

Fixed point theorems for some multivalued mappings

Cafer Aydin¹ and Seher Sultan Sepet²

¹Department of Mathematics, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, 46040, Turkey

²Institute of Science and Technology, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, 46040, Turkey

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Abstract: In this article, we establish some common fixed point theorems for multi-valued mappings satisfying g -weak compatible F -contraction conditions in complete partial metric spaces. These theorems generalize many previously obtained fixed point results. Also an example is given to illustrate the main theorems.

Keywords: Fixed point, g -weak compatible, multi-valued mappings.

1 Introduction

In 1922 Banach [5], introduced the famous fundamental fixed point theorem, also known as the Banach contraction principle. The Banach contraction principle is the simplest and one of the most adaptable elementary results in fixed point theory. Throughout the years, several extensions and generalizations of this principle have appeared in the literature. Matthews [15], introduced the partial metric spaces and presented a fixed point theorem on partial metric space. Following this various fixed point results were proved in these spaces, for more details, [1, 4, 7, 8, 9, 10, 12, 13, 14].

In the other part Wardowski [20], indicated the concept of a F -contraction mapping and investigated the existence of fixed points for such mappings. Consistent with Wardowski, we represented the \mathcal{F} set of all functions, $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following cases:

- (F1) F is strictly increasing. That is, $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}_+$
- (F2) For every sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ in \mathbb{R}_+ we have $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$
- (F3) There exists a number $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

The conclusions of Wardowski became the new attention of several authors (see [11, 16]).

Recently, Nadler [17], introduced the notion of multi-valued contraction mapping and proved well known Banach contraction principle. Many authors followed Nadler's idea and gave their contributions in that sense, see for example [3, 4]. Aydi et al., [4] proved the Banach type fixed point results for set valued mapping in partial metric spaces.

Now, consistent with Aydi et al. [4], we state the following: Let (X, p) be partial metric space. We shall denote $CB^p(X)$ the family of all nonempty closed bounded subsets of X .

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We recall that partial Hausdorff metric induced by a partial metric on a set X is given by

$$H_p(A, B) = \max \left\{ \sup_{x \in A} p(x, B), \sup_{y \in B} p(y, A) \right\}$$

for every $A, B \in CB^p(X)$, where $p(x, B) = \inf\{p(x, y) : y \in B\}$, $\delta_p(A, B) = \sup\{p(x, B) : x \in A\}$, $CB^p(X)$ is the collection of the closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X . Our results is related to mappings $T : X \rightarrow K^p(X)$. Then, we will use the following lemma.

Lemma 1. [2]. *Let (X, p) partial metric spaces and A compact subsets of X . Then, for $x \in X$, there exists $a \in A$ such that*

$$p(x, a) = p(x, A).$$

Lemma 2. [4]. *Let A and B be nonempty closed and bounded subsets of a partial metric space and $h > 1$. Then, for all $a \in A$, there exists $b \in B$ such that*

$$p(a, b) \leq hH_p(A, B).$$

Lemma 3. *Let A and B be nonempty closed and bounded subsets of a partial metric space. Then, For any $a \in A$,*

$$p(a, B) \leq H_p(A, B).$$

Below, we briefly mention a number of fundamental definitions.

Definition 1. [15]. *A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ (nonnegative reals) such that, for all $x, y, z \in X$:*

- (p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$ (T_0 -separation axiom),
- (p2) $p(x, x) \leq p(x, y)$ (small self-distance axiom),
- (p3) $p(x, y) = p(y, x)$ (symmetry),
- (p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (modified triangular inequality).

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X . It is clear that if $p(x, y) = 0$, then, from (p1) and (p2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair (\mathbb{R}^+, p) , where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all closed intervals $[a, b]$ for any real numbers $a \leq b$. Let $p : I \times I \rightarrow \mathbb{R}^+$ be the function such that $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (I, p) is a partial metric space. Other examples of partial metric space which are interesting from a computational point of view may be found in [6, 15].

