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Split monotone variational inclusion, mixed equilibrium problem and common fixed point for finite families of demicontractive mappings

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Abstract: In this paper we introduce an iterative scheme for approximating a common element in the set of solution of split monotone variational inclusion, mixed equilibrium problem and common fixed point for finite families of demicontractive mappings. We prove a strong convergence theorem for the sequence generated by the scheme. The results presented generalize and improve some recently announced ones.

Keywords: Split variational inequality problem; Mix equilibrium problem; Fixed points; Demicontractive mappings.

1 Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : C \to C$ be a map. A point $x \in C$ is called a fixed point of S if $Sx = x$, and the set of all fixed points of S is denoted by $F(S) := \{x \in C : Sx = x\}$. The mapping S is said to be quasi nonexpansive if $F(S) \neq \emptyset$ and $||Sx - x^*|| \leq ||x - x^*||$ for all $x \in C$ and $x^* \in F(S)$. S is said be k-demicontractive if for $k \in (0,1)$,

$$||Sx - x^*||^2 \leq ||x - x^*||^2 + k||Sx - x||^2 \ \ \forall x \in \ C \ and \ x^* \in F(S). \tag{1}$$

We can easily see that (1) is equivalent to

$$\langle Sx - x^*, x - x^* \rangle \leq ||x - x^*||^2 - \frac{1-k}{2}||Sx - x||^2. \tag{2}$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $A : C \to H$ be a nonlinear mapping. The mixed equilibrium problem (MEP) is: Find $x \in C$ such that

$$F(x,y) + \langle Ax, y - x \rangle \geq 0, \ \ \forall y \in C. \tag{3}$$

Mixed equilibrium problem (MEP)(3) was first studied by Moudafi and Thera [17]. The set of solution of MEP(3) is denoted by Sol(MEP(3)). If $F = 0$, then MEP(3) reduces to the classical variational inequality problem (VIP), which is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \ \forall y \in C. \tag{4}$$

The (VIP) was introduced and studied by Hartmann and Stampacchia [11] . If $A = 0$, MEP(3) reduces to the equilibrium problem (EP): find $x \in C$ such that

$$F(x,y) \geq 0, \ \ \forall y \in C. \tag{5}$$

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which was introduced and studied by Blum and Oettli [2]. The set of solutions of the equilibrium problem (5) is denoted by Sol(EP(5)).

Numerous problems in optimization, economics and physics reduce to finding a solution of equilibrium problems. Some methods have been proposed to solve equilibrium problems in Hilbert spaces, for example Blum and Oettli [2], Combettes and Hirstoaga [8]; Tada and Takahashi [22,23]. Takahashi and Takahashi [21] obtained weak and strong convergence theorems for finding a common element in the set of solutions of an equilibrium problem and a set of fixed points of nonexpansive mappings in a Hilbert space.

It is known that if H is a Hilbert space, then for every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \ \ \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. It is a common knowledge that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2, \ \ \forall x, y \in H.$$

Further, for $x \in H$ the following always hold

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C,$$

which implies that

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \ \ \forall x \in H, \ y \in C. \tag{6}$$

Definition 1. *A mapping $T : H \to H$ is said to be*

(1) *Monotone, if*

$$\langle Tx - Ty, x - y \rangle \geq 0, \ \ \forall x\, y \in H;$$

(2) *α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tx - Ty, x - y \rangle \geq \alpha ||Tx - Ty||^2, \ \ \forall x\, y \in H;$$

(3) *β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$||Tx - Ty|| \leq \beta ||x - y||, \ \ \forall x\, y \in H.$$

Remark. If T is α-inverse strongly monotone mapping, then T is monotone and $\frac{1}{\alpha}$Lipschitz continuous.

Definition 2. *A multi-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, with $u \in Mx$ and $v \in My$, $\langle x - y, u - v \rangle \geq 0$ hold.*

Definition 3. *A monotone mapping $M : H \to 2^H$ is maximal, if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mapping define on H.*

It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in$ G(M) implies that $u \in Mx$.

Definition 4. *Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the resolvent mapping $J_\lambda^M : H \to H$ associated with M and λ is defined by*

$$J_\lambda^M(x) = (I+\lambda M)^{-1}x, \ \ x \in H, \ \lambda > 0. \tag{7}$$

Remark. [12] The resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive.

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be inverse strongly monotone mappings and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings.

Let $B : H_1 \to H_2$ be a bounded linear mapping. The split monotone variational inclusion problem(SpMVIP) is to find $x^* \in H_1$ such that

$$0 \in f(x^*) + M_1(x^*), \tag{8}$$

and

$$y^* = Bx^* \in H_2 \ \ solves \ \ 0 \in g(y^*) + M_2(y^*). \tag{9}$$

If we consider (8) separately, we have a monotone variational inclusion problem (MVIP) with its solution set Sol(MVIP(8) and (9) is a monotone variational inclusion problem (MVIP) with its solution set Sol(MVIP(9).

The solution set of SpMVIP(8)-(9) is denoted by Sol(SpMVIP)= $\{x^* \in H_1 : x^* \in$ Sol(MVIP(8) and $Bx^* \in$ Sol(MVIP(9)$\}$.

Censor *et al*. [5] introduced the following split variational inequality problem (SpVIP): Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be nonlinear singled-valued mappings and let $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. The SpVIP is then formulated as follows: Find a point $x^* \in C$ such that

$$\langle fx^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \tag{10}$$

and such that

$$y^* = Bx^* \in Q \ \ and \ solves \ \ 0 \in \langle g(y^*), y - y^* \rangle \geq 0, \ \ \forall y \in Q. \tag{11}$$

The solution set of SpVIP(10)-(11) is denoted by Sol(SpVIP(10)-(11))= $\{x^* \in C : x^* \in$ Sol(VIP(10)) and $Bx^* \in$ Sol(VIP(11))$\}$. SpVIP(10)-(11) is a special case of SpMVIP(8)-(9).

From SpVIP(10)-(11), if $C = H_1$, $Q = H_2$; and letting $x = x^* - f(x^*) \in H_1$ and $y = Bx^* - g(Bx^*) \in H_2$ then the result reduces to split null point problem (SpNPP)which was introduce by Censor *et al*. [5]. It is to find $x^* \in H_1$ such that $f(x^*) = 0$ and $g(Bx^*) = 0$.

Moudafi [18] introduced and studied the iterative method for solving SpMVIP(8)-(9) and noted that SpMVIP(8)-(9) include as special cases SpVIP(10)-(11), split null point problem, the split fixed point problem and split feasibility problem see [3,4,6,7,8,18]. These have been studied by several authors and applied to modelling of intensity-modulated radiation therapy treatment planning. Also for modelling of inverse problems arising from phase retrieval and many real life problems; for example in sensor networks in computerized tomography and data compression.

