M^{-1/q}I_{0t}^{\alpha}f(t) + (1-\alpha)[f(t)]^{1/p}[g(t)]^{1/q}.\end{aligned}$$

By using (11) we obtain

$$I_{0t}^{\alpha}\left[[f(t)]^{1/p}[g(t)]^{1/q}\right] \geq M^{-1/q}I_{0t}^{\alpha}f(t),$$

and consequently

$$\left(I_{0t}^{\alpha}\left[[f(t)]^{1/p}[g(t)]^{1/q}\right]\right)^{1/p} \geq M^{-1/pq}\left[I_{0t}^{\alpha}f(t)\right]^{1/p}. \tag{14}$$

On the other hand, since $mg(\tau) \leq f(\tau), \tau \in [0,t], t > 0$, then we have

$$[f(\tau)]^{1/p} \geq m^{1/p}[g(\tau)]^{1/p},$$

and so

$$[g(\tau)]^{1/q}[f(\tau)]^{1/p} \geq m^{1/p}[g(\tau)]^{1/p}[g(\tau)]^{1/q} = m^{1/p}g(\tau). \tag{15}$$

Now, multiplying both sides of (15) by α, then integrating the resulting inequality with respect to τ over $(0,t)$, we obtain

$$\begin{aligned}&(1-\alpha)[g(t)]^{1/q}[f(t)]^{1/p} + \alpha\int_0^t g^{1/q}(\tau)f^{1/p}(\tau)d\tau - (1-\alpha)[g(t)]^{1/q}[f(t)]^{1/p}\\ &\geq m^{1/p}[(1-\alpha)g(t) + \alpha\int_0^t g(\tau)d\tau - (1-\alpha)g(t)].\end{aligned}$$

Therefore

$$I_{0t}^{\alpha}\left[[g(t)]^{1/q}[f(t)]^{1/p}\right] + (1-\alpha)m^{1/p}g(t) \geq m^{1/p}I_{0t}^{\alpha}g(t) + (1-\alpha)[g(t)]^{1/q}[f(t)]^{1/p}.$$

By using (10), we can write

$$I_{0t}^{\alpha}\left[[g(t)]^{1/q}[f(t)]^{1/p}\right] \geq m^{1/p}I_{0t}^{\alpha}g(t).$$

Hence, we obtain

$$\left(I_{0t}^{\alpha}\left[[g(t)]^{1/q}[f(t)]^{1/p}\right]\right)^{1/q} \geq m^{1/pq}\left(I_{0t}^{\alpha}g(t)\right)^{1/q}. \tag{16}$$

Thanks to (14) and (16), we obtain (12).

Lemma 2. *Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1,$ *f and g be two positive and continuous functions on* $[0,\infty)$. *If*

$$0<m\leq \frac{(f(\tau))^p}{(g(\tau))^q}\leq M<\infty, \qquad \tau\in[0,t],$$

then we have

$$\left[I_{0t}^{\alpha}f^p(t)\right]^{1/p}\left[I_{0t}^{\alpha}g^q(t)\right]^{1/q}\leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}I_{0t}^{\alpha}\Big(f(t)g(t)\Big). \tag{17}$$

Proof. Replacing $f(\tau)$ and $g(\tau)$ reswpectively by $(f(\tau))^p$ and $(g(\tau))^q$, $\tau\in[0,t], t>0$ in Lemma 1, we obtain (17).

Lemma 3. *Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$ *and let fand g be two positive and continuous functions on* $[0,\infty)$. *If*

$$0<m\leq \frac{f(\tau)}{g(\tau)}\leq M<\infty,$$

then

$$I_{0t}^{\alpha}\left[\frac{(f(t))^p}{(g(t))^{p/q}}\right]\leq \left(\frac{M}{m}\right)^{1/q}\frac{(I_{0t}^{\alpha}f(t))^p}{(I_{0t}^{\alpha}g(t))^{p/q}}. \tag{18}$$

Proof. Using Lemma 1 we obtain

$$\begin{aligned}I_{0t}^{\alpha}(f(t)) &= I_{0t}^{\alpha}\left[\left(\frac{(f(t))^p}{(g(t))^{p/q}}\right)^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\right]\\ &\geq \left(\frac{m}{M}\right)^{\frac{1}{pq}}\left[I_{0t}^{\alpha}\frac{(f(t))^p}{(g(t))^{p/q}}\right]^{\frac{1}{p}}\left[I_{0t}^{\alpha}g(t)\right]^{1/q}.\end{aligned}$$

Hence, we can write

$$\left[I_{0t}^{\alpha}(f(t))\right]^p\geq \left(\frac{m}{M}\right)^{\frac{1}{q}}\left[I_{0t}^{\alpha}\left[\frac{(f(t))^p}{(g(t))^{p/q}}\right]\right]\left[I_{0t}^{\alpha}g(t)\right]^{p/q}. \tag{19}$$

Thanks to (19) we obtain (18).

