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Cubical Covers of Sets in $\mathbb{R}^n$

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Abstract: In this paper, we explore several methods of representing subsets of $\mathbb{R}^n$ using their geometric and analytic properties. We present a heuristic, expository approach to estimating the size of various sets and their boundaries, with the goal of preserving important features in the representations. The aim is to stimulate interest in and serve as an introduction to this topic as well as two other well known methods: Jones' β numbers and varifolds from geometric measure theory. We provide various computations and exercises, numerous illustrations, suggestions for further research, and an extensive list of current resources and references in these areas.

Keywords: Analysis, geometric measure theory, varifolds, cubical covers.

1 Introduction

In this paper we explain and illuminate a few ideas for (1) representing sets and (2) learning from those representations. Though some of the ideas and results we explain are likely written down elsewhere (though we are not aware of those references), our purpose is not to claim priority to those pieces, but rather to stimulate thought and exploration. Our primary intended audience is students of mathematics even though other, more mature mathematicians may find a few of the ideas interesting. We believe that cubical covers can be used at an earlier point in the student career and that both the β numbers idea introduced by Peter Jones and the idea of varifolds pioneered by Almgren and Allard and now actively being developed by Menne, Buet, and collaborators are still very much underutilized by all (young and old!). To that end, we have written this exploration, hoping that the questions and ideas presented here, some rather elementary, will stimulate others to explore the ideas for themselves.

We begin by briefly introducing cubical covers, Jones' β, and varifolds, after which we look more closely at questions involving cubical covers. Then both of the other approaches are explained in a little bit of detail, mostly as an invitation to more exploration, after which we close with problems for the reader and some unexplored questions.

2 Representing sets and their boundaries in $\mathbb{R}^n$

2.1 Cubical refinements: dyadic Cubes

In order to characterize various sets in $\mathbb{R}^n$, we explore the use of cubical covers whose cubes have side lengths which are positive integer powers of $\frac{1}{2}$, **dyadic cubes**, or more precisely, (closed) dyadic n-cubes with sides parallel to the axes. Thus the side length at the dth subdivision is $l(C) = \frac{1}{2^d}$, which can be made as small as desired.

Figure 1 illustrates this by looking at a unit cube in $\mathbb{R}^2$ lying in the first quadrant with a vertex at the origin. We then

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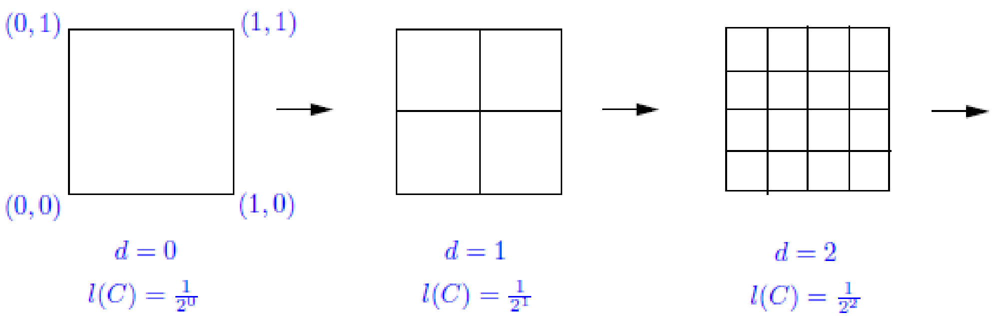

Fig. 1: Dyadic Cubes.

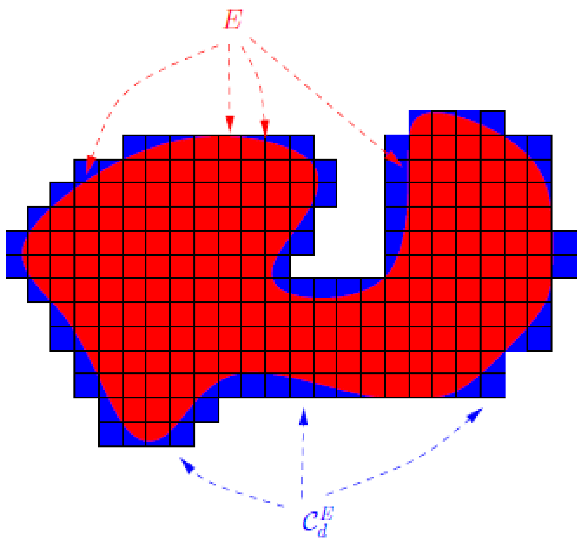

Fig. 2: Cubical cover $\mathscr{C}_d^E$ of a set E.

form a sequence of refinements by dividing each side length in half successively, and thus quadrupling the number of cubes each time.

Definition 1. *We shall say that the n-cube C (with side length denoted as $l(C)$) is dyadic if*

$$C=\prod_{j=1}^{n}[m_j2^{-d},(m_j+1)2^{-d}],\ m_j\in\mathbb{Z},\ d\in\mathbb{N}\cup\{0\}.$$

In this paper, we will assume C to be a dyadic n-cube throughout. We will denote the union of the dyadic n-cubes with edge length $\frac{1}{2^d}$ that intersect a set $E\subset\mathbb{R}^n$ by $\mathscr{C}_d^E$ and define $\partial\mathscr{C}_d^E$ to be the boundary of this union (see Figure 2). Two simple questions we will explore for their illustrative purposes are:

(1) ”If we know $\mathscr{L}^n(\mathscr{C}_d^E)$, what can we say about $\mathscr{L}^n(E)$” and similarly,
(2) ”If we know $\mathscr{H}^{n-1}(\partial\mathscr{C}_d^E)$, what can we say about $\mathscr{H}^{n-1}(\partial E)$”

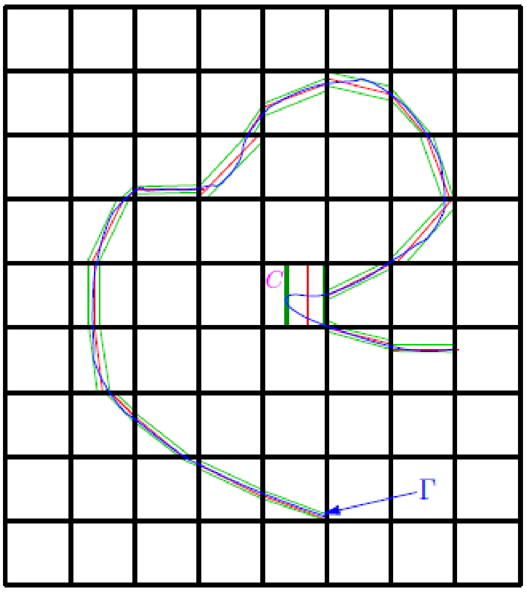

Fig. 3: Jones' β Numbers. The green lines indicate the thinnest cylinder containing Γ in the cube C. We see from this relatively large width that Γ is not very "flat" in this cube.

2.2 Jones' β numbers

Another approach to representing sets in $\mathbb{R}^n$, developed by Jones [23], and generalized by Okikiolu [33], Lerman [25], and Schul [35], involves the question of under what conditions a bounded set E can be contained within a rectifiable curve Γ, which Jones likened to the Traveling Salesman Problem taken over an infinite set. (See Definition 3 below for the definition of rectifiable.)

Jones showed that if the aspect ratios of the optimal containing cylinders in each dyadic cube go to zero fast enough, the set E is contained in a rectifiable curve. Jones' approach ends up providing one useful approach of defining a representation for a set in $\mathbb{R}^n$ similar to those discussed in the next section. We return to this topic in Section 5.1. The basic idea is illustrated in Figure 3.

2.3 Working upstairs: varifolds

A third way of representing sets in $\mathbb{R}^n$ uses *varifolds*. Instead of representing $E \subset \mathbb{R}^n$ by working in $\mathbb{R}^n$, we work in the *Grassmann Bundle*, $\mathbb{R}^n \times G(n,m)$.