If $f \equiv 0$ and $g \equiv 0$ the SpMVIP(8)-(9) reduces to the following split null point problem (SpNPP): find $x^* \in H_1$ such that

$$0 \in M_1(x^*) \tag{12}$$

and

$$y^* = Bx^* \in H_2 \ \ solves \ \ 0 \in M_2(y^*). \tag{13}$$

Byrne et al [3], introduced the following iterative scheme and obtained weak and strong convergence theorems for solving SpVIP(12)-(13); for a given $x_0 \in H_1$ the sequence $\{x_n\}$ was generated by

$$x_{n+1} = J_\lambda^{M_1}\big(x_n + \gamma B^*(J_\lambda^{M_2} - I)Bx_n\big), \qquad for \ \ \lambda > 0.$$

Motivated by the work of Byrne et al [3]. Kazmi and Rizvi [13] under some appropriate conditions, introduced and studied the following iterative scheme for approximation of solution of SpVIP(12)-(13) and fixed point of a nonexpansive mapping in the framework of real Hilbert space.

$$\begin{cases} u_n = J_\lambda^{M_1}\big(x_n + \gamma B^*(J_\lambda^{M_2} - I)Bx_n\big), \\ x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)Su_n. \end{cases} \tag{14}$$

Recently Shehu and Ogbuisi [20] introduced and studied the following iterative scheme for approximating a common solution of a fixed point problem for strictly pseudocontractive mappings and SpMVIP(8)-(9) without f and g being necessarily zero and obtained a strong convergence result under some appropriate conditions imposed on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$,

$$\begin{cases} w_n = (1-\alpha_n)x_n \\ y_n = J_\lambda^{M_1}(I - \lambda f_1)(w_n + \gamma B^*(J_\lambda^{M_2}(I - \lambda f_2) - I)Bw_n \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Sy_n, \ \ \forall n \geq 0. \end{cases} \tag{15}$$

Very recently Kazmi *et al*. [12] studied a hybrid-extragradient iterative method and approximated a common element of the set of solutions of split monotone variational inclusion, mixed equilibrium problem and fixed-point problem for a nonexpansive mapping. They studied under certain appropriate conditions imposed on $\{r_n\}, \lambda$ and $\{\alpha_n\}$, the convergence of the sequence define by the following scheme;

$$\begin{cases} x_0 = x \in H_1, \\ y_n = J_\lambda^{M_1}(I - \lambda f)x_n, \\ l_n = J_\lambda^{M_2}(I - \lambda g)By_n, \\ z_n = P_C[y_n + \gamma B^*(l_n - By_n)], \\ w_n = T_{r_n}(I - r_n A)z_n, \\ u_n = \alpha_n x_n + (1 - \alpha_n S_n T_{r_n}(z_n - r_n A w_n), \\ C_n = \{z \in H_1 : ||u_n - z||^2 \leq ||x_n - z||^2\}, \\ Q_n = \{z \in H_1 : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 1. \end{cases} \tag{16}$$

Motivated by the above mention results, we introduce an iterative scheme for approximating a common element in the set of solution of SpMVIP(8)-(9), (MEP(3)) and fixed point problem for demicontractive mappings. Furthermore a strong convergence theorem is established. Our result extends, generalized and improve the work of Kazmi [12] and many results announced recently.

2 Preliminaries

We present some important results needed in the sequel.

2.1 Asumption

The bifunction $F : C \times C \to \mathbb{R}$ is required to satisfies the following conditions:

(A) $F(x,x) = 0, \forall x \in C$;

(B) F is monotone, $i.e., F(x,y) + F(y,x) \leq 0 \quad \forall x,y \in C$;

(C) $\limsup_{t\to o} F(x+t(z-x),y) \leq F(x,y), \quad \forall x,y,z \in C$;

(D) The function $y \mapsto F(x,y)$ is convex and lower semi-continuous.

2.2 Asumption

For the bifunction $F : C \times C \to \mathbb{R}$ the inequality

$$F(x,y) + F(y,z) + F(z,x) \leq 0, \ \forall x,y,z \in C, holds. \tag{17}$$

Lemma 1. [8] *Let C be a nonempty closed convex subset of H_1. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1-A4). For $r > 0$ and for all $x \in H_1$, define a mapping $T_r : H_1 \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C\}, \tag{18}$$

then the following hold:

(i) *For each $x \in H_1$, $T_r(x) \neq \emptyset$;*
(ii) *T_r is single-valued;*
(iii) *T_r is firmly nonexpansive;*
(iv) *$Fix(T_r)$ =Sol(EP(5));*
(v) *Sol(EP(5))is closed and convex.*

Remark. From Lemma 1 (i)-(ii) we have

$$rF(T_r y) + \langle T_r x - x, y - T_r x\rangle \geq 0, \ \forall y \in C, \ x \in H_1. \tag{19}$$

Again, Lemma 1 (iii) implies

$$||T_r x - T_r y|| \leq ||x-y|| \ \forall x, \ y \in H_1. \tag{20}$$

Furthermore, inequality (19) implies

$$||T_r x - y||^2 \leq ||x-y||^2 - ||T_r x - x||^2 + 2rF(T_r x, y), \ \forall y \in C, \ x \in H_1. \tag{21}$$

Lemma 2.[9,10] *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping then for all $x,y \in H$,*

$$\langle (x-Tx)-(y-Ty), Ty-Tx\rangle \leq \frac{1}{2}||(Tx-x)-(Ty-y)||^2 \tag{22}$$

and consequently if $y \in Fix(T)$ then

$$\langle x-Tx, Ty-Tx\rangle \leq \frac{1}{2}||Tx-x||^2. \tag{23}$$

It is well known that a real Hilbert space H_1 satisfies the following identities

(1) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle, \ \forall x,y \in H.$

(2) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$. $\forall x,y \in H$ and $\alpha \in (0,1)$.

Lemma 3.[16] *(Demiclosedness principle) Let C be a nonempty,closed and convex subset of a Hilbert space H. Let $T : C \to C$ be k-strictly pseudocontractive mapping. Then $(I-T)$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $(x_n - Tx_n) \to 0$, then $x = Tx$.*

Lemma 4. [19,24] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \ \ n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(2) $\limsup_{n\to\infty}\delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5. [14] *Let $M : H \to 2^H$ be a maximal monotone mapping and $f : H \to H$ be a Lipschitz continuous mapping, then $G = M + f : H \to 2^H$ is a maximal monotone mapping.*

A mapping $T : H \to H$ is said be averaged if and only if it can be written as average of the identity mapping and a nonexpansive mapping. i.e

$$T := (1-\beta)I + \beta S,$$

where $\beta \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping and I is the identity mapping on H. Every averaged mapping is noneexpansive and every firmly nonexpansive mapping is averaged. Also since the resolvent of maximal monotone operators are nonexpansive, then they are averaged and therefore nonexpansive see [1,4,16,18].