Each partial metric p on X generates a T_0 topology τ_p on X which has a base as the family open p -balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

Lemma 4. [15]. *Let (X, p) be a complete partial metric space. Then,*

- (A) *If $p(x, y) = 0$, then $x = y$.*
- (B) *If $x \neq y$, then $p(x, y) > 0$.*

Example 1. [13, 19]. Let (X, d) and (X, p) be a metric space and a partial metric space, respectively. Functions $p : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{1, 2, 3\}$) given by

- (•) $\rho_1(x, y) = d(x, y) + p(x, y),$
- (•) $\rho_2(x, y) = d(x, y) + \max\{u(x), u(y)\},$
- (•) $\rho_3(x, y) = d(x, y) + a,$

define partial metrics on X , where $u : X \rightarrow \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

Definition 2. [15]. Let (X, p) be a partial metric space. Then,

- (i) a sequence $\{x_n\}$ in a partial metric space (X, p) converges with respect to τ_p to a point $x \in X$ if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n),$
- (ii) a sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ (and is finite),
- (iii) a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$; that is, $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$

We present an extension of the notion of f -weak compatibility of Pathak [18] on metric space in partial metric space.

Definition 3. Let (X, p) be partial metric space. The mapping $g : X \rightarrow X$ and $T : X \rightarrow CB^p(X)$ are g -weak compatible iff $gTx \in CB^p(X)$ for all $x \in X$ and the following limits exists and satisfy

- (i) $\lim_{n \rightarrow \infty} H_p(gTx_n, Tgx_n) \leq \lim_{n \rightarrow \infty} H_p(Tgx_n, Tx_n),$
- (ii) $\lim_{n \rightarrow \infty} p(gTx_n, gx_n) \leq \lim_{n \rightarrow \infty} H_p(Tgx_n, Tx_n),$

whenever $\{x_n\}$ is a sequence X such that $Tx_n \rightarrow M \in CB^p(X)$ and $gx_n \rightarrow z \in M$.

2 Main results

In this section, we give some common fixed point theorems using g -weak compatible F contractive of multi-valued mappings in partial metric space.

Definition 4. Let (X, p) be a complete partial metric space, $g : X \rightarrow X$ and $T : X \rightarrow K^p(X)$ be g -weak compatible 1. type F - contraction mappings, if there exists $\tau > 0$ such that;

$$\frac{1}{2}p(gx, Tx) < p(gx, gy) \tag{1}$$

then

$$\tau + F(H_p(Tx, Ty)) \leq F(p(gx, gy)) \tag{2}$$

for all $x, y \in X$.

Theorem 1. Let (X, p) be a complete partial metric space and g, T g -weak compatible 1. type F - contraction continuous mappings. Then there exists a point $z \in X$ such that $gz \in Tz$.

Proof. Let x_0 be an arbitrary point of X and choose $x_1 \in X$ such that $gx_1 \in Tx_0$. This is possible since $Tx_0 \subseteq g(X)$. In general, having chosen $x_n \in X$ we choose $x_{n+1} \in X$, so that $y_n = gx_{n+1} \in Tx_n$. Suppose that there exists $n \in \mathbb{N}$ for which

$y_n = y_{n+1}$. We obtain $gx_n \in Tx_n$ and so the proof is complete. Now assume that $n \in \mathbb{N}$ for which $y_n \neq y_{n+1}$. Since Tx_0 is compact, from Lemma 1 we obtain that $y_0 \in Tx_0$ such that $p(gx_0, y_0) = p(gx_0, Tx_0) > 0$. This implies that

$$\frac{1}{2}p(gx_0, Tx_0) < p(gx_0, gx_1) \tag{3}$$

From Lemma 3 and condition (F1), we have

$$F(p(gx_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1))$$

By the hypothesis of the theorem, we obtain,

$$\tau + F(p(gx_1, Tx_1)) \leq \tau + F(H_p(Tx_0, Tx_1)) \leq F(p(gx_0, gx_1)). \tag{4}$$

Also, since $gx_1 \in Tx_0$ and $gx_2 \in Tx_1$, we obtain

$$\tau + F(p(gx_1, gx_2)) \leq F(p(gx_0, gx_1)).$$

Moreover, $gx_{n+1} \in Tx_n$ we obtain

$$\frac{1}{2}p(gx_n, Tx_n) < p(gx_n, gx_{n+1})$$

for all $n \in \mathbb{N}$ and from (F1) and Lemma 3, we obtain

$$F(p(gx_n, Tx_n)) \leq F(H_p(Tx_{n-1}, Tx_n))$$

for all $n \in \mathbb{N}$. Since (2), we get

$$\tau + F(p(gx_n, Tx_n)) \leq \tau + F(H_p(Tx_{n-1}, Tx_n)) \leq F(p(gx_{n-1}, gx_n)),$$

$$\begin{aligned} F(p(gx_n, gx_{n+1})) &\leq F(p(gx_{n-1}, gx_n)) - \tau \leq F(p(gx_{n-2}, gx_{n-1})) - 2\tau \\ &\leq F(p(gx_{n-3}, gx_{n-2})) - 3\tau \dots \leq F(p(gx_0, gx_1)) - n\tau. \end{aligned}$$

