

Some notes on vector fields in the tangent bundle

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Abstract: Let (M, g) be a Riemannian manifold and $T(M)$ its tangent bundle with the horizontal lift ${}^H\nabla$ of the affine connection ∇ of M . The aims of the present paper are to study conditions of infinitesimal affine transformation for vertical and horizontal vector fields in the tangent bundle and to give necessary conditions for vertical, complete and horizontal vector fields in the tangent bundle to be a harmonic vector fields with respect to the horizontal lift ${}^H\nabla$ of the affine connection ∇ of M .

Keywords: Lift, Riemannian manifold, tangent bundle, infinitesimal Affine transformation, harmonic vector field.

1 Introduction

The theory of infinitesimal transformations was first given by Sophus Lie in [13]. In this work, he defined Lie groups and their accompanying Lie algebras and the identification of their role in geometry and especially the theory of differential equations. Afterwards, the term "Lie Algebra" as known the algebra of infinitesimal transformations of a Lie group was introduced in 1934 by Hermann Weyl. The problems of determining an infinitesimal affine transformation on M and $T(M)$ have been considered by several authors. Yano and Ishihara, in [15], investigated an infinitesimal affine transformations on $T(M)$ with respect to the complete lift connection ${}^C\nabla$. Hasegawa and Yamauchi, in [11], are improved the fiber-preserving hypothesis for non-affine infinitesimal projective transformations with horizontal lift connection.

Gezer and Akbulut, in [8] and [9], also studied infinitesimal affine transformations, geodesics and Killing vector fields on $T(M)$ with respect to the horizontal lift connections ${}^H\nabla$. Akbulut and Salimov, in [3], investigated infinitesimal affine transformations on $T(M)$ with respect to synectic lift connection ${}^S\nabla$. Similarly, Gezer and Akbulut, in [10], studied geodesics and Killing vector fields on $T(M)$ with respect to the Synectic lift connections ${}^S\nabla$.

Investigation on harmonic vector fields is a topic which was studied when the tangent bundle is equipped with the II , $I + II$, Sasaki metric and g -natural metrics ([1], [2],[4],[5],[7],[12],[15]). Bejan and Duruta, in [4] and [5], studied the harmonic almost complex structure and harmonic connections with respect to the general natural metrics. Abbasi, Calvaruso and Perrone, in [1] and [2], investigated harmonicity of unit vector fields and harmonic sections of tangent bundle with respect to Riemannian g - natural metrics. Similarly, Dragomir and Perrone, in [7], also studied harmonic vector fields.

The plan of the paper is as follows: In Section 2.1 we outline the necessary theory of tangent bundle to be applied later in the paper. In Sections 3.1 we study conditions of infinitesimal affine transformation for vertical and horizontal vector fields in the tangent bundle with respect to the horizontal lift ${}^H\nabla$ of the Levi-Civita connection ∇ of M .

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In Sections 3.2 we give necessary conditions for vertical, complete and horizontal vector fields in the tangent bundle to be a harmonic vector fields with respect to the horizontal lift ${}^H\nabla$ of the affine connection ∇ of M .

2 Preliminaries

2.1 Basic formulas on tangent bundles

Let M be a Riemannian manifold with metric g . We denote by $\mathfrak{S}_q^p(M)$ the set of all tensor fields of type (p, q) on M . Manifolds, tensor field and connections are always assumed to be differentiable and of class C^∞ .

Let $T(M)$ be a tangent bundle of M and π the natural projection $\pi : T(M) \rightarrow M$. Let the manifold M be covered by a system of coordinate neighbourhoods (U, x^i) , where $(x^i), i = 1, \dots, n$ is a local coordinate system defined in the neighbourhood U . Let (y^i) be the Cartesian coordinates in each tangent space $T_p(M)$ at $p \in M$ with respect to the natural base $\left\{ \frac{\partial}{\partial x^i} \right\}$, p being an arbitrary point in U whose coordinates are (x^i) . Then we can introduce local coordinates (x^i, y^i) on the open set $\pi^{-1}(U) \subset T(M)$. We call them coordinates induced on $\pi^{-1}(U)$ from (U, x^i) . The projection π is represented by $(x^i, y^i) \rightarrow (x^i)$. The indices I, J, \dots run from 1 to $2n$, the indices \bar{i}, \bar{j}, \dots run from $n+1$ to $2n$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M . Then the horizontal lift ${}^H X$ and the vertical lift ${}^V X$ of X are given by,

$${}^V X = X^i \partial_i, \quad (\partial_i = \frac{\partial}{\partial x^i}), \quad (1)$$

and

$${}^H X = X^i \partial_i - y^j \Gamma_{jk}^i X^k \partial_i, \quad (2)$$

with respect to the induced coordinates, where Γ_{jk}^i are the coefficients of the Levi-Civita connection ∇ .