3 Main results

Theorem 1. *Let H_1 and H_2 be real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with it's adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying assumption 2.1((A1),(A2),(A3) and (A4)) and assumption 2.2; let $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings; let $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively σ,θ_1,θ_2-inverse strongly monotone mappings and let $S_i : C \to C$ for $i = 1,2,...,N$ be finite family of k_i-demicontractive mappings such that $\Omega = Sol(SpMVIP) \cap Sol((MEP) \cap (\cap_{i=1}^{N} F(S_i) \neq \emptyset$, $k = min_{1\leq i\leq N}\{k_i\}$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ be generated by the following algorithm:*

$$\begin{cases} x_0 = x \in C, \\ y_n = J_\lambda^{M_1}(I-\lambda f)x_n, \\ l_n = J_\lambda^{M_2}(I-\lambda g)By_n, \\ z_n = P_C[y_n + \gamma B^*(l_n - By_n)], \\ w_n = T_{r_n}(I - r_n A)z_n, \\ u_n = (1-\alpha_n)x_n + \alpha_n S_{[n]} T_{r_n}(z_n - r_n A w_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S_{[n]} u_n, \quad n \geq 1, \end{cases} \tag{24}$$

for $i = 1, 2, ..., N$ where $[n] = n(mod\, N)$, $\{r_n\} \subset [a,b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := min\{\theta_1, \theta_2)\}$ and $\gamma \in \left(0, \frac{1}{||B^*||^2}\right)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ satisfying the following conditions;

(1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$,
(2) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - k$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to $p \in \Omega$.

*Proof.*The proof is divided into four steps.
Step I. We first show that the sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ are bounded. Let $\bar{x} \in \Omega$ then $\bar{x} \in$ Sol(SpMVIP) therefore $\bar{x} = J_\lambda^{M_1}(I-\lambda f)\bar{x}$ and $B\bar{x} = J_\lambda^{M_2}(I-\lambda g)B\bar{x}$, we have

$$\begin{aligned} ||y_n - \bar{x}||^2 &= ||J_\lambda^{M_1}(x_n - \lambda f x_n) - J_\lambda^{M_1}(\bar{x} - \lambda f\bar{x})||^2 \\ &\leq ||(x_n - \bar{x}) - \lambda(fx_n - f\bar{x})||^2 \\ &= ||x_n - \bar{x}||^2 + \lambda^2||fx_n - f\bar{x}||^2 - 2\lambda\langle x_n - \bar{x}, fx_n - f\bar{x}\rangle \\ &\leq ||x_n - \bar{x}||^2 - \lambda(2\theta_1 - \lambda)||fx_n - f\bar{x}||^2 \qquad (25) \\ &\leq ||x_n - \bar{x}||^2. \qquad (26) \end{aligned}$$

$$\begin{aligned} ||l_n - B\bar{x}||^2 &= ||J_\lambda^{M_2}(I-\lambda g)By_n - J_\lambda^{M_2}(I-\lambda g)B\bar{x}||^2 \\ &\leq ||By_n - B\bar{x}||^2 - \lambda(2\theta_2 - \lambda)||gBy_n - gB\bar{x}||^2 \qquad (27) \\ &\leq ||By_n - B\bar{x}||^2. \qquad (28) \end{aligned}$$

$$\begin{aligned} ||z_n - \bar{x}||^2 &= ||P_C[y_n + \gamma B^*(l_n - By_n)] - \bar{x}||^2 \\ &\leq ||y_n + \gamma B^*(l_n - By_n)] - \bar{x}||^2 \\ &= ||y_n - \bar{x}||^2 + ||\gamma B^*(l_n - By_n)||^2 \\ &\quad + 2\gamma\langle y_n - \bar{x}, B^*(l_n - By_n)\rangle \\ &\leq ||y_n - \bar{x}||^2 + \gamma^2||B^*||^2||l_n - By_n||^2 \\ &\quad + 2\gamma\langle B(y_n - \bar{x}), l_n - By_n\rangle \\ &= ||y_n - \bar{x}||^2 - \gamma(1 - \gamma||B^*||^2)||l_n - By_n||^2 \qquad (29) \\ &\leq ||y_n - \bar{x}||^2 \leq ||x_n - \bar{x}||^2. \qquad (30) \end{aligned}$$

$$\begin{aligned}||w_n-\bar{x}||^2 &= ||T_{r_n}(z_n-r_nAz_n)-T_{r_n}(\bar{x}-r_nA\bar{x})||^2\\ &\le ||(z_n-\bar{x}-r_n(Az_n-A\bar{x}||^2\\ &= ||z_n-\bar{x}||^2-2r_n\langle z_n-\bar{x},Az_n-A\bar{x}\rangle\\ &\quad +r_n^2||Az_n-A\bar{x}||^2\\ &\le ||z_n-\bar{x}||^2-r_n(2\sigma-r_n)||Az_n-A\bar{x}||^2\\ &\le ||z_n-\bar{x}||^2\le ||x_n-\bar{x}||^2.\end{aligned}$$

Using (21) with $t_n=T_{r_n}(z_n-r_nAw_n)$ and the fact that $\bar{x}\in\Omega$, we have the following estimates.

$$\begin{aligned}||t_n-\bar{x}||^2 &= ||T_{r_n}(z_n-r_nAw_n)-\bar{x}||^2\\ &\le ||z_n-r_nAw_n-\bar{x}||^2-||t_n-(z_n-r_nAw_n)||^2\\ &\quad +2r_nF(t_n,\bar{x})\\ &= ||z_n-\bar{x}||^2-||t_n-z_n||^2\\ &\quad +2r_n\langle Aw_n,\bar{x}-t_n\rangle+2r_nF(t_n,\bar{x})\\ &= ||z_n-\bar{x}||^2-||t_n-z_n||^2\\ &\quad +2r_n[\langle Aw_n-A\bar{x},\bar{x}-w_n\rangle+\langle A\bar{x},\bar{x}-w_n\rangle\\ &\quad -\langle Aw_n,t_n-w_n\rangle]+2r_nF(t_n,\bar{x}).\end{aligned} \tag{31}$$

Applying (19),(3) and monotonicity of A in (31), we have

$$\begin{aligned}||t_n-\bar{x}||^2 &\le ||z_n-\bar{x}||^2-||t_n-z_n||^2+2r_n\langle Aw_n,w_n-t_n\rangle\\ &\quad +2r_n[F(\bar{x},w_n)+F(t_n,\bar{x})]\\ &\le ||z_n-\bar{x}||^2-||z_n-w_n||^2-||w_n-t_n||^2\\ &\quad -2\langle z_n-w_n,w_n-t_n\rangle+2r_n\langle Aw_n,w_n-t_n\rangle\\ &\quad +2r_n[F(\bar{x},w_n)+F(t_n,\bar{x})]\\ &= ||z_n-\bar{x}||^2-||z_n-w_n||^2-||w_n-t_n||^2\\ &\quad -2\langle w_n-(z_n-r_nAz_n),t_n-w_n\rangle\\ &\quad +2r_n\langle Az_n-Aw_n,t_n-w_n\rangle\\ &\quad +2r_n[F(\bar{x},w_n)+F(t_n,\bar{x})]\\ &= ||z_n-\bar{x}||^2-||z_n-w_n||^2-||w_n-t_n||^2\\ &\quad +2r_n\langle Az_n-Aw_n,t_n-w_n\rangle\\ &\quad +2r_n[F(\bar{x},w_n)+F(w_n,t_n)+F(t_n,\bar{x})].\end{aligned} \tag{32}$$