Thus, we have

$$F(p(gx_n, gx_{n+1})) \leq F(p(gx_0, gx_1)) - n\tau \tag{5}$$

for $n = 1, 2, \dots$. Let $t_n = p(gx_n, gx_{n+1})$, we get

$$\lim_{n \rightarrow \infty} F(p(gx_n, gx_{n+1})) = \lim_{n \rightarrow \infty} F(t_n) = -\infty.$$

Hence, from (F2), we obtain $\lim_{n \rightarrow \infty} p(gx_n, gx_{n+1}) = 0$. Since (F3), there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} t_n^k F(t_n) = 0$.

Using (5), we have

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k n\tau \leq 0. \tag{6}$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} t_n^k = 0. \tag{7}$$

There exist for all $n > n_1$ and $n_1 \in \mathbb{N}$ such that $t_n^k \leq 1$. Thus, we have

$$t_n \leq \frac{1}{(n)^{\frac{1}{k}}}. \tag{8}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$ with $m > n > n_1$, we get

$$\begin{aligned} p(gx_n, gx_m) &\leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2}) + \dots + p(gx_{m-1}, gx_m) \\ &\quad - [p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_{n+2}) + \dots + p(gx_{m-1}, gx_{m-1})] \\ &= t_n + t_{n+1} + \dots + t_{m-1} = \sum_{i=n}^{m-1} t_i \leq \sum_{i=n}^{\infty} t_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have $p(gx_n, gx_m) \rightarrow 0$. Thus, $\{gx_n\}$ sequence is a Cauchy sequence. Also, by the completeness of (X, p) there exists $z \in X$ such that

$$p(z, z) = \lim_{n \rightarrow \infty} p(gx_n, z) = \lim_{n, m \rightarrow \infty} p(gx_n, gx_m) = 0.$$

$$\tau + F(H_p(Tx_{n-1}, Tx_n)) \leq F(p(gx_{n-1}, gx_n)).$$

These imply that $\{Tx_n\}$ is a Cauchy sequence in the complete partial metric space $(K^p(X), H_p)$. Thus, let $Tx_n \rightarrow M \in K^p(X)$. Now we have

$$\begin{aligned} p(z, M) &\leq p(z, gx_n) + p(gx_n, M) - p(gx_n, gx_n) \\ &\leq p(z, gx_n) + p(Tx_{n-1}, M) \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned}$$

Since M is closed, we obtain $z \in M$. g -weak compatibility and the continuity of g and T implies that

$$H_p(gM, Tz) \leq H_p(Tz, M), \tag{9}$$

$$p(gz, z) \leq H_p(Tz, M). \tag{10}$$

Now we have

$$\begin{aligned} p(gz, Tz) &\leq p(gz, ggx_{n+1}) + p(ggx_{n+1}, Tz) - p(ggx_{n+1}, ggx_{n+1}) \\ &\leq p(gz, ggx_{n+1}) + H_p(gTx_n, Tz) \\ &\leq p(gz, ggx_{n+1}) + H_p(gTx_n, Tgx_n) + H_p(Tgx_n, Tz) - \inf_{z \in Tgx_n} \{p(z, z)\}. \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$p(gz, Tz) \leq H_p(Tz, M).$$

Now using (2), we have

$$\tau + F(H_p(Tx_n, Tz)) \leq F(p(gx_n, gz)).$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$\tau + F(H_p(M, Tz)) \leq F(p(z, gz)).$$

Using (10) and (F1), we obtain

$$\tau + F(H_p(M, Tz)) \leq F(p(z, gz)) \leq F(H_p(Tz, M)).$$

Since $\tau > 0$, we obtain $H_p(Tz, M) = 0$. Then $p(gz, Tz) = 0$, $gz \in Tz$ since T closed.

Corollary 1. Let (X, p) be a complete partial metric space and $T : X \rightarrow K^p(X)$ be F -contraction mappings, if there exists $\tau > 0$ such that;

$$\frac{1}{2}p(x, Tx) < p(x, y) \quad (11)$$

then

$$\tau + F(H_p(Tx, Ty)) \leq F(p(x, y)) \quad (12)$$

for all $x, y \in X$.