Suppose that we are given in M a tensor field

$$S = S_{lk\dots j}^{i\dots h} \frac{\partial}{\partial x^i} \otimes \dots \otimes \frac{\partial}{\partial x^h} \otimes dx^l \otimes dx^k \otimes \dots \otimes dx^j, \quad (3)$$

other than a function and a vector field $X = X^i \left(\frac{\partial}{\partial x^i} \right)$. We then define a tensor field $\gamma_X S$ in $\pi^{-1}(U)$ by

$$\gamma_X S = (X^l S_{lk\dots j}^{i\dots h}) \frac{\partial}{\partial y^i} \otimes \dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots \otimes dx^j, \quad (4)$$

and a tensor field γS in $\pi^{-1}(U)$ by

$$\gamma S = (y^l S_{lk\dots j}^{i\dots h}) \frac{\partial}{\partial y^i} \otimes \dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots \otimes dx^j, \quad (5)$$

with respect to the induced coordinates (x^h, y^h) , U being an arbitrary coordinate neighborhood in M [15].

Let M be a Riemannian manifold with metric g whose components in a coordinate neighborhood U are g_{ji} and Γ_{ji}^h the Christoffel symbols formed with g_{ji} . If in the neighborhood $\pi^{-1}(U)$ of the tangent bundle $T(M)$, U being neighborhood M , then ${}^H g$ has components given by

$$({}^H g_{JI}) = \begin{pmatrix} \Gamma_j^i g_{ti} + \Gamma_i^t g_{jt} & g_{ji} \\ g_{ji} & 0 \end{pmatrix} \quad (6)$$

with respect to the induced coordinates (x^i, y^i) in $T(M)$ and $\Gamma_i^h = y^j \Gamma_{ji}^h$, Γ_{ji}^h being components of affine connection ∇ in M . From (6), ${}^H g$ has contravariant components

$$({}^H \tilde{g}^{IJ}) = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & -\Gamma_h^j g^{hi} - \Gamma_h^i g^{hj} \end{pmatrix} \tag{7}$$

with respect to the induced coordinates in $T(M)$, where g^{ij} denote the contravariant components of ${}^H \tilde{g}$ in M , i.e., ${}^H g_{IJ}^H \tilde{g}^{IS} = \delta_J^S$ [15].

Proposition 1. *Let g and ∇ be, respectively, a pseudo-Riemannian metric and an affine connection such that $\nabla g = 0$. Then ${}^H \nabla^H g = 0$, where the horizontal lift ${}^H g$ of g with respect to ${}^H \nabla$ is a pseudo-Riemannian metric in $T(M)$ [15].*

Proposition 2. *Let g be a pseudo-Riemannian metric and ∇ the Riemannian connection determined by g in M such that $\nabla g = 0$. Then, ${}^C g$ coincides with the horizontal lift ${}^H g$ of g to $T(M)$ with respect to ∇ [15].*

Proposition 3. *If we write $ds^2 = g_{ji} dx^j dx^i$ the pseudo-Riemannian metric in M given by g , then the pseudo-Riemannian metric in $T(M)$ given by the ${}^H g$ of g to $T(M)$ with respect to an affine connection ∇ in M is*

$$ds^2 = 2g_{ji} \tilde{\delta} y^j dx^i, \tag{8}$$

where $\tilde{\delta} y^j = dy^j + \tilde{\Gamma}_{lk}^j y^l dx^k$ and $\tilde{\Gamma}_{ji}^h = \Gamma_{ij}^h$ are components of the connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_Y X + [X, Y], \tag{9}$$

for $X, Y \in \mathfrak{S}_0^1(M)$ [15].