We parameterize the Grassmannian $G(2,1)$ by taking the upper unit semicircle in $\mathbb{R}^2$ (including the point $(1,0)$, but not including $(1,\pi)$, where both points are given in polar coordinates) and straightening it out into a vertical axis (as in Figure 4). The bundle $\mathbb{R}^2 \times G(2,1)$ is then represented by $\mathbb{R}^2 \times [0,\pi)$.
Figure 5 illustrates how the tangents are built into this representation of subsets of $\mathbb{R}^n$, giving us a sense of why this representation might be useful. A circular curve in $\mathbb{R}^2$ becomes two half-spirals upstairs (in the Grassmann bundle representation, as shown in the first image of Figure 5). Other curves in $\mathbb{R}^2$ are similarly illuminated by their Grassmann bundle representations. We return to this idea in Section 5.2.

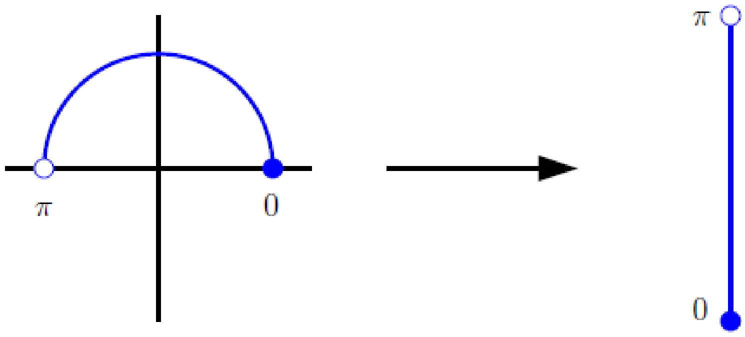

Fig. 4: The vertical axis for the "upstairs."

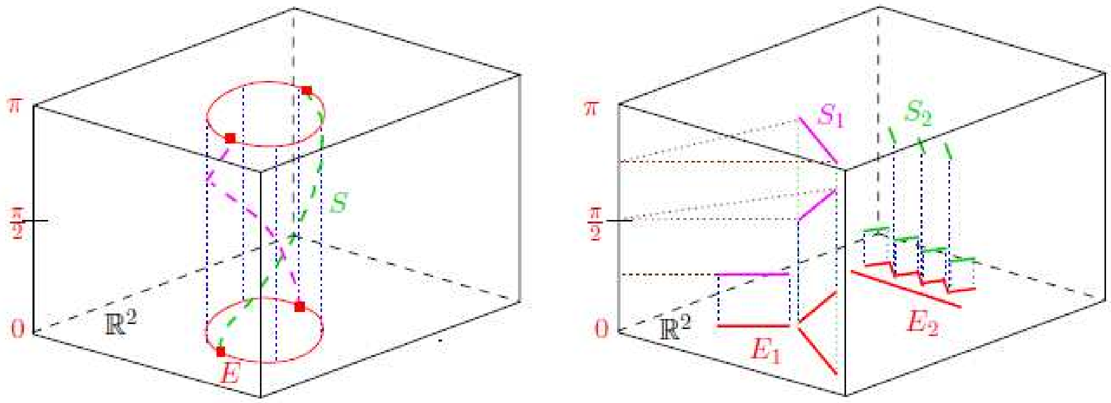

Fig. 5: Working Upstairs in the Grassmann bundle.

3 Simple questions

Let $E \subset \mathbb{R}^n$ and C be any dyadic n-cube as before. Define

$$\mathscr{C}(E,d) = \{C \mid C \cap E \neq \emptyset,\ l(C) = 1/2^d\}$$

and, as above,

$$\mathscr{C}_d^E \equiv \bigcup_{C \in \mathscr{C}(E,d)} C.$$

Here are two questions:

(1) Given $E \subset \mathbb{R}^n$, when is there a d_0 such that for all $d \geq d_0$, we have

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq M(n)\mathscr{L}^n(E) \tag{1}$$

for some constant $M(n)$ independent of E?

(2) Given $E \subset \mathbb{R}^n$, and any $\delta > 0$, when does there exists a d_0 such that for all $d \geq d_0$, we have

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq (1+\delta)\mathscr{L}^n(E)? \tag{2}$$

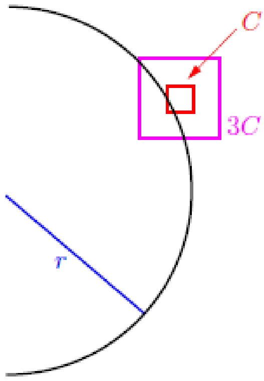

Fig. 6: Concentric Cubes.

Remark. Of course using the fact that Lebesgue measure is a Radon measure, we can very quickly get that for d large enough (i.e. 2^{-d} small enough), the measure of the cubical cover is as close to the measure of the set as you want, as long as the set is compact and has positive measure. But the focus of this paper is on what we can get in a much more transparent, barehanded fashion, so we explore along different paths, getting answers that are, by some metrics, suboptimal.

Example 1. If $E = \mathbb{Q}^n \cap [0,1]^n$, then $\mathscr{L}^n(E) = 0$, but $\mathscr{L}^n(\mathscr{C}_d^E) = 1\ \forall d \geq 0$.

Example 2. Let E be as in Example 1. Enumerate E as $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \hat{\mathbf{q}}_3, \ldots$. Now let $D_i = B(\hat{\mathbf{q}}_i, \frac{\varepsilon}{2^i})$ and $E_\varepsilon \equiv \{\cup D_i\} \cap [0,1]^n$ with ε chosen small enough so that $\mathscr{L}^n(E_\varepsilon) \leq \frac{1}{100}$. Then $\mathscr{L}^n(E_\varepsilon) \leq \frac{1}{100}$, but $\mathscr{L}^n(\mathscr{C}_d^{E_\varepsilon}) = 1\ \forall d > 0$.

3.1 A Union of balls

For a given set $F \subseteq \mathbb{R}^n$, suppose $E = \cup_{x \in F} \bar{B}(x,r)$, a union of closed balls of radius r centered at each point x in F. Then we know that E is **regular** (locally Ahlfors n-regular or *locally* n-regular), and thus there exist $0 < m < M < \infty$ and an $r_0 > 0$ such that for all $x \in E$ and for all $0 < r < r_0$, we have

$$mr^n \leq \mathscr{L}^n(\bar{B}(x,r) \cap E) \leq Mr^n.$$

This is all we need to establish a sufficient condition for Equation (1) above.

Remark. The upper bound constant M is immediate since E is a union of n-balls, so $M = \alpha_n$, the n-volume of the unit n-ball, works. However, this is not the case for k-regular sets in $\mathbb{R}^n$, $k < n$, since we are now asking for a bound on the k-dimensional measure of an n-dimensional set which could easily be infinite.

(1) Suppose $E = \cup_{x \in F} \bar{B}(x,r)$, a union of closed balls of radius r centered at each point x in F.

(2) Let $\mathscr{C} = \mathscr{C}(E,d)$ for some d such that $\frac{1}{2^d} \ll r$, and let $\hat{\mathscr{C}} = \{3C \mid C \in \mathscr{C}\}$, where $3C$ is an n-cube concentric with C with sides parallel to the axes and $l(3C) = 3l(C)$, as shown in Figure 6.

(3) This implies that for $3C \in \hat{\mathscr{C}}$

$$\frac{\mathscr{L}^n(3C \cap E)}{\mathscr{L}^n(3C)} > \theta > 0, \quad \text{with } \theta \in \mathbb{R}. \tag{3}$$

(4) We then make the following observations:
   (a) Note that there are 3^n different tilings of the plane by $3C$ cubes whose vertices live on the $\frac{1}{2^d}$ lattice. (This can be seen by realizing that there are 3^n shifts you can perform on a $3C$ cube and both (1) keep the originally central cube C in the $3C$ cube and (2) keep the vertices of the $3C$ cube in the $\frac{1}{2^d}$ lattice.)