Using Assumption 2.2 and the fact that A is $\frac{1}{\sigma}$-Lipschitz continous in (32) we obtain

$$\begin{aligned}||t_n-\bar{x}|| &\le ||z_n-\bar{x}||^2-||z_n-w_n||^2-||w_n-t_n||^2\\ &\quad +2r_n\frac{1}{\sigma}||z_n-w_n||||t_n-w_n|| \qquad (33)\\ &\le ||z_n-\bar{x}||^2-||z_n-w_n||^2-||w_n-t_n||^2\\ &\quad +||w_n-t_n||^2+\big(\frac{r_n}{\sigma}\big)^2||z_n-w_n||^2\\ &\le ||x_n-\bar{x}||^2-\big(1-\big(\frac{r_n}{\sigma}\big)^2\big)||z_n-w_n||^2, \qquad (34)\end{aligned}$$

since $r_n \in [a,b]$, we have

$$||t_n - \bar{x}|| \leq ||z_n - \bar{x}||^2 \leq ||y_n - \bar{x}||^2 \leq ||x_n - \bar{x}||^2. \tag{35}$$

For n large enough we have

$$\begin{aligned}
||u_n - \bar{x}||^2 &= ||(1-\alpha_n)x_n + \alpha_n S_{[n]}t_n - \bar{x}||^2 \\
&= ||(1-\alpha_n)(x_n - \bar{x}) + \alpha_n(S_{[n]}t_n - \bar{x})||^2 \\
&= (1-\alpha_n)^2||x_n - \bar{x}||^2 + \alpha_n^2||S_{[n]}t_n - \bar{x}||^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle x_n - \bar{x}, S_{[n]}t_n - \bar{x}\rangle \\
&\leq (1-\alpha_n)^2||x_n - \bar{x}||^2 + \alpha_n^2\big[||t_n - \bar{x}||^2 \\
&\quad +k||t_n - S_{[n]}t_n||^2\big] + 2\alpha_n(1-\alpha_n)\Big[||x_n - \bar{x}||^2 \\
&\quad -\frac{1-k}{2}||t_n - S_{[n]}t_n||^2\Big] \qquad (36)\\
&= (1-2\alpha_n+\alpha_n^2)||x_n - \bar{x}||^2 \\
&\quad +\alpha_n^2\big[||x_n - \bar{x}||^2 + k||t_n - S_{[n]}t_n||^2\big] \\
&\quad +2\alpha_n||x_n - \bar{x}||^2 - 2\alpha_n^2||x_n - \bar{x}||^2 \\
&\quad +\alpha_n(1-\alpha_n)(1-k)||t_n - S_{[n]}t_n||^2 \\
&= ||x_n - \bar{x}||^2 \\
&\quad +\big[\alpha_n^2 k - \alpha_n(1-\alpha_n)(1-k)\big]||t_n - S_{[n]}t_n||^2 \\
&= ||x_n - \bar{x}||^2 + \alpha_n\big[k + \alpha_n - 1\big]||t_n - S_{[n]}t_n||^2 \\
&\leq ||x_n - \bar{x}||^2, \qquad (37)
\end{aligned}$$

$$\begin{aligned}
||x_{n+1} - \bar{x}||^2 &= ||(1-\beta_n)u_n + \beta_n S_{[n]}u_n - \bar{x}||^2 \\
&= ||(1-\beta_n)(u_n - \bar{x}) + \beta_n(S_{[n]}u_n - \bar{x})||^2 \\
&= (1-\beta_n)^2||u_n - \bar{x}||^2 + \beta_n^2||S_{[n]}u_n - \bar{x}||^2 \\
&\quad +2\beta_n(1-\beta_n)\langle u_n - \bar{x}, S_{[n]}u_n - \bar{x}\rangle \\
&\leq (1-\beta_n)^2||u_n - \bar{x}||^2 + \beta_n^2\big[||u_n - \bar{x}||^2 \\
&\quad +k||u_n - S_{[n]}u_n||^2\big] + 2\beta_n(1-\beta_n)\Big[||u_n - \bar{x}||^2 \\
&\quad -\frac{1-k}{2}||u_n - S_{[n]}u_n||\Big] \\
&= (1-2\beta_n+\beta_n^2)||u_n - \bar{x}||^2 + \beta_n^2||u_n - \bar{x}||^2 \\
&\quad +k\beta_n^2||u_n - S_{[n]}u_n||^2\big] + 2\beta_n||u_n - \bar{x}||^2 \\
&\quad -2\beta_n^2||u_n - \bar{x}||^2 \\
&\quad -\beta_n(1-\beta_n)(1-k)||u_n - S_{[n]}u_n||^2 \\
&= ||u_n - \bar{x}||^2 \\
&\quad +\big[\beta_n^2 k - \beta_n(1-\beta_n)(1-k)\big]||u_n - S_{[n]}u_n||^2 \\
&= ||u_n - \bar{x}||^2 + \beta_n[k + \beta_n - 1]||u_n - S_{[n]}u_n||^2 \qquad (38)\\
&\leq ||u_n - \bar{x}||^2 \leq ||x_n - \bar{x}||^2. \qquad (39)
\end{aligned}$$

Step II. We show that $\lim_{n\to\infty}||x_{n+1}-x_n||^2 = \lim_{n\to\infty}||z_n-x_n||^2 = \lim_{n\to\infty}||u_n-x_n||^2 = \lim_{n\to\infty}||x_n-y_n||^2 = \lim_{n\to\infty}||x_n-t_n||^2 = \lim_{n\to\infty}||S_i^n t_n - t_n||^2 = 0$.
From (38) we have

$$\beta_n((1-k)-\beta_n)||u_n - S_{[n]}u_n||^2 \le ||x_n-\bar{x}||^2 - ||x_{n+1}-\bar{x}||^2, \tag{40}$$

thus, as $n\to\infty$

$$||S_{[n]}u_n - u_n|| \to 0. \tag{41}$$

Since $\{||x_n-\bar{x}||\}$ is Cauchy for any $k\in\mathbb{N}$ we have,
$||x_{n+k}-x_n|| = ||x_{n+k}-\bar{x}|| - ||x_n-\bar{x}|| \to 0$ as $n\to\infty$,
and in particular,

$$\lim_{n\to\infty}||x_{n+1}-x_n|| = 0. \tag{42}$$

From (63) and (41) we have

$$\begin{aligned}||x_{n+1}-u_n|| &= ||(1-\beta_n)u_n + \beta_n S_{[n]}u_n - u_n||\\ &= \beta_n||S_{[n]}u_n - u_n|| \to 0 \; as \; n\to\infty.\end{aligned} \tag{43}$$

Since

$$||u_n-x_n|| \le ||u_n-x_{n+1}|| + ||x_{n+1}-x_n||,$$

then using (42) and (43) we have

$$\lim_{n\to\infty}||u_n-x_n|| = 0. \tag{44}$$

Substituting (34) in (36) and simplifying we get

$$\begin{aligned}||z_n-w_n||^2 &\le \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}\left(||x_n-\bar{x}||^2 - ||u_n-\bar{x}||^2\right)\\ &= \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}\big(||x_n-\bar{x}||\\ &\quad -||u_n-\bar{x}||\big)\big(||x_n-\bar{x}|| + ||u_n-\bar{x}||\big)\\ &\le \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}||x_n-u_n||\big(||x_n-\bar{x}||\\ &\quad +||u_n-\bar{x}||\big).\end{aligned}$$