The horizontal lift ${}^H \nabla$ of affine connection ∇ in M_n to $T(M)$ is defined by the conditions

$${}^H \nabla_{V_X} V Y = 0, {}^H \nabla_{V_X} H Y = 0, {}^H \nabla_{H_X} V Y = (\nabla_X Y)^V, {}^H \nabla_{H_X} H Y = (\nabla_X Y)^H, \tag{10}$$

for $X, Y \in \mathfrak{S}_0^1(M)$. From (10) the horizontal lift ${}^H \nabla$ of ∇ has components ${}^H \Gamma_{IJ}^K$ such that

$$\begin{aligned} {}^H \Gamma_{ij}^k &= \Gamma_{ij}^k, {}^H \Gamma_{i\bar{j}}^k = {}^H \Gamma_{ij}^k = {}^H \Gamma_{i\bar{j}}^k = {}^H \Gamma_{i\bar{j}}^{\bar{k}} = 0, \\ {}^H \Gamma_{i\bar{j}}^{\bar{k}} &= y^s \partial_s \Gamma_{ij}^k - y^s R_{sij}^k, {}^H \Gamma_{i\bar{j}}^{\bar{k}} = {}^H \Gamma_{i\bar{j}}^{\bar{k}} = \Gamma_{ij}^k \end{aligned} \tag{11}$$

with respect to the induced coordinates in $T(M_n)$, where Γ_{ij}^k are components of ∇ in M [15].

Remark. The connection ${}^H \nabla$ in Proposition 1 has nontrivial torsion even for the Riemannian connection ∇ determined by g , unless g is locally flat [15].

Explicit expressions for the Lie bracket $[\cdot, \cdot]$ of the tangent bundle $T(M)$ are given in [6] and [15]. The Lie bracket of vertical and horizontal vector fields on $T(M)$ is given by the following propositions.

Proposition 4. *Let (M, g) be a pseudo-Riemannian manifold, ∇ be a Levi-Civita connection. Then the Lie bracket on the tangent bundle $T(M)$ satisfies the following identities:*

$$\begin{aligned} [{}^V X, {}^V Y] &= 0, \\ [{}^H X, {}^V Y] &= {}^V (\nabla_X Y), \\ [{}^H X, {}^H Y] &= {}^H [X, Y] - {}^V (R(X, Y)y) \end{aligned} \tag{12}$$

for all $X, Y \in \mathfrak{S}_0^1(M)$, R is the Riemannian curvature tensor of ∇ defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ and ${}^V(R(X, Y))y$ is a tensor field of type $(1, 0)$ of $T(M)$ such that ${}^V(R(X, Y))y = \gamma(R(X, Y))$ which is locally expressed as $\gamma(R(X, Y)) = y^s R_{jks}^i X^j Y^k \partial_i$ with respect to the induced coordinates [6].

Let there be given an affine connections ∇ and a vector field in $X \in \mathfrak{S}_0^1(M)$. Then the Lie derivative $L_X \nabla$ with respect to X is, by definition, an element of $\mathfrak{S}_2^1(M)$ such that

$$(L_X \nabla)(Y, Z) = L_X(\nabla_Y Z) - \nabla_Y(L_X Z) - \nabla_{[X, Y]} Z = [L_X, \nabla_Y] Z - \nabla_{[X, Y]} Z, \quad (13)$$

for any $Y, Z \in \mathfrak{S}_0^1(M)$ [14],[15].

In a manifold M with affine connection ∇ , an infinitesimal affine transformation $x^{h'} = x^h + X^h(x^1, \dots, x^n) \delta t$ defined by a vector field $X \in \mathfrak{S}_0^1(M)$ is called an infinitesimal affine transformation if $L_X \nabla = 0$, [15].

3 Main results

3.1 Infinitesimal transformation in the tangent bundle

Suppose that $X \in \mathfrak{S}_0^1(M)$, so that X is a vector field in M . We define a vector field ${}^V X$ in $T(M)$ by ${}^V X(\omega) = (\omega(X))^V$, where ω being an arbitrary 1-form in M . Then, we call ${}^V X$ the vertical lift of X in M to $T(M)$.

Additionally, let there be given an element X of $\mathfrak{S}_0^1(M)$. Then, we define the horizontal lift ${}^H X$ of X by ${}^H X = X^C - \nabla_\gamma X$ in $T(M)$ where $\nabla_\gamma X = \gamma(\nabla X)$.