(b) Denote the $3C$ cubes in these tilings $\mathscr{T}_i, i = 1, ..., 3^n$.
(c) Define $\hat{\mathscr{C}_i} \equiv \hat{\mathscr{C}} \cap \mathscr{T}_i$.
(d) Note now that by Step (3), the number of $3C$ cubes in $\hat{\mathscr{C}_i}$ cannot exceed

$$N_i \equiv \frac{\mathscr{L}^n(E)}{\theta \mathscr{L}^n(3C)}.$$

(e) Denote the total number of cubes in $\mathscr{C}$ by $N_{\mathscr{C}_d^E}$.
(f) The number of cubes in $\mathscr{C}$, $N_{\mathscr{C}_d^E}$, cannot exceed

$$\sum_{i=1}^{3^n} N_i = 3^n \frac{\mathscr{L}^n(E)}{\theta \mathscr{L}^n(3C)}.$$

(g) Putting it all together, we get

$$\mathscr{L}^n(\mathscr{C}_d^E) = \mathscr{L}^n(\cup_{C \in \mathscr{C}} C) = N_{\mathscr{C}_d^E} \mathscr{L}^n(C) \leq 3^n \frac{\mathscr{L}^n(E)}{\theta \mathscr{L}^n(3C)} \mathscr{L}^n(C) = \frac{\mathscr{L}^n(E)}{\theta}. \tag{4}$$

(5) This shows that if $E = \cup_{x \in F} \overline{B}(x,r)$, then

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq \frac{1}{\theta} \mathscr{L}^n(E).$$

We now have two conclusions:

(1) (Regularized sets) We notice that for any fixed $r_0 > 0$, as long as we pick d_0 big enough, then $r < r_0$ and $d > d_0$ imply that $E = \cup_{x \in F} \overline{B}(x,r)$ satisfies

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq \frac{1}{\theta(n)} \mathscr{L}^n(E),$$

for a $\theta(n) > 0$ that depends on n, but not on F.

(2) (Regular sets) Now suppose that

$$F \in \mathscr{R}_m \equiv \{W \subset \mathbb{R}^n \mid mr^n < \mathscr{L}^n(W \cap \overline{B}(x,r)), \ \forall\, x \in W \text{ and } r < r_0\}.$$

Then we immediately get the same result: for a big enough d (depending only on r_0),

$$\mathscr{L}^n(\mathscr{C}_d^F) \leq \frac{1}{\theta(m)} \mathscr{L}^n(F),$$

where $\theta(m) > 0$ depends only on the regularity class that F lives in and not on which subset in that class we cover with the cubes.

3.2 Minkowski content

Definition 2. (Minkowski content). *Let $W \subset \mathbb{R}^n$, and let $W_r \equiv \{x \mid d(x,W) < r\}$. The $(n-1)$-dimensional Minkowski Content is defined as $\mathscr{M}^{n-1}(W) \equiv \lim_{r \to 0} \frac{\mathscr{L}^n(W_r)}{2r}$, when the limit exists (see Figure 7).*

Definition 3. ($(\mathscr{H}^m, m)$-rectifiable set). *A set $W \subset \mathbb{R}^n$ is called* **$(\mathscr{H}^m, m)$-rectifiable** *if $\mathscr{H}^m(W) < \infty$ and $\mathscr{H}^m$-almost all of W is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. We will use* **rectifiable** *and* **$(\mathscr{H}^m, m)$-rectifiable** *interchangeably when the dimension of the sets are clear from the context.*

Definition 4. *[m-rectifiable] We will say that $E \subset \mathbb{R}^n$ is m-***rectifiable** *if there is a Lipschitz function mapping a bounded subset of $\mathbb{R}^m$ onto E.*

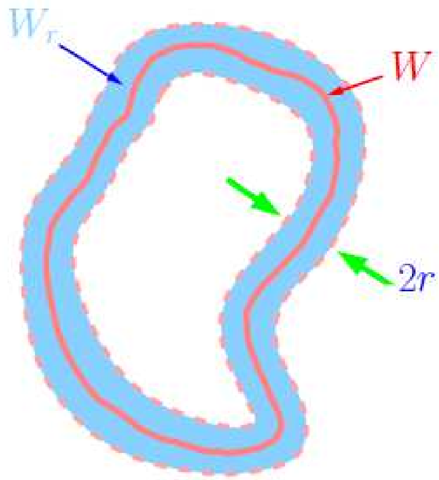

Fig. 7: Minkowski Content.

Theorem 1. $\mathscr{M}^{n-1}(W) = \mathscr{H}^{n-1}(W)$ *when* W *is a closed,* $(n$-1$)$*-rectifiable set.*

See Theorem 3.2.39 in [21] for a proof.

Remark. Notice that m-rectifiable is more restrictive that $(\mathscr{H}^m, m)$-rectifiable. In fact, Theorem 1 is false for $(\mathscr{H}^m, m)$-rectifiable sets. See the notes at the end of section 3.2.39 in [21] for details.

Now, let W be (n-1)-rectifiable, set $r_d \equiv \sqrt{n}\left(\frac{1}{2^d}\right)$, and choose r_δ small enough so that

$$\mathscr{L}^n(W_{r_d}) \leq \mathscr{M}^{n-1}(W)2r_d + \delta,$$

for all $d \in \mathbb{N} \cup \{0\}$ such that $r_d \leq r_\delta$. (Note: Because the diameter of an n-cube with edge length $\frac{1}{2^d}$ is $r_d = \sqrt{n}\left(\frac{1}{2^d}\right)$, no point of $\mathscr{C}_d^W$ can be farther than r_d away from W. Thus $\mathscr{C}_d^W \in W_{r_d}$.)

Assume that $\mathscr{L}^n(E) \neq 0$ and ∂E is (n-1)-rectifiable. Letting $W \equiv \partial E$, we have

$$\mathscr{L}^n(\mathscr{C}_d^E) - \mathscr{L}^n(E) \leq \mathscr{L}^n(W_{r_d}) \leq \mathscr{M}^{n-1}(\partial E)2r_d + \delta \leq \mathscr{M}^{n-1}(\partial E)2r_\delta + \delta$$

so that

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq (1+\hat{\delta})\mathscr{L}^n(E), \text{ where } \hat{\delta} = \frac{\mathscr{M}^{n-1}(\partial E)2r_\delta + \delta}{\mathscr{L}^n(E)}. \quad (5)$$

Since we control r_δ and δ, we can make $\hat{\delta}$ as small as we like, and we have a sufficient condition to establish Equation (2) above.

The result: let $\hat{\delta}$ be as in Equation (5) and $E \subset \mathbb{R}^n$ such that $\mathscr{L}^n(E) \neq 0$. Suppose that ∂E (which is automatically closed) is (n-1)-rectifiable and $\mathscr{H}^{n-1}(\partial E) < \infty$, then, for every $\delta > 0$ there exists a d_0 such that for all $d \geq d_0$,

$$\mathscr{L}^n(\mathscr{C}_d^E) \leq (1+\hat{\delta})\mathscr{L}^n(E).$$

Problem 1. Suppose that $E \subset \mathbb{R}^n$ is bounded. Show that for any $r > 0$, E_r, the set of points that are at most a distance r from E, has a $(\mathscr{H}^{n-1}, n-1)$-rectifiable boundary. Show this by showing that ∂E_r is contained in a finite number of graphs of Lipschitz functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Hint: cut E_r into small chunks F_i with common diameter $D \ll r$ and prove that $(F_i)_r$ is the union of a finite number of Lipschitz graphs.

Problem 2. Can you show that in fact the boundary of E_r, ∂E_r, is actually (n-1)-rectifiable? See if you can use the results of the previous problem to help you.

Remark. We can cover a union E of open balls of radius r, whose centers are bounded, with a cover $\mathscr{C}_d^E$ satisfying Equation (2). In this case, $\partial\mathscr{C}_d^E$ certainly meets the requirements for the result just shown.

3.3 Smooth boundary, positive reach

In this section, we show that if ∂E is *smooth* (at least $C^{1,1}$), then E has positive *reach* allowing us to get an even cleaner bound, depending in a precise way on the curvature of ∂E.

We will assume that E is closed. Define $E_r = \{x \in R^n \,|\, \mathrm{dist}(x,E) \leq r\}$, $\mathrm{cls}(x) \equiv \{y \in E \mid d(x,E) = |x-y|\}$ and $\mathrm{unique}(E) = \{x \mid \mathrm{cls}(x) \text{ is a single point}\}$.

Definition 5. *[Reach] The **reach** of E,* $\mathrm{reach}(E)$*, is defined*

$$\mathrm{reach}(E) \equiv \sup\{r \mid E_r \subset \mathrm{unique}(E)\}$$

Remark. Sets of positive reach were introduced by Federer in 1959 [20] in a paper that also introduced the famous coarea formula.