Definition 5. Let (X, p) be a complete partial metric space, $g : X \rightarrow X$ and $T : X \rightarrow K^p(X)$ be g -weak compatible 2. type F -contraction mappings, if there exists $\tau > 0$ such that;

$$\frac{1}{2}p(gx, Tx) < p(gx, gy) \quad (13)$$

then

$$\tau + F(H_p(Tx, Ty)) \leq \alpha F(p(gx, gy)) + \beta F(p(gx, Tx)) + \gamma F(p(gy, Ty)) \quad (14)$$

for all $x, y \in X$ where $\alpha, \beta \in (0, 1]$ are real numbers such that $0 \leq \gamma < 1$ and $\alpha + \beta + \gamma = 1$.

Theorem 2. Let (X, p) be a complete partial metric space and g, T g -weak compatible 2. type F -contraction continuous mappings. Then there exists a point $z \in X$ such that $gz \in Tz$.

Proof. Let x_0 be an arbitrary point of X and choose $x_1 \in X$ such that $gx_1 \in Tx_0$. This is possible since $Tx_0 \subseteq g(X)$. In general, having chosen $x_n \in X$ we choose $x_{n+1} \in X$, so that $y_n = gx_{n+1} \in Tx_n$. Suppose that there exists $n \in \mathbb{N}$ for which $y_n = y_{n+1}$. We obtain $gx_n \in Tx_n$ and so the proof is complete. Now assume that $n \in \mathbb{N}$ for which $y_n \neq y_{n+1}$. Since Tx_0 is compact, from Lemma 1, we obtain that $y_0 \in Tx_0$ such that $p(gx_0, y_0) = p(gx_0, Tx_0) > 0$. This implies that

$$\frac{1}{2}p(gx_0, Tx_0) < p(gx_0, gx_1) \quad (15)$$

From Lemma 3 and condition (F1), we have

$$F(p(gx_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1))$$

By the hypothesis of the theorem, we obtain,

$$\tau + F(p(gx_1, Tx_1)) \leq \tau + F(H_p(Tx_0, Tx_1)) \leq \alpha F(p(gx_0, gx_1)) + \beta F(p(gx_0, Tx_0)) + \gamma F(p(gx_1, Tx_1)). \quad (16)$$

Also, since $gx_1 \in Tx_0$ and $gx_2 \in Tx_1$, we obtain

$$\tau + F(p(gx_1, gx_2)) \leq \alpha F(p(gx_0, gx_1)) + \beta F(p(gx_0, gx_1)) + \gamma F(p(gx_1, gx_2))$$

and

$$\tau + (1 - \gamma)F(p(gx_1, gx_2)) \leq (\alpha + \beta)F(p(gx_0, gx_1)).$$

Since $\alpha + \beta + \gamma = 1$, we get

$$F(p(gx_1, gx_2)) \leq F(p(gx_0, gx_1)) - \frac{\tau}{\alpha + \beta} \quad (17)$$

Moreover, $gx_{n+1} \in Tx_n$ we obtain

$$\frac{1}{2}p(gx_n, Tx_n) < p(gx_n, gx_{n+1})$$

for all $n \in \mathbb{N}$. From (F1) and Lemma 3, we obtain

$$F(p(gx_n, Tx_n)) \leq F(H_p(Tx_{n-1}, Tx_n))$$

for all $n \in \mathbb{N}$. Since (14), we get

$$\tau + F(p(gx_n, Tx_n)) \leq \tau + F(H_p(Tx_{n-1}, Tx_n)) \leq \alpha F(p(gx_{n-1}, gx_n)) + \beta F(p(gx_{n-1}, Tx_{n-1})) + \gamma F(p(gx_n, Tx_n)),$$

and

$$\tau + (1 - \gamma)F(p(gx_n, gx_{n+1})) \leq (\alpha + \beta)F(p(gx_{n-1}, gx_n))$$

$$\begin{aligned} F(p(gx_n, gx_{n+1})) &\leq F(p(gx_{n-1}, gx_n)) - \frac{\tau}{\alpha + \beta} \\ &\leq F(p(gx_{n-2}, gx_{n-1})) - \frac{2\tau}{\alpha + \beta} \\ &\leq F(p(gx_{n-3}, gx_{n-2})) - \frac{3\tau}{\alpha + \beta} \\ &\vdots \\ &\leq F(p(gx_0, gx_1)) - \frac{n\tau}{\alpha + \beta}. \end{aligned}$$