Lemma 1. Let ${}^V X$ and ${}^H X$ be the vertical and horizontal lifts of X to $T(M)$, respectively. Then

$$\begin{aligned} {}^H \nabla_{\gamma(R(X, Y))} {}^V Z &= 0, \\ {}^H \nabla_{\gamma(R(X, Y))} {}^H Z &= 0, \\ {}^H \nabla_{V_Y} \gamma(R(X, Z)) &= {}^V(R(X, Z)Y), \\ {}^H \nabla_{H_Y} \gamma(R(X, Z)) &= \gamma(\nabla_Y R(X, Z)) \end{aligned}$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M)$, where $\gamma(R(X, Y))$ is a tensor field of type $(1, 0)$ in $T(M)$.

Let ∇ be a Riemannian connection in M and the vertical lift ${}^V X$ of X to $T(M)$. Using (10) and (12) in (13) we obtain

$$\begin{aligned} (L_{V_X} {}^H \nabla)({}^V Y, {}^H Z) &= L_{V_X} ({}^H \nabla_{V_Y} {}^H Z) - {}^H \nabla_{V_Y} (L_{V_X} {}^H Z) - {}^H \nabla_{[V_X, V_Y]} {}^H Z \\ &= -{}^H \nabla_{V_Y} [{}^V X, {}^H Z] \\ &= -{}^H \nabla_{V_Y} ({}^V [X, Z] - {}^V (\nabla_X Z)) \\ &= -{}^H \nabla_{V_Y} {}^V [X, Z] + {}^H \nabla_{V_Y} {}^V (\nabla_X Z) \\ &= 0, \end{aligned}$$

$$\begin{aligned} (L_{V_X} {}^H \nabla)({}^H Y, {}^V Z) &= L_{V_X} ({}^H \nabla_{H_Y} {}^V Z) - {}^H \nabla_{H_Y} (L_{V_X} {}^V Z) - {}^H \nabla_{[V_X, H_Y]} {}^V Z \\ &= L_{V_X} {}^V (\nabla_Y Z) - {}^H \nabla_{H_Y} [{}^V X, {}^V Z] - {}^H \nabla_{[X, Y]} {}^V (\nabla_X Z) \\ &= [{}^V X, {}^V (\nabla_Y Z)] - {}^H \nabla_{V_{[X, Y]}} {}^V Z + {}^H \nabla_{V_{(\nabla_X Y)}} {}^V Z \\ &= 0, \end{aligned} \quad (14)$$

$$\begin{aligned} (L_{V_X}{}^H \nabla)({}^V Y, {}^V Z) &= L_{V_X}({}^H \nabla_{V_Y} {}^V Z) - {}^H \nabla_{V_Y}(L_{V_X} {}^V Z) - {}^H \nabla_{[{}^V X, {}^V Y]} {}^V Z \\ &= -{}^H \nabla_{V_Y} [{}^V X, {}^V Z] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} (L_{V_X}{}^H \nabla)({}^H Y, {}^H Z) &= L_{V_X}({}^H \nabla_{H_Y} {}^H Z) - {}^H \nabla_{H_Y}(L_{V_X} {}^H Z) - {}^H \nabla_{[{}^V X, {}^H Y]} {}^H Z \\ &= L_{V_X}({}^H \nabla_Y Z) - {}^H \nabla_{H_Y} [{}^V X, {}^H Z] - {}^H \nabla_{[{}^V X, Y] - {}^V \nabla_X Y} {}^H Z \\ &= [{}^V X, {}^H \nabla_Y Z] - {}^H \nabla_{H_Y} ({}^V [X, Z] - {}^V \nabla_X Z) - {}^H \nabla_{[{}^V X, Y]} {}^H Z + {}^H \nabla_{V \nabla_X Y} {}^H Z \\ &= {}^V [X, \nabla_Y Z] - {}^V \nabla_X (\nabla_Y Z) - {}^H \nabla_{H_Y} {}^V [X, Z] + {}^H \nabla_{H_Y} {}^V \nabla_X Z \\ &= {}^V [X, \nabla_Y Z] - {}^V \nabla_X (\nabla_Y Z) - {}^V \nabla_Y [X, Z] + {}^V \nabla_Y \nabla_X Z \\ &= {}^V (L_X \nabla_Y Z) - {}^V \nabla_X (\nabla_Y Z) - {}^V \nabla_Y (L_X Z) + {}^V \nabla_Y \nabla_X Z \\ &= {}^V (L_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_Y (L_X Z) + \nabla_Y \nabla_X Z) - \nabla_{[X, Y]} Z + \nabla_{[X, Y]} Z \\ &= {}^V (L_X \nabla_Y Z - \nabla_Y (L_X Z) - \nabla_{[X, Y]} Z - (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z)) \\ &= {}^V ((L_X \nabla)(Y, Z) - R(X, Y)Z) \end{aligned}$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M)$, where R is curvature tensor of Riemannian connection of ∇ . Thus, from (14) we have,