Remark. If $E \subset \mathbb{R}^n$ is (n-1)-dimensional and E is closed, then $E = \partial E$.

Another equivalent definition involves rolling balls around the boundary of E. The closed ball $\bar{B}(x,r)$ **touches** E if $\partial\bar{B}(x,r) \cap \partial E \neq \emptyset$ and

$$\bar{B}(x,r) \cap E \subset \partial\bar{B}(x,r) \cap \partial E$$

Definition 6. *The **reach** of E,* $\mathrm{reach}(E)$*, is defined*

$$\mathrm{reach}(E) \equiv \sup\{r \mid \textit{every ball of radius } r \textit{ touching } E \textit{ touches at a single point}\}.$$

Put a little more informally, $\mathrm{reach}(E)$ is the supremum of radii r of the balls such that each ball of that radius rolling around E touches E at only one point (see Figure 8). As mentioned above, if ∂E is $C^{1,1}$, then it has positive reach (see Remark 4.20 in [20]). That is, if for all $x \in \partial E$, there is a neighborhood of x, $U_x \subset \mathbb{R}^n$, such that after a suitable change of coordinates, there is a $C^{1,1}$ function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\partial E \cap U_x$ is the graph of f. (Recall that a function is $C^{1,1}$ if its derivative is Lipschitz continuous.) This implies, among other things, that the (symmetric) second fundamental form of ∂E exists $\mathscr{H}^{n-1}$-almost everywhere on ∂E. The fact that ∂E is $C^{1,1}$ implies that at $\mathscr{H}^{n-1}$-almost every point of ∂E, the $n-1$ principal curvatures κ_i of our set exist and $|\kappa_i| \leq \frac{1}{\mathrm{reach}(\partial E)}$ for $1 \leq i \leq n-1$.

We will use this fact to determine a bound for the $(n-1)$-dimensional change in area as the boundary of our set is expanded outwards or contracted inwards by ε (see Figure 9, Diagram 1). Let us first look at this in $\mathbb{R}^2$ by examining the following ratios of lengths of expanded or contracted arcs for sectors of a ball in $\mathbb{R}^2$ as shown in Diagram 2 in Figure 9 below.

$$\frac{\mathscr{H}^1(l_\varepsilon)}{\mathscr{H}^1(l)} = \frac{(r+\varepsilon)\theta}{r\theta} = 1 + \frac{\varepsilon}{r} = 1 + \varepsilon\kappa$$

$$\frac{\mathscr{H}^1(l_{-\varepsilon})}{\mathscr{H}^1(l)} = \frac{(r-\varepsilon)\theta}{r\theta} = 1 - \frac{\varepsilon}{r} = 1 - \varepsilon\kappa,$$

Fig. 8: Positive and Non-positive reach.

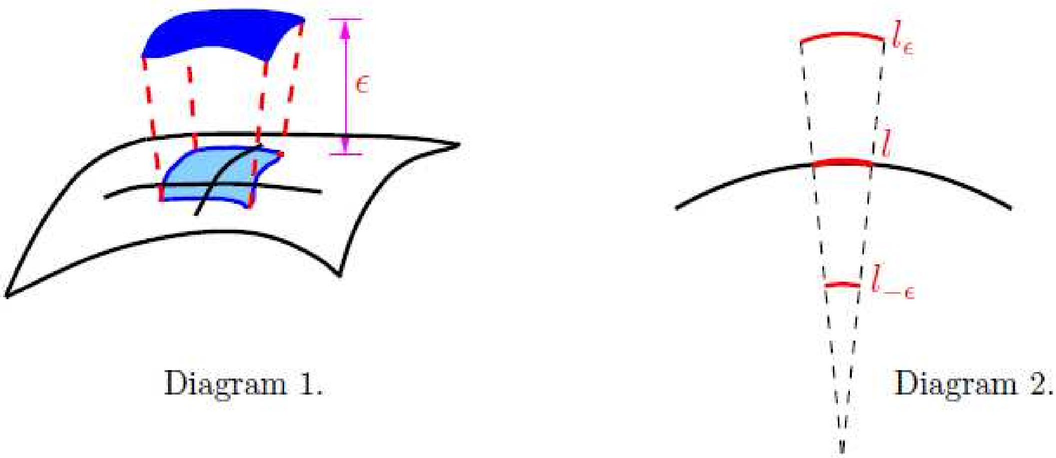

Fig. 9: Moving out and sweeping in.

where κ is the principal curvature of the circle (the boundary of the 2-ball), which we can think of as defining the reach of a set $E \subset \mathbb{R}^2$ with $C^{1,1}$-smooth boundary.

The Jacobian for the normal map pushing in or out by ε, which by the area formula is the factor by which the area changes, is given by $\prod_{i=1}^{n-1}(1 \pm \varepsilon\kappa_i)$ (see Figure 9, Diagram 1). If we define $\hat{\kappa} \equiv \max\{|\kappa_1|, |\kappa_2|, \ldots, |\kappa_{n-1}|\}$, then we have the following ratios:

Max Fractional Increase of $\mathscr{H}^{n-1}$ boundary "area" Moving Out:

$$\prod_{i=1}^{n-1}(1+\varepsilon\kappa_i) \le (1+\varepsilon\hat{\kappa})^{n-1}.$$

Max Fractional Decrease of $\mathscr{H}^{n-1}$ boundary "area" Sweeping In:

$$\prod_{i=1}^{n-1}(1-\varepsilon\kappa_i) \ge (1-\varepsilon\hat{\kappa})^{n-1}.$$

Remark. Notice that $\hat{\kappa} = \frac{1}{\text{reach}(\partial E)}$.

For a ball, we readily find the value of the ratio

$$\frac{\mathscr{L}^n(B(0,r+\varepsilon))}{\mathscr{L}^n(B(0,r))} = \left(\frac{r+\varepsilon}{r}\right)^n = (1+\varepsilon\kappa)^n \quad (\text{setting } \delta = \varepsilon\kappa) = (1+\delta)^n, \tag{6}$$

where $\kappa = \frac{1}{r}$ is the curvature of the ball along any geodesic. Now we calculate the bound we are interested in for E, assuming ∂E is $C^{1,1}$. Define $E_\varepsilon \subset \mathbb{R}^n \equiv \{x \mid d(x,E) < \varepsilon\}$. We first compute a bound for

$$\frac{\mathscr{L}^n(E_\varepsilon)}{\mathscr{L}^n(E)} = \frac{\mathscr{L}^n(E) + \mathscr{L}^n(E_\varepsilon \setminus E)}{\mathscr{L}^n(E)} = 1 + \frac{\mathscr{L}^n(E_\varepsilon \setminus E)}{\mathscr{L}^n(E)}. \tag{7}$$

Since κ_i is a function of $x \in \partial E$ defined $\mathscr{H}^{n-1}$-almost everywhere, we may set up the integral below over ∂E and do the actual computation over $\partial E \setminus K$, where $K \equiv \{\text{the set of measure 0 where } \kappa_i \text{ is not defined}\}$. Computing bounds for the numerator and denominator separately in the second term in (7), we find, by way of the Area Formula [31],

$$\begin{aligned}
\mathscr{L}^n(E_\varepsilon \setminus E) &= \int_0^\varepsilon \int_{\partial E} \prod_{i=1}^{n-1}(1+r\kappa_i)\,d\mathscr{H}^{n-1}dr \\
&\le \int_0^\varepsilon \int_{\partial E} (1+r\hat{\kappa})^{n-1}\,d\mathscr{H}^{n-1}dr \\
&= \mathscr{H}^{n-1}(\partial E)\left.\frac{(1+r\hat{\kappa})^n}{n\hat{\kappa}}\right|_0^\varepsilon \\
&= \mathscr{H}^{n-1}(\partial E)\left(\frac{(1+\varepsilon\hat{\kappa})^n}{n\hat{\kappa}} - \frac{1}{n\hat{\kappa}}\right)
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathscr{L}^n(E) &\ge \int_0^{r_0} \int_{\partial E} \prod_{i=1}^{n-1}(1-r\kappa_i)\,d\mathscr{H}^{n-1}dr \\
&\ge \int_0^{r_0} \int_{\partial E} (1-r\hat{\kappa})^{n-1}\,d\mathscr{H}^{n-1}dr \\
&= \mathscr{H}^{n-1}(\partial E)\left.\frac{-(1-r\hat{\kappa})^n}{n\hat{\kappa}}\right|_0^{r_0} \\
&= \frac{\mathscr{H}^{n-1}(\partial E)}{n\hat{\kappa}}, \quad \text{when } r_0 = \frac{1}{\hat{\kappa}}.
\end{aligned} \tag{9}$$