But $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n\to\infty$ in the above inequality we have

$$\lim_{n\to\infty}||z_n-w_n|| = 0. \tag{45}$$

Using the same argument as in (33), we have

$$\begin{aligned}||t_n-\bar{x}|| &\le ||z_n-\bar{x}||^2 - ||z_n-w_n||^2 - ||w_n-t_n||^2\\ &\quad +2r_n\frac{1}{\sigma}||z_n-w_n||||t_n-w_n||\\ &\le ||z_n-\bar{x}||^2 - ||z_n-w_n||^2 - ||w_n-t_n||^2\\ &\quad +||z_n-w_n||^2 + (\frac{r_n}{\sigma})^2||t_n-w_n||^2\\ &\le ||x_n-\bar{x}||^2 - \left(1-(\frac{r_n}{\sigma})^2\right)||t_n-v_n||^2.\end{aligned} \tag{46}$$

Substituting (46) in (36) and simplifying we get

$$||u_n-\bar{x}||^2 \leq ||x_n-\bar{x}||^2-\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)||t_n-v_n||^2.$$

$$\begin{aligned}||t_n-w_n||^2 &\leq \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}\left(||x_n-\bar{x}||^2-||u_n-\bar{x}||^2\right)\\ &= \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}\big(||x_n-\bar{x}||\\ &\quad -||u_n-\bar{x}||\big)\left(||x_n-\bar{x}||+||u_n-\bar{x}||\right)\\ &\leq \left[\alpha_n^2\left(1-(\frac{r_n}{\sigma})^2\right)\right]^{-1}||x_n-u_n||\big(||x_n-\bar{x}||\\ &\quad +||u_n-\bar{x}||\big).\end{aligned}$$

Again, $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n\to\infty$ in the above inequality we have

$$\lim_{n\to\infty}||t_n-w_n||=0. \tag{47}$$

Substituting (35) in (36)and then substituting (25) in the result, simplifying we get

$$||u_n-\bar{x}||^2 \leq ||x_n-\bar{x}||^2-\alpha_n^2\lambda(2\theta_1-\lambda)||fx_n-f\bar{x}||^2,$$

which gives

$$\begin{aligned}||fx_n-f\bar{x}||^2 &\leq \left[\alpha_n^2\lambda(2\theta_1-\lambda)\right]^{-1}\left(||x_n-\bar{x}||^2-||u_n-\bar{x}||^2\right)\\ &= \left[\alpha_n^2\lambda(2\theta_1-\lambda)\right]^{-1}\big(||x_n-\bar{x}||\\ &\quad -||u_n-\bar{x}||\big)\left(||x_n-\bar{x}||+||u_n-\bar{x}||\right)\\ &\leq \left[\alpha_n^2\lambda(2\theta_1-\lambda)\right]^{-1}||x_n-u_n||\big(||x_n-\bar{x}||\\ &\quad +||u_n-\bar{x}||\big).\end{aligned}$$

But $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n\to\infty$ in the above inequality we have

$$\lim_{n\to\infty}||fx_n-f\bar{x}||=0. \tag{48}$$

Substituting (35) in (36) and then substituting (29) in the result, simplifying we get

$$||u_n-\bar{x}||^2 \leq ||x_n-\bar{x}||^2-\alpha_n^2\gamma(1-\gamma||B^*||^2)||l_n-By_n||^2,$$

which gives

$$\begin{aligned}||l_n-By_n||^2 &\leq \left[\alpha_n^2\gamma(1-\gamma||B^*||^2)\right]^{-1}\left(||x_n-\bar{x}||^2-||u_n-\bar{x}||^2\right)\\ &= \left[\alpha_n^2\gamma(1-\gamma||B^*||^2)\right]^{-1}\big(||x_n-\bar{x}||\\ &\quad -||u_n-\bar{x}||\big)\left(||x_n-\bar{x}||+||u_n-\bar{x}||\right)\\ &\leq \left[\alpha_n^2\gamma(1-\gamma||B^*||^2)\right]^{-1}||x_n-u_n||\big(||x_n-\bar{x}||\\ &\quad +||u_n-\bar{x}||\big).\end{aligned}$$

But $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n \to \infty$ in the above inequality we have

$$\lim_{n\to\infty} ||l_n - By_n|| = 0. \tag{49}$$

Furthermore, it follows from (27) that

$$||l_n - B\bar{x}||^2 \le ||By_n - B\bar{x}||^2 - \lambda(2\theta_2 - \lambda)||gBy_n - gB\bar{x}||^2,$$

which gives

$$\begin{aligned}
||gBy_n - gB\bar{x}||^2 &\le \left[\lambda(2\theta_2 - \lambda)\right]^{-1}(||By_n - B\bar{x}||^2 - ||l_n - B\bar{x}||^2 \\
&= \left[\lambda(2\theta_2 - \lambda)\right]^{-1}(||By_n - B\bar{x}||^2 \\
&\quad -||l_n - B\bar{x}||^2)(||By_n - B\bar{x}||^2 + ||l_n - B\bar{x}||^2) \\
&\le \left[\lambda(2\theta_2 - \lambda)\right]^{-1}||By_n - l_n||^2(||By_n - B\bar{x}||^2 \\
&\quad +||l_n - B\bar{x}||^2).
\end{aligned}$$

But $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n \to \infty$ in the above inequality we have

$$\lim_{n\to\infty} ||gBy_n - gB\bar{x}|| = 0. \tag{50}$$

By the firmly nonexpansivity of $J_\lambda^{M_1}$ and the arguments in (26), we have

$$\begin{aligned}
||y_n - \bar{x}||^2 &= ||J_\lambda^{M_1}(I - \lambda f)x_n - J_\lambda^{M_1}(I - \lambda f)\bar{x}||^2 \\
&\le \langle (I - \lambda f)x_n - (I - \lambda f)\bar{x}, y_n - \bar{x}\rangle \\
&= \frac{1}{2}\big[||(I - \lambda f)x_n - \bar{x}||^2 + ||(I - \lambda f)\bar{x} - y_n||^2 \\
&\quad -||(I - \lambda f)x_n - y_n||^2 - ||(I - \lambda f)\bar{x} - \bar{x}||^2 \\
&= \frac{1}{2}\big[||(I - \lambda f)x_n - (I - \lambda f)\bar{x}||^2 \\
&\quad +||y_n - \bar{x}||^2 - ||x_n - y_n||^2 \\
&\quad +2\lambda\langle x_n - y_n, fx_n - f\bar{x}\rangle - \lambda^2||fx_n - f\bar{x}||^2\big] \\
&\le \frac{1}{2}\big[||x_n - \bar{x}||^2 + ||y_n - \bar{x}||^2 - ||x_n - y_n||^2 \\
&\quad +2\lambda\langle x_n - y_n, fx_n - f\bar{x}\rangle - \lambda^2||fx_n - f\bar{x}||^2\big] \\
&\le \frac{1}{2}\big[||x_n - \bar{x}||^2 + ||y_n - \bar{x}||^2 - ||x_n - y_n||^2 \\
&\quad +2\lambda||x_n - y_n||||fx_n - f\bar{x}||\big],
\end{aligned}$$

which gives

$$\begin{aligned}
||y_n - \bar{x}||^2 &\le ||x_n - \bar{x}||^2 - ||x_n - y_n||^2 \\
&\quad +2\lambda||x_n - y_n||||fx_n - f\bar{x}||
\end{aligned} \tag{51}$$