Theorem 16. *Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$ *and let f be a positive and continuous function on* $[0,\infty)$. *If* $0<m\leq f(\tau)\leq M<\infty$ *and*

$$I_{0t}^{\alpha}f(t)\geq(1-\alpha+\alpha t)^{-\frac{p}{q}}, \tag{20}$$

then the inequality

$$I_{0t}^{\alpha}\left[(f(t))^p\right]\leq\left(\frac{M}{m}\right)^{1/q}\left[I_{0t}^{\alpha}f(t)\right]^{p+1},$$

holds.

Proof. Using Lemma 3 and the condition (20), we obtain

$$\begin{aligned}I_{0t}^{\alpha}\left[(f(t))^p\right] &= I_{0t}^{\alpha}\left[\frac{(f(t))^p}{(1)^{p/q}}\right]\\ &\leq \left(\frac{M}{m}\right)^{1/q}\cdot\frac{\left(I_{0t}^{\alpha}f(t)\right)^p}{(I_{0t}^{\alpha}1)^{\frac{p}{q}}}\\ &\leq \left(\frac{M}{m}\right)^{1/q}(1-\alpha+\alpha t)^{-\frac{p}{q}}\left(I_{0t}^{\alpha}f(t)\right)^p\\ &\leq \left(\frac{M}{m}\right)^{1/q}\left(I_{0t}^{\alpha}f(t)\right)^{p+1},\end{aligned}$$

as required.

The following theorem gives conditions under which the **open problem 1** holds.

Theorem 17. *Let $a,b \in \mathbb{R}(a<b)$ and $f(x)$ be a function such that $f(x) \geq x-a$ for all $x \in (a,b)$ and*

$$(1/(\alpha+\beta+1)) \cdot (b-a)^{\alpha+\beta+1} > 1, \tag{21}$$

for α, β and γ positive real numbers with $0<\gamma<1$. Then

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \left(\int_a^b (x-a)^{\alpha} \cdot f^{\beta}(x)dx\right)^{\gamma}. \tag{22}$$

Proof. From $f(x) \geq x-a > 0$, it is easy to see that

$$(f(x))^{\alpha+\beta} \geq (x-a)^{\alpha+\beta} > 0, \tag{23}$$
$$(f(x))^{\beta} \geq (x-a)^{\beta} > 0, \tag{24}$$
$$(f(x))^{\alpha} \geq (x-a)^{\alpha} > 0. \tag{25}$$

On the one hand, by using inequalities (21) and (23), we obtain

$$\int_a^b (f(x))^{\alpha+\beta}dx \geq \int_a^b (x-a)^{\alpha+\beta}dx = (1/(\alpha+\beta+1)) \cdot (b-a)^{\alpha+\beta+1} > 1. \tag{26}$$

On the other hand, from (24) and (21), we obtain

$$\begin{aligned}\int_a^b (x-a)^{\alpha} \cdot f^{\beta}(x)dx &\geq \int_a^b (x-a)^{\alpha} \cdot (x-a)^{\beta}dx = \int_a^b (x-a)^{\alpha+\beta}dx \\ &= (1/(\alpha+\beta+1)) \cdot (b-a)^{\alpha+\beta+1} > 1.\end{aligned} \tag{27}$$

Moreover, from (25) and (27), yields

$$\int_a^b f^{\alpha+\beta}(x)dx = \int_a^b f^{\alpha}(x) \cdot f^{\beta}(x)dx \geq \int_a^b (x-a)^{\alpha} \cdot f^{\beta}(x)dx > 1. \tag{28}$$

Combining (28) with the fact that $0<\gamma<1$, we obtain

$$\begin{aligned}\int_a^b f^{\alpha+\beta}(x)dx &= \int_a^b f^{\alpha}(x) \cdot f^{\beta}(x)dx \\ &\geq \int_a^b (x-a)^{\alpha} \cdot f^{\beta}(x)dx \geq \left(\int_a^b (x-a)^{\alpha} \cdot f^{\beta}(x)dx\right)^{\gamma}.\end{aligned} \tag{29}$$

From (29), we obtain (22).

4 Conclusion

In this paper, we have used the Caputo-Fabrizio fractional integral to develop some interesting fractional inequalities. These results have been obtained with the help of functions with the same sense of variation. Also an open problem concerning an integral inequality has been discussed. As a future work, author is planning to use these inequalities to prove the existence and uniqueness of some ordinary differential equations containing the Caputo-Fabrizio operator.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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