From 7, 8, and 9, we have

$$\begin{aligned}
\frac{\mathscr{L}^n(E_\varepsilon)}{\mathscr{L}^n(E)} &\le 1 + \frac{\mathscr{H}^{n-1}(\partial E)\left(\frac{(1+\varepsilon\hat{\kappa})^n}{n\hat{\kappa}} - \frac{1}{n\hat{\kappa}}\right)}{\frac{\mathscr{H}^{n-1}(\partial E)}{n\hat{\kappa}}} \\
&= (1+\varepsilon\hat{\kappa})^n \quad (\text{setting } \delta = \varepsilon\hat{\kappa}) \\
&= (1+\delta)^n.
\end{aligned} \tag{10}$$

From this we get that

$$\mathscr{L}^n(E_\varepsilon) \le (1+\varepsilon\hat{\kappa})^n \mathscr{L}^n(E)$$

so that

$$\mathscr{L}^n(\mathscr{C}^E_{d(\varepsilon)}) \le (1+\varepsilon\hat{\kappa})^n \mathscr{L}^n(E)$$

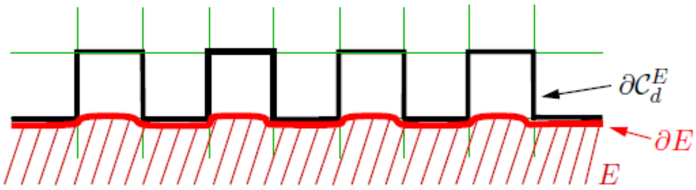

Fig. 10: Cubes on the boundary.

where $d(\varepsilon) = \log_2(\frac{\sqrt{n}}{\varepsilon})$ is found by solving $\sqrt{n}\frac{1}{2^d} = \varepsilon$. Thus, when ∂E is smooth enough to have positive reach, we find a nice bound of the type in Equation (2), with a precisely known dependence on curvature.

4 A boundary conjecture

What can we say about boundaries? Can we bound

$$\frac{\mathscr{H}^{n-1}(\partial \mathscr{C}_d^E)}{\mathscr{H}^{n-1}(\partial E)}?$$

Conjecture 1. If $E \subset \mathbb{R}^n$ is compact and ∂E is $C^{1,1}$,

$$\limsup_{d\to\infty} \frac{\mathscr{H}^{n-1}(\partial \mathscr{C}_d^E)}{\mathscr{H}^{n-1}(\partial E)} \le n.$$

Proof. [Brief Sketch of Proof for $n = 2$]

(1) Since ∂E is $C^{1,1}$, we can zoom in far enough at any point $x \in \partial E$ so that it looks flat.

(2) Let C be a cube in the cover $\mathscr{C}(E,d)$ that intersects the boundary near x and has faces in the boundary $\partial \mathscr{C}_d^E$. Define $F = \partial C \cap \partial \mathscr{C}_d^E$.

(3) (Case 1) Assume that the tangent at x, $T_x\partial E$, is not parallel to either edge direction of the cubical cover (see Figure 11).

  (1) Let Π be the projection onto the horizontal axis and notice that $\frac{\mathscr{H}^1(F)}{\Pi(F)} \le 2+\varepsilon$ for any epsilon.

  (2) This is stable to perturbations which is important since the actual piece of the boundary ∂E we are dealing with is not a straight line.

(4) (Case 2) Suppose that the tangent at x, $T_x\partial E$, is parallel to one of the two faces of the cubical cover, and let U_x be a neighborhood of $x \in \partial E$.

  (1) Zooming in far enough, we see that the cubical boundary can only oscillate up and down so that the maximum ratio for any horizontal tangent is (locally) 2.

  (2) But we can create a sequence of examples that attain ratios as close to 2 as we like by finding a careful sequence of perturbations that attains a ratio locally of $2-\varepsilon$ for any ε (see Figure 10).

  (3) That is, we can create perturbations that, on an unbounded set of d's, $\{d_i\}_{i=1}^\infty$, yield a ratio $\frac{\mathscr{H}^1(\mathscr{C}_{d_i}^E \cap U_x)}{\partial E} > 2-\varepsilon$, and we can send $\varepsilon \to 0$.

(5) Use the compactness of ∂E to put this all together into a complete proof.

Problem 3. Suppose we exclude C's that contain less than some fraction θ of E (as defined in Conjecture 1) from the cover to get the reduced cover $\hat{\mathscr{C}}_d^E$. In this case, what is the optimal bound $B(\theta)$ for the ratio of boundary measures

$$\limsup_{d\to\infty} \frac{\mathscr{H}^{n-1}(\partial \hat{\mathscr{C}}_d^E)}{\mathscr{H}^{n-1}(\partial E)} \le B(\theta)?$$

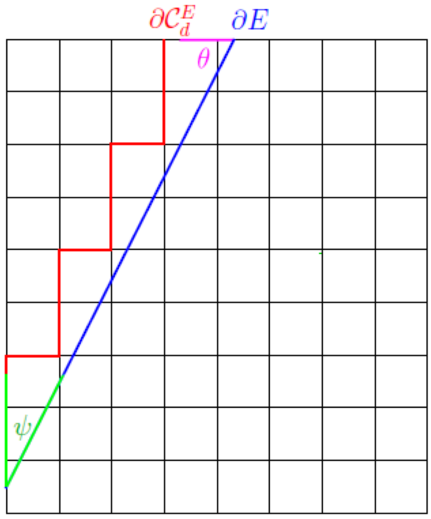

Fig. 11: The case in which θ, the angle between $T_x\partial E$ and the x-axis, is neither 0 nor $\pi/2$.

5 Other representations

5.1 The Jones' β approach

As mentioned above, another approach to representing sets in $\mathbb{R}^n$, developed by Jones [23], and generalized by Okikiolu [33], Lerman [25], and Schul [35], involves the question of under what conditions a bounded set E can be contained within a rectifiable curve Γ, which Jones likened to the Traveling Salesman Problem taken over an infinite set. While Jones worked in $\mathbb{C}$ in his original paper, the work of Okikiolu, Lerman, and Schul extended the results to $\mathbb{R}^n$ $\forall n \in \mathbb{N}$ as well as infinite dimensional space.

Recall that a compact, connected set $\Gamma \subset \mathbb{R}^2$ is rectifiable if it is contained in the image of a countable set of Lipschitz maps from $\mathbb{R}$ into $\mathbb{R}^2$, except perhaps for a set of $\mathscr{H}^1$ measure zero. We have the result that if Γ is compact and connected, then $l(\Gamma) = \mathscr{H}^1(\Gamma) < \infty$ implies it is rectifiable (see pages 34 and 35 of [19]).

Let W_C denote the width of the thinnest cylinder containing the set E in the dyadic n-cube C (see Figure 12), and define the β number of E in C to be

$$\beta_E(C) \equiv \frac{W_C}{l(C)}.$$

Jones' main result is this theorem:

Theorem 2. *[23] Let E be a bounded set and Γ be a connected set both in $\mathbb{R}^2$. Define $\beta_\Gamma(C) \equiv \frac{W_C}{l(C)}$, where W_C is the width of the thinnest cylinder in the 2-cube C containing Γ. Then, summing over all possible C,*

$$\beta^2(\Gamma) \equiv \sum_C (\beta_\Gamma(3C))^2 l(C) < \eta\, l(\Gamma) < \infty, \text{ where } \eta \in \mathbb{R}.$$

Conversely, if $\beta^2(E) < \infty$ there is a connected set Γ, with $E \subset \Gamma$, such that

$$l(\Gamma) \leq (1+\delta)\operatorname{diam}(E) + \alpha_\delta \beta^2(E),$$

where $\delta > 0$ and $\alpha_\delta = \alpha(\delta) \in \mathbb{R}$.