Substituting (35) in (36) and then substituting (51) in the result, simplifying we get

$$||u_n - \bar{x}||^2 \le ||x_n - \bar{x}||^2 - \alpha_n^2||x_n - y_n||^2 + 2\lambda\alpha_n^2||x_n - y_n||||fx_n - f\bar{x}||,$$

which gives

$$\begin{aligned}
||x_n-y_n||^2 \le & \left(\alpha_n^2\right)^{-1}\left[||x_n-\bar{x}||^2-||u_n-\bar{x}||^2\right.\\
&+2\lambda\alpha_n^2||x_n-y_n||||fx_n-f\bar{x}||\\
=& \left(\alpha_n^2\right)^{-1}\left[\left(||x_n-\bar{x}||-||u_n-\bar{x}||\right)\left(||x_n-\bar{x}||\right.\right.\\
&+||u_n-\bar{x}||+2\lambda\alpha_n^2||x_n-y_n||||fx_n-f\bar{x}||\\
\le & \left(\alpha_n^2\right)^{-1}\left[\left(||x_n-u_n||\left(||x_n-\bar{x}||+||u_n-\bar{x}||\right)\right.\right.\\
&+2\lambda\alpha_n^2||x_n-y_n||||fx_n-f\bar{x}||. \qquad (52)
\end{aligned}$$

But $\{x_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n\to\infty$ in the above inequality we have

$$\lim_{n\to\infty}||x_n-y_n||=0. \qquad (53)$$

Using the firmly nonexpansivity of P_C, we have

$$\begin{aligned}
||z_n-\bar{x}||^2 =& ||P_C[y_n+\gamma B^*(l_n-By_n)]-\bar{x}||^2\\
\le & \langle y_n+\gamma B^*(l_n-By_n)-\bar{x}, z_n-\bar{x}\rangle\\
=& \frac{1}{2}\Big[||(y_n-\bar{x})+\gamma B^*(l_n-By_n)||^2+||z_n-\bar{x}||^2\\
&-||y_n+\gamma B^*(l_n-By_n)-\bar{x}-z_n+\bar{x}||^2\Big]\\
=& \frac{1}{2}\Big[||y_n-\bar{x}||^2+||z_n-\bar{x}||^2+||\gamma B^*(l_n-By_n)||^2\\
&+2\gamma\langle By_n-B\bar{x}, l_n-By_n\rangle\\
&-||y_n-z_n+\gamma B^*(l_n-By_n)||^2\Big]\\
\le & \frac{1}{2}\Big[||y_n-\bar{x}||^2+||z_n-\bar{x}||^2\\
&+2\gamma||By_n-B\bar{x}||\,||l_n-By_n||\\
&-||y_n-z_n||^2-2\gamma\langle y_n-z_n, B^*(l_n-By_n)\rangle\Big],
\end{aligned}$$

which gives

$$\begin{aligned}
||z_n-\bar{x}||^2 \le & ||y_n-\bar{x}||^2-||y_n-z_n||^2+2\gamma||By_n-B\bar{x}||\,||l_n-By_n||\\
&-2\gamma||y_n-z_n||\,||B^*||\,||l_n-By_n||\\
\le & ||y_n-\bar{x}||^2-||y_n-z_n||^2+2\gamma||l_n-By_n||\left(||By_n-B\bar{x}||\right.\\
&\left.-||B^*||\,||y_n-z_n||\right). \qquad (54)
\end{aligned}$$

Substituting (35) in (36) and then substituting (54) in the result, simplifying we get

$$\begin{aligned}
||u_n-\bar{x}||^2 \le & ||x_n-\bar{x}||^2-\alpha_n^2||y_n-z_n||^2\\
&+2\gamma\alpha_n^2\Big[||l_n-By_n||\left(||By_n-B\bar{x}||\right.\\
&\left.-||B^*||\,||y_n-z_n||\right)\Big],
\end{aligned}$$

which gives

$$\begin{aligned}
||y_n - z_n||^2 \leq & \left(\alpha_n^2\right)^{-1}\Big[||x_n - \bar{x}||^2 - ||u_n - \bar{x}||^2 \\
& +2\gamma\alpha_n^2||l_n - By_n||\,\big(||By_n - B\bar{x}|| \\
& -||B^*||\,||y_n - z_n||\big)\Big] \\
= & \left(\alpha_n^2\right)^{-1}\Big[\big(||x_n - \bar{x}|| - ||u_n - \bar{x}||\big)\big(||x_n - \bar{x}|| \\
& +||u_n - \bar{x}||\big) + 2\gamma\alpha_n^2||l_n - By_n||\,\big(||By_n - B\bar{x}|| \\
& -||B^*||\,||y_n - z_n||\big)\Big] \\
\leq & \left(\alpha_n^2\right)^{-1}\Big[||x_n - u_n||\big(||x_n - \bar{x}|| + ||u_n - \bar{x}||\big) \\
& +2\gamma\alpha_n^2||l_n - By_n||\,\big(||By_n - B\bar{x}|| \\
& -||B^*||\,||y_n - z_n||\big)\Big].
\end{aligned}$$

But $\{x_n\}$, $\{y_n\}$,$\{z_n\}$ and $\{u_n\}$ are bounded and taking the limit as $n \to \infty$ in the above inequality we have,

$$\lim_{n\to\infty} ||y_n - z_n|| = 0. \tag{55}$$

From (53) and (55), we have

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \tag{56}$$

Also, from (45) and (56), we have

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{57}$$

From (47) and (57), we have

$$\lim_{n\to\infty} ||t_n - x_n|| = 0. \tag{58}$$

Since $\{x_n\}$ is bounded, $u_n \to x^*$ for some $x^* \in H$. By Lemma 3 and (44) $x^* \in F(S_{[n]})\ \forall n \in \mathbb{N}$. From this we get $x^* \in (\cap_{i=1}^N F(S_i)$. Consequently $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{t_n\}$ converge weakly to x^*.
Step III: We show that $\{x_n\}$ converges strongly to $\bar{x}$,

$$\begin{aligned}
||x_{n+1} - x^*||^2 = & ||(1-\beta_n)u_n + \beta_n S_{[n]}P_C u_n - x^*||^2 \\
\leq & ||u_n - \bar{x}||^2 \\
= & ||(1-\alpha_n)(x_n - x^*) + \alpha_n(S_{[n]}P_C t_n - x^*)||^2 \\
\leq & (1-\alpha_n)^2||x_n - x^*||^2 + \alpha_n^2\left[||t_n - x^*||^2\right. \\
& \left.+k||t_n - S_{[n]}P_C t_n||^2\right] \\
& +2\alpha_n(1-\alpha_n)\Big[||x_n - x^*||^2 \\
& -\frac{1-k}{2}||t_n - S_{[n]}P_C t_n||\Big] \\
\leq & (1-\alpha_n)||x_n - x^*||^2 + \alpha_n\Big[\alpha_n||t_n - x^*||^2 \\
& +\alpha_n k||t_n - S_{[n]}P_C t_n||^2 \\
& +2(1-\alpha_n)\Big(||x_n - x^*||^2 \\
& -\frac{1-k}{2}||t_n - S_{[n]}P_C t_n||\Big)\Big],
\end{aligned}$$