Theorem 1. *Let ∇ be a Riemannian connection in M . If X is an infinitesimal affine transformation of M and ∇ has zero curvature, then the vertical lift ${}^V X$ of X to $T(M)$ is also infinitesimal affine transformation of $T(M)$ with respect to ${}^H \nabla$.*

Similarly, let ∇ be a Riemannian connection in M and the horizontal lift ${}^H X$ of X to $T(M)$. Using Lemma, (10) and (12) in (13) we have

$$\begin{aligned} (L_{H_X}{}^H \nabla)({}^H Y, {}^V Z) &= L_{H_X}({}^H \nabla_{H_Y} {}^V Z) - {}^H \nabla_{H_Y}(L_{H_X} {}^V Z) - {}^H \nabla_{[{}^H X, {}^H Y]} {}^V Z \\ &= L_{H_X}({}^V \nabla_Y Z) - {}^H \nabla_{H_Y} [{}^H X, {}^V Z] - {}^H \nabla_{[{}^H X, Y] - \gamma(R(X, Y))} {}^V Z \\ &= [{}^H X, {}^V \nabla_Y Z] - {}^H \nabla_{H_Y} ({}^V \nabla_X Z) - {}^H \nabla_{[{}^H X, Y]} {}^V Z + {}^H \nabla_{\gamma(R(X, Y))} {}^V Z \\ &= {}^V \nabla_X \nabla_Y Z - {}^V \nabla_Y \nabla_X Z - {}^V \nabla_{[X, Y]} Z \\ &= {}^V \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ &= {}^V (R(X, Y)Z), \end{aligned}$$

$$\begin{aligned} (L_{H_X}{}^H \nabla)({}^V Y, {}^V Z) &= L_{H_X}({}^H \nabla_{V_Y} {}^V Z) - {}^H \nabla_{V_Y}(L_{H_X} {}^V Z) - {}^H \nabla_{[{}^H X, {}^V Y]} {}^V Z \\ &= -{}^H \nabla_{V_Y} [{}^H X, {}^V Z] - {}^H \nabla_{[{}^H X, V_Y]} {}^V Z \\ &= -{}^H \nabla_{V_Y} {}^V \nabla_X Z - {}^H \nabla_{V \nabla_X Y} {}^V Z \\ &= 0, \end{aligned}$$

$$\begin{aligned} (L_{H_X}{}^H \nabla)({}^V Y, {}^H Z) &= L_{H_X}({}^H \nabla_{V_Y} {}^H Z) - {}^H \nabla_{V_Y}(L_{H_X} {}^H Z) - {}^H \nabla_{[{}^H X, {}^V Y]} {}^H Z \\ &= -{}^H \nabla_{V_Y} [{}^H X, {}^H Z] - {}^H \nabla_{V \nabla_X Y} {}^H Z \\ &= -{}^H \nabla_{V_Y} ({}^H [X, Z] - \gamma(R(X, Z))) - {}^H \nabla_{[{}^H X, Y]} {}^H Z + {}^H \nabla_{\gamma(R(X, Y))} {}^H Z \\ &= {}^V (R(X, Z)Y), \end{aligned} \tag{15}$$