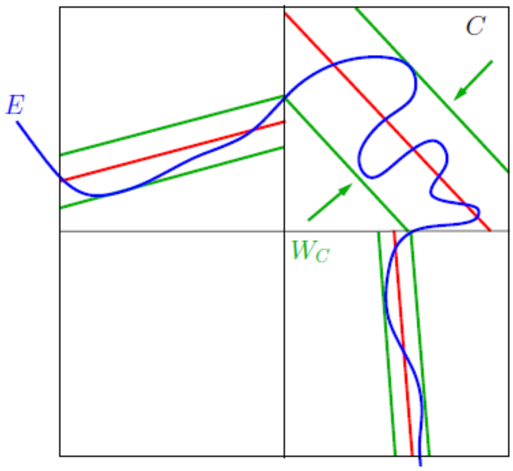

Fig. 12: Jones' β Numbers and W_C. Each of the two green lines in a cube C is an equal distance away from the red line and is chosen so that the green lines define the thinnest cylinder containing $E\cap C$. Then the red lines are varied over all possible lines in C to find that red line whose corresponding cylinder is the thinnest of all containing cylinders. In this sense, the minimizing red lines are the best fit to E in each C.

Jones' main result, generalized to $\mathbb{R}^n$, is that a bounded set $E\subset\mathbb{R}^n$ is contained in a rectifiable curve Γ if and only if

$$\beta^2(E)\equiv\sum_C(\beta_E(3C))^2 l(C)<\infty,$$

where the sum is taken over all dyadic cubes.

Note that each β number of E is calculated over the dyadic cube $3C$, as defined in Section 3.1. Intuitively, we see that in order for E to lie within a rectifiable curve Γ, E must look flat as we zoom in on points of E since Γ has tangents at $\mathscr{H}^1$-almost every point $x\in\Gamma$. Since both W_C and $l(C)$ are in units of length, $\beta_E(C)$ is a scale-invariant measure of the flatness of E in C. In higher dimensions, the analogous cylinders' widths and cube edge lengths are also divided to get a scale-invariant $\beta_E(C)$.

The notion of local linear approximation has been explored by many researchers. See for example the work of Lerman and collaborators [15,5,40,6]. While distances other than the sup norm have been considered when determining closeness to the approximating line, see [25], there is room for more exploration there. In the section below, *Problems and Questions*, we suggest an idea involving the multiscale flat norm from geometric measure theory.

5.2 A varifold approach

As mentioned above, a third way of representing sets in $\mathbb{R}^n$ uses *varifolds*. Instead of representing $E\subset\mathbb{R}^n$ by working in $\mathbb{R}^n$, we work in the *Grassmann Bundle*, $\mathbb{R}^n\times G(n,m)$. Advantages include, for example, the automatic encoding of tangent information directly into the representation. By building into the representation this tangent information, we make set comparisons where we care about tangent structure easy and natural.

Definition 7. *[Grassmannian] The m-dimensional Grassmannian in* $\mathbb{R}^n$*,*

$$G(n,m)=G(\mathbb{R}^n,m),$$

is the set of all m-dimensional planes through the origin.

For example, $G(2,1)$ is the space of all lines through the origin in $\mathbb{R}^2$, and $G(3,2)$ is the space of all planes through the origin in $\mathbb{R}^3$. The Grassmann bundle $\mathbb{R}^n \times G(n,m)$ can be thought of as a space where $G(n,m)$ is attached to each point in $\mathbb{R}^n$.

Definition 8. *[Varifold] A varifold is a* Radon measure μ *on the Grassmann bundle* $\mathbb{R}^n \times G(n,m)$.

Suppose $\pi : (x,g) \in \mathbb{R}^n \times G(n,m) \to x$. One of the most common appearances of varifolds are those that arise from rectifiable sets E. In this case the measure μ_E on $\mathbb{R}^n \times G(n,m)$ is the pushforward of m-Hausdorff measure on E by the tangent map $T : x \to (x, T_x E)$.

Let $E \subset \mathbb{R}^n$ be an $(\mathscr{H}^m, m)$-rectifiable set (see Definition 3). We know the approximate m-dimensional tangent space $T_x E$ exists $\mathscr{H}^m$-almost everywhere since E is $(\mathscr{H}^m, m)$-rectifiable, which in turn implies that, except for an $\mathscr{H}^m$-measure 0 set, E is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.

The measure of $A \subset \mathbb{R}^n \times G(n,m)$ is given by $\mu(A) = \mathscr{H}^m(T^{-1}\{A\})$. Let $S \equiv \{(x, T_x E) \,|\, x \in E\}$, the section of the Grassmann bundle defining the varifold. S, intersected with each fiber $\{x\} \times G(n,m)$, is the single point $(x, T_x E)$, and so we could just as well use the projection π in which case we would have $\mu_E(A) = \mathscr{H}^m(\pi(A \cap S))$.

Definition 9. *A* rectifiable varifold *is a radon measure* μ_E *defined on an* $(\mathscr{H}^m, m)$*-rectifiable set* $E \subset \mathbb{R}^n$. *Recalling* $S \equiv \{(x, T_x E) \,|\, x \in E\}$, *let* $A \subset \mathbb{R}^n \times G(n,m)$ *and define*

$$\mu_E(A) = \mathscr{H}^m(\pi(A \cap S)).$$

We will call $E = \pi(S)$ the "downstairs" representation of S for any $S \subset \mathbb{R}^n \times G(n,m)$, and we will call $S = T(E) \subset \mathbb{R}^n \times G(n,m)$ the "upstairs" representation of any rectifiable set E, where T is the tangent map over the rectifiable set E.

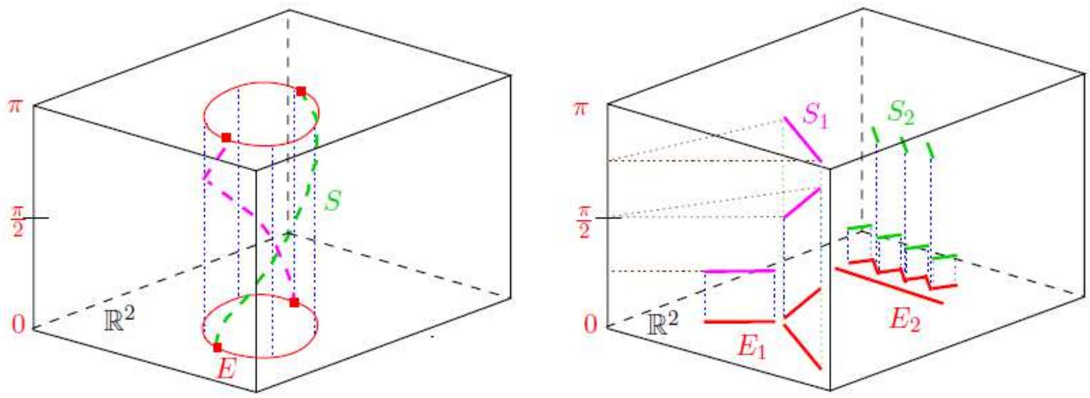

Fig. 13: Working upstairs.

Figure 13, repeated from above, illustrates how the tangents are built into this representation of subsets of $\mathbb{R}^n$, giving us a sense of why this representation might be useful. Suppose we have three line segments almost touching each other, i.e. appearing to touch as subsets of $\mathbb{R}^2$. The upstairs view puts each segment at a different height corresponding to the angle of the segment. So, these segments are not close in any sense in $\mathbb{R}^2 \times G(n,m)$. Or consider a straight line segment and a very fine sawtooth curve that may look practically indistinguishable, but will appear drastically different upstairs.

We can use varifold representations in combination with a cubical cover to get a quantized version of a curve that has tangent information as well as position information. If, for example, we cover a set $S \subset \mathbb{R}^2 \times G(2,1)$ with cubes of edge length $\frac{1}{2^d}$ and use this cover as a representation for S, we know the position and angle to within $\frac{\sqrt{3}}{2^{d+1}}$. In other words, we can approximate our curve $S \subset \mathbb{R}^2 \times G(2,1)$ by the union of the centers of the cubes (with edge length $\frac{1}{2^d}$) intersecting S. This simple idea seems to merit further exploration.