Hence, by Lemma 4, we have $x_n \to x^*$ as $n \to \infty$.
Step IV: Now we show that $x^* \in$ Sol(SpMVIP). From (63) we have

$$\frac{1}{\lambda}\big((x_n - y_n) - \lambda f(x_n)\big) \in M_1 y_n. \tag{59}$$

Taking the limit as $n \to \infty$ in (59) and using the fact that f is $\frac{1}{\theta_1}$-Lipschitz continuous mapping, then by Lemma 5 we conclude $M_1(x^*) + f(x^*)$ is maximal monotone, therefore we have

$$0 \in M_1(x^*) + f(x^*),$$

which implies that $\bar{x} \in$ Sol(SpMVIP(8).
Since $||y_n - x_n|| \to 0$ as $n \to \infty$ we have that By_n converges weakly to Bx^* and by (49), using the fact $J_\lambda^{M_2}(I - \lambda g)$ is nonexpansive and Lemma 3, we have

$$0 \in M_2(Bx^*) + g(Bx^*),$$

which implies that $Bx^* \in$ Sol(SpMVIP(9).
Next, we show $x^* \in$ MEP(3). From (63) we obtain

$$F(w_n, y) + \langle Az_n, y - w_n\rangle + \frac{1}{r_n}\langle y - w_n, w_n - z_n\rangle \geq 0, \ \ \forall y \in C.$$

Using the fact that F is a monotone operator, we have

$$\langle Az_n, y - w_n\rangle + \frac{1}{r_n}\langle y - w_n, w_n - z_n\rangle \geq F(y, w_n), \ \ \forall y \in C.$$

Let $y_t = ty + (1-t)x^* \in C$. for $t \in (0,1]$ using the inequality above, we have

$$\begin{aligned}
\langle y_t - w_n, Ay_t\rangle &\geq \langle y_t - w_n, Ay_t\rangle - \langle y_t - w_n, Az_n\rangle \\
&\quad - \langle y_t - t_n, \frac{t_n - z_n}{r_n}\rangle + F(y_t, t_n) \\
&= \langle y_t - w_n, Ay_t - Aw_n\rangle + \langle y_t - w_n, Aw_n - Az_n\rangle \\
&\quad - \langle y_t - w_n, \frac{w_n - z_n}{r_n}\rangle + F(y_t, w_n).
\end{aligned}$$

Since $||w_n - z_n|| \to 0$ as $n \to \infty$ and A is Lipschitz continuous, then $||Aw_n - Az_n|| \to 0$ as $n \to \infty$. Again, since A is monotone and F is convex and lower semicontinuous, $\frac{w_n - z_n}{r_n} \to 0$ as $n \to \infty$ and w_n converges weakly to x^*, we obtain as $n \to \infty$

$$\langle y_t - x^*, Ay_t\rangle \geq F(y_t, x^*). \tag{60}$$

Again, we have

$$\begin{aligned}
0 &= F(y_t, y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, x^*) \\
&\leq tF(y_t, y) + (1-t)\langle y_t - x^*, Ay_t\rangle \\
&= tF(y_t, y) + (1-t)t\langle y - x^*, Ay_t\rangle,
\end{aligned} \tag{61}$$

therefore

$$0 \leq F(y_t, y) + (1-t)\langle y - x^*, Ay_t\rangle.$$

For each $y \in C$ and setting $t \to 0^+$ we have

$F(x^*,y)+(1-t)\langle y-x^*,Ax^*\rangle \geq 0.$

This implies that $x^* \in$ MEP(3). Hence $x^* \in \Omega$.
This completes the proof.

Corollary 1.*Let H_1 and H_2 be real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with it's adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying assumption 2.1((A1),(A3) and (A4)) and assumption 2.2; let $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings; let $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively σ,θ_1,θ_2-inverse strongly monotone mappings and let $S_i : C \to C$ for $i = 1,2,...,N$ be a finite family of nonexpansive mappings such that $\Omega = Sol(SpMVIP) \cap Sol((MEP) \cap (\cap_{i=1}^{N} F(S_i) \neq \emptyset$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_0 = x \in C, \\ y_n = J_{\lambda}^{M_1}(I-\lambda f)x_n, \\ l_n = J_{\lambda}^{M_2}(I-\lambda g)By_n, \\ z_n = P_C[y_n+\gamma B^*(l_n - By_n)], \\ w_n = T_{r_n}(I-r_nA)z_n, \\ u_n = (1-\alpha_n)x_n + \alpha_n S_{[n]}T_{r_n}(z_n - r_nAw_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n S_{[n]}u_n, \quad n \geq 1. \end{cases}$$

for $i = 1,2,...,N$ where $[n] = n(mod\,N)$ and $\{r_n\} \subset [a,b]$ for some $a,b \in (0,\sigma)$, $\lambda \subset [a',b']$ for some $a',b' \in (0,\theta)$, where $\theta := min\{\theta_1,\theta_2)\}$ and and $\gamma \in \left(0, \frac{1}{||B^*||^2}\right)$. $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

(1) $0 < \liminf \beta_n \leq \limsup \beta_n < 1-k$,
(2) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,

Then the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ converges strongly to $p \in \Omega$.

3.1 Application

In this section we present some application of Theorem 1

3.1.1 Split variational inequality problem, mixed equilibrium problem and common fixed point for finite families of demicontractive mappings

Theorem 2.*Let H_1 and H_2 be real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with it's adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying assumption 2.1((A1),(A3) and (A4)) and assumption 2.2; let $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively σ,θ_1,θ_2-inverse strongly monotone mappings and let $S_i : C \to C$ for $i = 1,2,...,N$ be a finite family of k_i-demicontractive mappings such that $\Omega = Sol(SpVIP) \cap Sol((MEP) \cap (\cap_{i=1}^{N} F(S_i) \neq \emptyset$., $k = min_{1\leq i\leq N}\{k_i\}$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(I - \lambda f)x_n, \\ l_n = P_C(I - \lambda g)By_n, \\ z_n = P_C[y_n + \gamma B^*(l_n - By_n)], \\ w_n = T_{r_n}(I - r_n A)z_n, \\ u_n = (1 - \alpha_n)x_n + \alpha_n S_{[n]} T_{r_n}(z_n - r_n A w_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n S_{[n]} u_n, \quad n \geq 1. \end{cases} \tag{62}$$

for $i = 1, 2, ..., N$ where $[n] = n(mod\, N)$ and $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := min\{\theta_1, \theta_2)\}$ and and $\gamma \in \left(0, \frac{1}{||B^*||^2}\right)$. $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions

(1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,

(2) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - k$,

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to $p \in \Omega$.