Thus, we have

$$F(p(gx_n, gx_{n+1})) \leq F(p(gx_0, gx_1)) - \frac{n\tau}{\alpha + \beta} \quad (18)$$

for $n = 1, 2, \dots$. Let $t_n = p(gx_n, gx_{n+1})$, we get

$$\lim_{n \rightarrow \infty} F(p(gx_n, gx_{n+1})) = \lim_{n \rightarrow \infty} F(t_n) = -\infty.$$

Hence, from (F2), we obtain $\lim_{n \rightarrow \infty} p(gx_n, gx_{n+1}) = 0$. Since (F3), there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} t_n^k F(t_n) = 0$. Using (18), we have

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k \frac{n\tau}{\alpha + \beta} \leq 0. \quad (19)$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} t_n^k \frac{n}{\alpha + \beta} = 0. \quad (20)$$

There exist for all $n > n_1$ and $n_1 \in \mathbb{N}$ such that $t_n^k \frac{n}{\alpha + \beta} \leq 1$. Thus, we have

$$t_n \leq \frac{1}{\left(\frac{n}{\alpha + \beta}\right)^{\frac{1}{k}}}. \quad (21)$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}$ with $m > n > n_1$, we get

$$\begin{aligned} p(gx_n, gx_m) &\leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2}) + \dots + p(gx_{m-1}, gx_m) \\ &\quad - [p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_{n+2}) + \dots + p(gx_{m-1}, gx_{m-1})] \\ &= t_n + t_{n+1} + \dots + t_{m-1} = \sum_{i=n}^{m-1} t_i \leq \sum_{i=n}^{\infty} t_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have $p(gx_n, gx_m) \rightarrow 0$. Thus, $\{gx_n\}$ sequence is a Cauchy sequence. Also, by the completeness of (X, p) there exists $z \in X$ such that

$$p(z, z) = \lim_{n \rightarrow \infty} p(gx_n, z) = \lim_{n, m \rightarrow \infty} p(gx_n, gx_m) = 0.$$

$$\tau + F(H_p(Tx_{n-1}, Tx_n)) \leq \alpha F(p(gx_{n-1}, gx_n)) + \beta F(p(gx_{n-1}, gx_n)) + \gamma F(p(gx_n, gx_{n+1})).$$

These imply that $\{Tx_n\}$ is a Cauchy sequence in the complete partial metric space $(K^p(X), H_p)$. Thus, let $Tx_n \rightarrow M \in K^p(X)$. Now we have

$$p(z, M) \leq p(z, gx_n) + p(gx_n, M) - p(gx_n, gx_n) \leq p(z, gx_n) + p(Tx_{n-1}, M) \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

Since M is closed, we obtain $z \in M$. g -weak compatibility and the continuity of g and T implies that

$$H_p(gM, Tz) \leq H_p(Tz, M), \quad (22)$$

$$p(gz, z) \leq H_p(Tz, M). \quad (23)$$

Now we have

$$\begin{aligned} p(gz, Tz) &\leq p(gz, ggx_{n+1}) + p(ggx_{n+1}, Tz) - p(ggx_{n+1}, ggx_{n+1}) \\ &\leq p(gz, ggx_{n+1}) + H_p(gTx_n, Tz) \\ &\leq p(gz, ggx_{n+1}) + H_p(gTx_n, Tgx_n) + H_p(Tgx_n, Tz) - \inf_{z \in Tgx_n} \{p(z, z)\}. \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$p(gz, Tz) \leq H_p(Tz, M).$$

Now using (14), we have

$$\begin{aligned} \tau + F(H_p(Tx_n, Tz)) &\leq \alpha F(p(gx_n, gz)) + \beta F(p(gx_n, Tx_n)) + \gamma F(p(gz, Tz)), \\ \tau + F(H_p(M, Tz)) &\leq \alpha F(p(z, gz)) + \beta F(p(z, M)) + \gamma F(p(gz, Tz)), \\ \tau + F(H_p(M, Tz)) &\leq (\alpha + \gamma)F(H_p(Tz, M)). \end{aligned}$$

Since $\tau > 0$, and $\alpha + \gamma < 1$, we obtain $H_p(Tz, M) = 0$. Then $p(gz, Tz) = 0$, $gz \in Tz$ since T closed.