6 Problems and questions

Problem 4. Find a smooth ∂E, with $E \subset \mathbb{R}^n$, such that

$$\mathscr{H}^{n-1}(\partial \mathscr{C}_d^E)/\mathscr{H}^{n-1}(\partial E) = 0 \; \forall d.$$

Hint: Look at unbounded $E \subset \mathbb{R}^2$ such that $\mathscr{L}^2(E^c) < \infty$.

Problem 5. Suppose that E is open and $\mathscr{H}^{n-1}(\partial E) < \infty$. Show that if the **reach** of ∂E is positive, then

$$\liminf_{d\to\infty} \frac{\mathscr{H}^{n-1}(\partial \mathscr{C}_d^E)}{\mathscr{H}^{n-1}(\partial E)} \geq 1.$$

Hint: First show that ∂E has unique inward and outward pointing normals. (Takes a bit of work!) Next, examine the map $F : \partial E \to \mathbb{R}^n$, where $F(x) = x + \eta(x)N(x)$, $N(x)$ is the normal to ∂E at x, and $\eta(x)$ is a positive real-valued function chosen so that *locally* $F(\partial E) = \partial \mathscr{C}_d^E$. Use the Binet-Cauchy Formula to find the Jacobian, and then apply the Area Formula. To do this calculation, notice that at any point $x_0 \in \partial E$ we can choose coordinates so that $T_{x_0}\partial E$ is horizontal (i.e. $N(x_0) = e_n$). Calculate using $F : T_{x_0}\partial E = \mathbb{R}^{n-1} \to \mathbb{R}^n$ where $F(x) = x + \eta(x)N(x)$. (See Chapter 3 of [18] for the Binet-Cauchy formula and the Area Formula.)

Problem 6. Suppose E has dimension $n-1$, positive reach, and is locally regular (in $\mathbb{R}^n$).
(a) Find bounds for $\mathscr{H}^n(\mathscr{C}_d^E)/\frac{1}{2^d}$.
(b) How does this ratio relate to $\mathscr{H}^{n-1}(E)$?

Hint: Use the ideas in Section 3.3 to calculate a bound on the volume of the tube with thickness $2\frac{\sqrt{n}}{2^d}$ centered on E.

*Question 1.*Can we use the "upstairs" version of cubical covers to find better representations for sets and their boundaries? (Of course, "better" depends on your objective!)

For the following question, we need the notion of the *multiscale flat norm* [32]. The basic idea of this distance, which works in spaces of oriented curves and surfaces of any dimension (known as currents), is that we can decompose the curve or surface T into $(T - \partial S) + \partial S$, *but* we measure the cost of the decomposition by adding the volumes of $T - \partial S$ and S (not ∂S!). By volume, we mean the m-dimensional volume, or m-volume of an m-dimensional object, so if T is m-dimensional, we would add the m-volume of $T - \partial S$ and the (m+1)-volume of S (scaled by the parameter λ). We get that

$$\mathbb{F}_\lambda(T) = \min_S M_m(T - \partial S) + \lambda M_{m+1}(S).$$

It turns out that $T - \partial S$ is the best approximation to T that has curvature bounded by λ [2]. We exploit this in the following ideas and questions.

Remark. Currents can be thought of as generalized oriented curves or surfaces of any dimension k. More precisely, they are members of the dual space to the space of k-forms. For the purposes of this section, thinking of them as (perhaps unions of pieces of) oriented k-dimensional surfaces W, so that W and $-W$ are simply oppositely oriented and cancel if we add them, will be enough to understand what is going on. For a nice introduction to the ideas, see for example the first few chapters of [31].

Question 2. Choose $k \in \{1,2,3\}$. In what follows we focus on sets Γ which are one-dimensional, the interior of a cube C will be denoted C^o, and we will work at some scale d, i.e. the edge length of the cube will be $\frac{1}{2^d}$.

Consider the piece of Γ in C^o, $\Gamma \cap C^o$. Inside the cube C with edge length $\frac{1}{2^d}$, we will use the flat norm to

(1) find an approximation of $\Gamma \cap C^o$ with curvature bounded by $\lambda = 2^{d+k}$ and
(2) find the distance of that approximation from $\Gamma \cap C^o$.

This decomposition is then obtained by minimizing

$$M_1((\Gamma \cap C^o) - \partial S) + 2^{d+k} M_2(S) = \mathscr{H}^1((\Gamma \cap C^o) - \partial S) + 2^{d+k} \mathscr{L}^2(S).$$

The minimal S will be denoted S_d (see Figure 14).

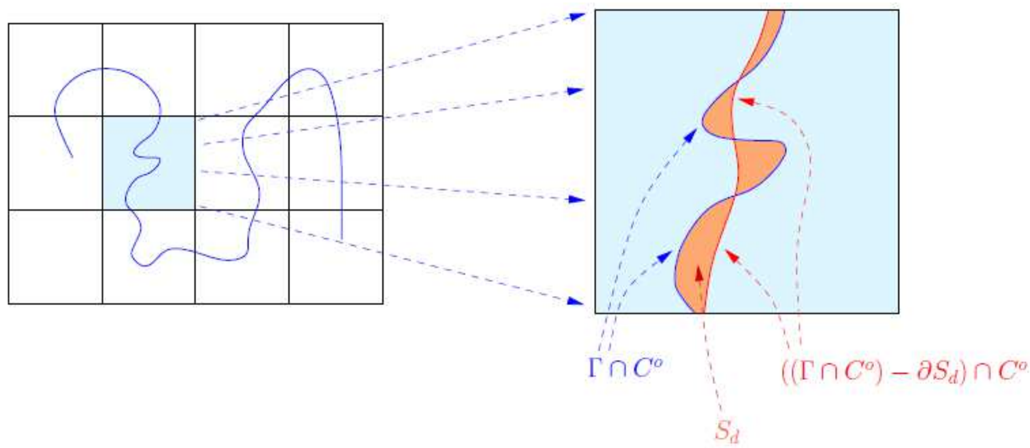

Fig. 14: Multiscale flat norm decomposition inspiring the definition of $\beta_\Gamma^{\mathbb{F}}$.

Suppose that we define $\beta_\Gamma^{\mathbb{F}}(C)$ by

$$\beta_\Gamma^{\mathbb{F}}(C) l(C) = 2^{d+k} \mathscr{L}^2(S_d)$$

so that

$$\beta_\Gamma^{\mathbb{F}}(C) = 2^{2d+k} \mathscr{L}^2(S_d).$$

What can we say about the properties (e.g. rectifiability) of Γ given the finiteness of $\sum_C \left(\beta_\Gamma^{\mathbb{F}}(3C)\right)^2 l(C)$?

Question 3. Can we get an advantage by using the flat norm decomposition as a preconditioner before we find cubical cover approximations? For example, define

$$\mathscr{F}_d^\Gamma \equiv \mathscr{C}_d^{\Gamma_d} \text{ and } \Gamma_d \equiv \Gamma - \partial S_d,$$

$$\text{where } S_d = \underset{S}{\operatorname{argmin}} \left(\mathscr{H}^1(\Gamma - \partial S) + 2^{d+k} \mathscr{L}^2(S) \right).$$

Since the flat norm minimizers have bounded mean curvature, is this enough to force the cubical covers to give us better quantitative information on Γ? How about in the case in which $\Gamma = \partial E$, $E \subset \mathbb{R}^2$?

7 Further exploration

There are a number of places to begin in exploring these ideas further. Some of these works require significant dedication to master, and it is always recommended that you have someone who has mastered a path into pieces of these areas that you can ask questions of when you first wade in. Nonetheless, if you remember that the language can always be translated into pictures, and you make the effort to do that, headway towards mastery can always be made. Here is an annotated list with comments:

(1) (Primary Varifold References) Almgren's little book [4] and Allard's founding contribution [1] are the primary sources for varifolds. Leon Simon's book on geometric measure theory [36] (available for free online) has a couple of excellent chapters, one of which is an exposition of Allard's paper.

(2) (Recent Varifold Work) Both Buet and collaborators [8,9,7,10,11] and Charon and collaborators [12,13,14] have been digging into varifolds with an eye to applications. While these papers are a good start, there is still a great deal of opportunity for the use and further development of varifolds. On the theoretical front, there is the work of Menne and collaborators [29,30]. We want to call special attention to the recent introduction to the idea of a varifold that appeared in the December 2017 AMS Notices [30].