and

$$\begin{aligned}
 (L_{H_X} {}^H \nabla)({}^H Y, {}^H Z) &= L_{H_X} ({}^H \nabla_{H_Y} {}^H Z) - {}^H \nabla_{H_Y} (L_{H_X} {}^H Z) - {}^H \nabla_{[H_X, H_Y]} {}^H Z \\
 &= L_{H_X} ({}^H \nabla_Y Z) - {}^H \nabla_{H_Y} ({}^H (L_X Z) - \gamma(R(X, Z))) - {}^H \nabla_{H_{[X, Y] - \gamma(R(X, Y))}} {}^H Z \\
 &= {}^H (L_X (\nabla_Y Z)) - \gamma(R(X, \nabla_Y Z)) - {}^H (\nabla_Y (L_X Z)) + {}^H \nabla_{H_Y} \gamma(R(X, Z)) - {}^H \nabla_{H_{[X, Y]}} {}^H Z + {}^H \nabla_{\gamma(R(X, Y))} {}^H Z \\
 &= {}^H (L_X (\nabla_Y Z) + \nabla_Y (L_X Z) - \nabla_{[X, Y]} Z) - \gamma(R(X, \nabla_Y Z)) + \gamma(\nabla_Y R(X, Z)) \\
 &= {}^H ((L_X \nabla)(Y, Z)) - \gamma(R(X, \nabla_Y Z)) + \gamma(\nabla_Y R(X, Z))
 \end{aligned}$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M)$, where $\gamma(R(X, Y))$ is a tensor field of type $(1, 0)$ in $T(M)$ and R is curvature tensor field of Riemannian connection of ∇ . Thus, from (15) we have

Theorem 2. *Let ∇ be a Riemannian connection in M . If X is an infinitesimal affine transformation of M and ∇ has zero curvature, then the horizontal lift ${}^H X$ of X to $T(M)$ is also infinitesimal affine transformation of $T(M)$ with respect to ${}^H \nabla$.*

3.2 Harmonic vector fields in the tangent bundle

Let \tilde{X} be a vector field in $T(M)$. The 1-form ω defined by $\omega(\tilde{Y}) = g(\tilde{X}, \tilde{Y})$, \tilde{Y} being an arbitrary element of $\mathfrak{S}_0^1(T(M))$, is called the covector field associated with \tilde{X} and denoted by \tilde{X}^* . If \tilde{X} has local components \tilde{X}^A , then the associated covector field \tilde{X}^* has local components

$$\tilde{X}_C^* = \tilde{g}_{CB} \tilde{X}^B. \quad (16)$$

A vector field $X \in \mathfrak{S}_0^1(M)$ is said to be a Harmonic vector field of a Riemannian manifold with metric g , if $\nabla_j X_i - \nabla_i X_j = 0$ and $\nabla_i X^i = 0$, where $X_j = g_{ji} X^i$ are components of the covector field X^* associated with X and ∇ is an affine connection of the metric g [14].

We now consider a vector field $X \in \mathfrak{S}_0^1(M)$, then its vertical lift ${}^V X \in \mathfrak{S}_0^1(T(M))$, complete lift ${}^C X \in \mathfrak{S}_0^1(T(M))$ and horizontal lift ${}^H X \in \mathfrak{S}_0^1(T(M))$ have respectively components of the form

$${}^V X = \begin{pmatrix} 0 \\ X^h \end{pmatrix}, {}^C X = \begin{pmatrix} X^h \\ y^s \partial_s X^h \end{pmatrix}, {}^H X = \begin{pmatrix} X^h \\ -y^s \Gamma_{si}^h X^i \end{pmatrix} \quad (17)$$

with respect to the induced coordinates in $T(M)$, where $\Gamma_i^h x^i = y^s \Gamma_{si}^h x^i$ [15].

Let \tilde{X} be a vector field in $T(M)$ and $(\tilde{X}^A) = \begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ its components with respect to the induced coordinates. Then the covariant derivative ${}^H \nabla \tilde{X}$ has components

$${}^H \nabla_I \tilde{X}^J = \partial_I \tilde{X}^J + {}^H \Gamma_{IM}^J \tilde{X}^M \quad (18)$$

$${}^H \nabla_I \tilde{X}_J = \partial_I \tilde{X}_J - {}^H \Gamma_{IJ}^M \tilde{X}_M \quad (19)$$

where ${}^H \Gamma_{IM}^J$ being Christoffel symbols constructed with ${}^H g$ and $\tilde{X}_j = \tilde{g}_{ji} \tilde{X}^i$ are components of the covector field \tilde{X}^* associated with \tilde{X} [15].