Proof. By setting $M_1 = \partial I_C$ and $M_2 = \partial I_Q$ in Theorem 1.

3.1.2 Split null point problem, mixed equilibrium problem and common fixed point for finite families of demicontractive mappings

Theorem 3. *Let H_1 and H_2 be real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with it's adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying assumption 2.1((A1),(A3) and (A4)) and assumption 2.2; let $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings; let $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively $\sigma, \theta_1, \theta_2$-inverse strongly monotone mappings and let $S_i : C \to C$ for $i = 1, 2, ..., N$ be a finite family of k_i-demicontractive mappings such that $\Omega = Sol(SpNPP) \cap Sol((MEP) \cap (\cap_{i=1}^{N} F(S_i) \neq \emptyset$, $k = min_{1\leq i\leq N}\{k_i\}$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_0 = x \in C, \\ y_n = J_{\lambda}^{M_1} x_n, \\ l_n = J_{\lambda}^{M_2} By_n, \\ z_n = P_C[y_n + \gamma B^*(l_n - By_n)], \\ w_n = T_{r_n}(I - r_n A)z_n, \\ u_n = (1 - \alpha_n)x_n + \alpha_n S_{[n]} T_{r_n}(z_n - r_n A w_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n S_{[n]} u_n, \quad n \geq 1. \end{cases} \tag{63}$$

for $i = 1, 2, ..., N$ where $[n] = n(mod\, N)$ and $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := min\{\theta_1, \theta_2)\}$ and and $\gamma \in \left(0, \frac{1}{||B^*||^2}\right)$. $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions

(1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,

(2) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - k$,

Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to $p \in \Omega$.

Proof. By setting $f = 0$ and $g = 0$ in Theorem 1.

4 Numerical Example

We give the following numerical example to justify Theorem 1

Example 1. Let $H_1 = H_2 = \mathbb{R}$ with an inner product defined by $\langle x,y\rangle = xy$, $\forall x,y \in \mathbb{R}$, and induced norm $|.|$. Let $C = [0,1]$ and $Q = (-\infty,0]$; Let $F : C\times C \to \mathbb{R}$ be a bifunction defined by $F(x,y) = x(y-x)$, $\forall x,y \in C$; Let $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ be defined by $M_1(x) = 2x$ and $M_2(x) = 4x$ $\forall x \in \mathbb{R}$ Let the mapping $A : C \to \mathbb{R}, B : \mathbb{R} \to \mathbb{R}$ and $S : C \to C$ be defined by $A(x) = 2x$, $B(x) = -2x$ and $S(x) = \frac{x}{2}$ $\forall x \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ $\forall x \in \mathbb{R}$ and $g(y) = 0$ $\forall y \in \mathbb{R}$. Clearly F is a bifunction satisfying Assumption 2.1 and Assumption 2.2 M_1 and M_2 are maximal monotone; A is $\frac{1}{2}-$inverse strongly monotone, S is k-demicontractive mapping and B is a bounded linear operator with its adjoint B^* such that $||B|| = ||B^*|| = 2$. The iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{u_n\}$ generated by 63 are reduced to the following iterative scheme.

$$\begin{cases} y_n = \frac{1}{3}x_n; \\ l_n = \frac{-2}{5}y_n; \\ z_n = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 1, \\ [y_n + 0.4(l_n - 2y_n)] & \text{otherwise}; \end{cases} \\ w_n = z_n; \\ u_n = \left(1 - \frac{1}{n+1}\right)x_n + \frac{1}{2}\left(\frac{1}{n+1}\right)\left(z_n - 2\right)w_n; \\ x_{n+1} = \left(1 - \frac{n}{2n+1}\right)u_n + \frac{1}{2}\left(\frac{n}{2n+1}\right)u_n. \end{cases}$$

where $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2n+1}$ and $r_n = 1$. Then $\{x_n\}$ converges strongly to $0 \in \Omega = \{0\}$

Next, using Matlab software we have the following figures which shows that the sequence $\{x_n\}$ converges to strongly to 0.

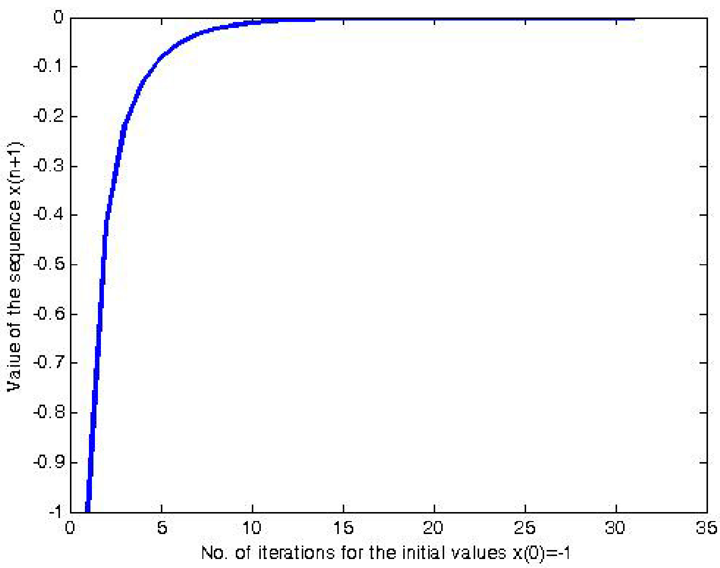

Fig. 1

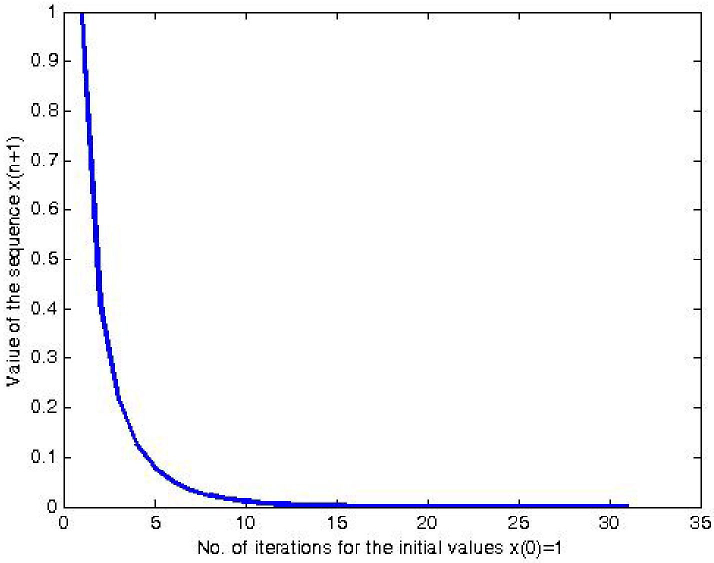

Fig. 2

4.1 Conclusion

In our work, we removed C_n and Q_n in the scheme of Kazmi et al. [12] and still obtain strong convergence theorem. Corollary 3.2, generalized the result of Kazmi et al. [12]. Hence our result improved, extends and generalized the result of Kazmi et al. [12] and many others.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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