(3) (Geometric Measure Theory I) The area underlying the ideas here are those from geometric measure theory. The fundamental treatise in the subject is still Federer's 1969 *Geometric Measure Theory* [21] even though most people start by reading Morgan's beautiful introduction to the subject, *Geometric Measure Theory: A Beginner's Guide* [31] and Evans' *Measure Theory and Fine Properties of Functions* [18]. Also recommended are Leon Simon's lecture notes [36], Francesco Maggi's book that updates the classic Italian approach [27], and Krantz and Parks' *Geometric Integration Theory* [24].

(4) (Geometric Measure Theory II) The book by Mattila [28] approaches the subject from the harmonic-analysis-flavored thread of geometric measure theory. Some use this as a first course in geometric measure theory, albeit one that does not touch on minimal surfaces, which is the focus of the other texts above. De Lellis' exposition *Rectifiable Sets, Densities, and Tangent Measures* [17] or Priess' 1987 paper *Geometry of Measures in* $\mathbb{R}^n$*: Distribution, Rectifiability, and Densities* [34] is also very highly recommended.

(5) (Jones' β) In addition to the papers cited in the text [23,33,25,35], there are related works by David and Semmes that we recommended. See for example [16]. There is also the applied work by Gilad Lerman and his collaborators that is often inspired by Jones' β and his own development of Jones' ideas in [25]. See also [15,40,39,5]. See also the work by Maggioni and collaborators [26,3].

(6) (Multiscale Flat Norm) The flat norm was introduced by Whitney in 1957 [38] and used to create a topology on currents that permitted Federer and Fleming, in their landmark paper in 1960 [22], to obtain the existence of minimizers. In 2007, Morgan and Vixie realized that a variational functional introduced in image analysis was actually computing a multiscale generalization of the flat norm [32]. The ideas are beginning to be explored in these papers [37].

A Measures: a brief reminder

In this section we remind the reader of a handful of concepts used in the text.

(1) (Measure) One way to think of a measure is as a generalization of the familiar notions of length, area, and volume in a way that allows us to define how we assign "size" to a given subset of a set X, the most common being that of n-dimensional Lebesgue measure $\mathscr{L}^n$. Formally, let X be a nonempty set and 2^X be the collection of all subsets of X. A *measure* is defined [18] to be a mapping $\mu : 2^X \to [0,\infty]$ such that
   (a) $\mu(\emptyset) = 0$ and
   (b) if

$$A \subset \cup_{i=1}^{\infty} A_i,$$

then

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Note that in most texts, this definition is known as an *outer measure*, but we use this definition with the advantage that we can still "measure" non-measurable sets.

(2) (μ-measurable) A subset $S \subset X$ is called μ-measurable if and only if it satisfies the Carathéodory condition for each set $A \subset X$:

$$\mu(A) = \mu(A \cap S) + \mu(A \setminus S).$$

(3) (Radon Measure) Let us define the Borel sets in $\mathbb{R}^n$ to be those sets that are derived from the set of all open sets in $\mathbb{R}^n$ through the operations of countable union, countable intersection, and set difference. Then a measure μ on $\mathbb{R}^n$ is a *Radon* measure if [18]

(a) every Borel set is μ-measurable; i.e. μ is a **Borel measure**.

(b) for each $A \subset \mathbb{R}^n$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$; i.e. μ is **Borel regular.**

(c) for each compact set $K \in \mathbb{R}^n$, $\mu(K) < \infty$; i.e. μ is **locally finite.**

(4) (Hausdorff Measure) With this outer (radon) measure, we can measure k-dimensional subsets of $\mathbb{R}^n$ $(k \leq n)$. While it is true that $\mathscr{L}^n = \mathscr{H}^n$ for $n \in \mathbb{N}$ (see section 2.2 of [18]), Hausdorff measure $\mathscr{H}^k$ is also defined for $k \in [0, \infty)$ so that even sets as wild as fractals are *measurable* in a meaningful way (see Figure 15). To compute the k-dimensional Hausdorff measure of $A \subset \mathbb{R}^n$:

(a) Cover A with a collection of sets $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$, where $\text{diam}(E_i) \leq d \ \forall i$.

(b) Compute the k-dimensional measure of that cover:

$$\mathscr{V}_{\mathscr{E}}^k(A) = \sum_i \alpha(k) \left(\frac{\text{diam}(E_i)}{2}\right)^k,$$

where $\alpha(k)$ is the k-volume of the unit k-ball.

(c) Define $\mathscr{H}_d^k(A) = \inf_{\mathscr{E}} \mathscr{V}_{\mathscr{E}}^k(A)$, where the infimum is taken over all covers whose elements have maximal diameter d.

(d) Finally, we define $\mathscr{H}^k(A) = \lim_{d \downarrow 0} \mathscr{H}_d^k(A)$.

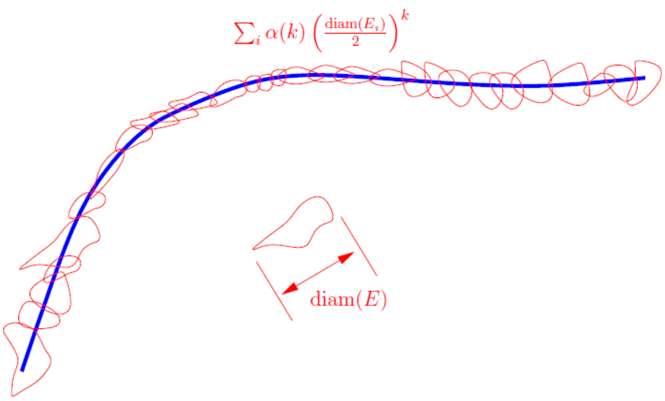

Fig. 15: The Hausdorff Measure is derived from a cover of arbitrary sets.

(5) (Approximate Tangent Plane) We present here an approximate tangent k-plane based on integration. (The one-dimensional version is of course an approximate tangent line.) We start with the fact that we can integrate functions defined on $\mathbb{R}^n$ over k-dimensional sets using k-dimensional measures μ (typically $\mathscr{H}^k$). We zoom in on the point p through dilation of the set F:

$$F_\rho(p) = \{x \in \mathbb{R}^n \mid x = \frac{y-p}{\rho} + \text{ p for some } y \in F\}.$$

We will say that the set F has an approximate tangent k-plane L at p if the dilation of $F_\rho(p)$ converges weakly to L; i.e. if

$$\int_{F_\rho} \phi d\mu \rightarrow \int_L \phi d\mu \quad \text{as } \rho \rightarrow 0$$

for all continuously differentiable, compactly supported $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$. In the next two figures, we note that the solid green lines are the level sets of ϕ while the dashed green line indicates the boundary of the support of ϕ. Note also that the ρ's of 0.4, 0.1, and 0.02 are approximate.

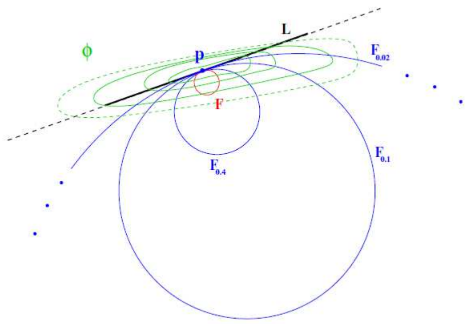

Fig. 16: The case of 1-planes (lines) where L is the weak limit of the dilations of F.

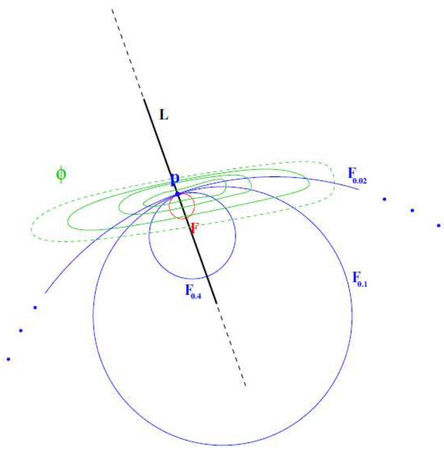

Fig. 17: The case of 1-planes (lines) where L is *not* the weak limit of the dilations of F.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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