Corollary 2. Let (X, p) be a complete partial metric space and $T : X \rightarrow K^p(X)$ be F - contraction mappings, if there exists $\tau > 0$ such that:

$$\frac{1}{2}p(x, Tx) < p(x, y) \tag{24}$$

then

$$\tau + F(H_p(Tx, Ty)) \leq \alpha F(p(x, y)) + \beta F(p(x, Tx)) + \gamma F(p(y, Ty)) \tag{25}$$

for all $x, y \in X$ where $\alpha, \beta \in (0, 1]$ are real numbers such that $0 \leq \gamma < 1$ and $\alpha + \beta + \gamma = 1$.

Example 2. Let $X = [0, \infty)$ and define $p : X \times X \rightarrow \mathbb{R}^+$ by (X, p) be a partial metric space where $p(x, y) = |x - y| + \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a complete partial metric space. We define $g : X \rightarrow X$, $T : X \rightarrow K^p(X)$ a multivalued mapping where $gx = x$ and $T(x) = [0, \frac{x}{8}]$ for all $x \in X$. Let $F(\alpha) = -\frac{1}{\alpha}$, and $y = \frac{x}{8} \in T(x)$ for all $x \in X$. Then, we have,

$$\begin{aligned} \frac{1}{2}p\left(x, \left[0, \frac{x}{8}\right]\right) &< p\left(x, \frac{x}{8}\right) \\ \frac{1}{2}\left(\left|x - \left[0, \frac{x}{8}\right]\right| + \max\left\{x, \left[0, \frac{x}{8}\right]\right\}\right) &< \left(\left|x - \frac{x}{8}\right| + \max\left\{x, \frac{x}{8}\right\}\right) \\ \frac{1}{2}\left(x + \frac{7x}{8}\right) &< x + \frac{7x}{8} \end{aligned}$$

that is, in (1) is satisfied.

Now, we may to show that satisfying in (2) of the Theorem 1.

$$\tau + F(H_p(Tx, Ty)) \leq F(p(gx, gy))$$

$$\tau - \frac{1}{H_p(Tx, Ty)} \leq -\frac{1}{p(gx, gy)} \tag{26}$$

$$H_p(Tx, Ty) = H_p\left(\left[0, \frac{x}{8}\right], \left[0, \frac{x}{64}\right]\right) = \max\left\{\delta_p\left(\left[0, \frac{x}{8}\right], \left[0, \frac{x}{64}\right]\right), \delta_p\left(\left[0, \frac{x}{64}\right], \left[0, \frac{x}{8}\right]\right)\right\}.$$

Then, we have

$$\delta_p\left(\left[0, \frac{x}{8}\right], \left[0, \frac{x}{64}\right]\right) = \sup\left\{p\left(a, \left[0, \frac{x}{64}\right]\right) : a \in \left[0, \frac{x}{8}\right]\right\}.$$

Here, let $a \in \left[0, \frac{x}{64}\right]$, then $\sup p\left(a, \left[0, \frac{x}{64}\right]\right) = 0$. Also, let $a \in \left[0, \frac{x}{8}\right] \setminus \left[0, \frac{x}{64}\right]$, then $\sup p\left(a, \left[0, \frac{x}{64}\right]\right) = \frac{7x}{64}$. Hence, $\delta_p\left(\left[0, \frac{x}{64}\right], \left[0, \frac{x}{8}\right]\right) = \sup\left\{p\left(b, \left[0, \frac{x}{8}\right]\right) : b \in \left[0, \frac{x}{64}\right]\right\} = 0$. Also, we have $\delta_p\left(\left[0, \frac{x}{8}\right], \left[0, \frac{x}{64}\right]\right) = \frac{7x}{64}$. Therefore,

$$H_p(Tx, Ty) = \max\left\{0, \frac{7x}{64}\right\} = \frac{7x}{64}.$$

Also, we have,

$$p(gx, gy) = |x - y| + \max\{x, y\} = \left|x - \frac{x}{8}\right| + \max\left\{x, \frac{x}{8}\right\} = \frac{7x}{8} + x = \frac{15x}{8}$$

also using (26), we obtain

$$\tau \leq \frac{1}{\frac{7x}{64}} - \frac{1}{\frac{15x}{8}}$$

$$\tau \leq \frac{64}{7x} - \frac{8}{15x} = \frac{904}{105x}.$$

Obviously, this is satisfying the condition of Theorem 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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