Using (17) and (18) we see that, for the covariant derivatives of the vertical, complete and horizontal lifts of \tilde{X} to $T(M)$ with the horizontal lift ${}^H\nabla$ of the affine connection ∇ ,

$$\begin{aligned} {}^H\nabla_I^V X^J &= \begin{pmatrix} 0 & 0 \\ \nabla_i X^j & 0 \end{pmatrix}, \\ {}^H\nabla_I^C X^J &= \begin{pmatrix} \nabla_i X^j & 0 \\ y^s \partial_s \nabla_i X^j - y^s R_{sij}^h X^h & \nabla_i X^j \end{pmatrix}, \\ {}^H\nabla_I^H X^J &= \begin{pmatrix} \nabla_i X^j & 0 \\ -\Gamma_i^j \nabla_i X^t & 0 \end{pmatrix} \end{aligned} \tag{20}$$

with respect to the induced coordinates in $T(M)$. Thus, from (20) we have

Theorem 3. *The vertical, complete and horizontal lifts of a vector field in M to $T(M)$ with the horizontal lift ${}^H\nabla$ of the affine connection ∇ are parallel if and only if the vector field given in M is parallel.*

Using (16) and (17), we see that the covector fields respectively associated with ${}^V X$, ${}^C X$ and ${}^H X$ have in $T(M)$ with the metric ${}^H g$ components,

$${}^V X_I = (X_i, 0), {}^C X_I = (-\hat{X}_i + y^s \partial_s X_i, X_i), {}^H X_I = (y^s \Gamma_{si}^t X_t, X_i) \tag{21}$$

with respect to the induced coordinates where $X_j = g_{ji} X^i$ are components of the covector field \tilde{X}^* associated with \tilde{X} and $\hat{X}_j = y^s (\nabla_s g_{kj}) g^{kt} X_t$.

Using (19) and (21) we obtain

$$\begin{aligned} {}^H\nabla_I^V X_J &= \begin{pmatrix} \nabla_i X_j & 0 \\ 0 & 0 \end{pmatrix}, \\ {}^H\nabla_I^H X_J &= \begin{pmatrix} y^s \Gamma_{sj}^t \nabla_i X_t & \nabla_i X_j \\ 0 & 0 \end{pmatrix}, \\ {}^H\nabla_I^C X_J &= \begin{pmatrix} y^s \partial_s \nabla_i X_j + y^s R_{sij}^h X_h - \nabla_i \hat{X}_j & \nabla_i X_j \\ \nabla_i X_j - (\nabla_i g_{hj}) g^{ht} X_t & 0 \end{pmatrix}. \end{aligned} \tag{22}$$

From (22), we have

$$\begin{aligned} {}^H\nabla_I^V X_J - {}^H\nabla_J^V X_I &= \begin{pmatrix} \nabla_i X_j - \nabla_j X_i & 0 \\ 0 & 0 \end{pmatrix}, {}^H g^{IJH} \nabla_I^V X_J = 0, \\ {}^H\nabla_I^H X_J - {}^H\nabla_J^H X_I &= \begin{pmatrix} \Gamma_j^t \nabla_i X_t - \Gamma_i^t \nabla_j X_t & \nabla_i X_j \\ -\nabla_j X_i & 0 \end{pmatrix}, {}^H g^{IJH} \nabla_I^H X_J = g^{ij} \nabla_i X_j, \\ {}^H\nabla_I^C X_J - {}^H\nabla_J^C X_I &= \begin{pmatrix} \partial (\nabla_i X_j - \nabla_j X_i) + y^s (R_{sij}^h - R_{sji}^h) X_h - (\nabla_i \hat{X}_j - \nabla_j \hat{X}_i) & \nabla_i X_j - \nabla_j X_i + (\nabla_j g_{hi}) g^{ht} X_t \\ \nabla_i X_j - \nabla_j X_i - (\nabla_i g_{hj}) g^{ht} X_t & 0 \end{pmatrix}, \\ {}^H g^{IJH} \nabla_I^C X_J &= g^{ij} \nabla_i X_j + g^{ij} (\nabla_i X_j - \nabla_i g_{hj}) g^{ht} X_t. \end{aligned} \tag{23}$$

From (23), we have

Theorem 4. *Necessary and sufficient conditions in order that (a) the vertical (b) complete and (c) horizontal lifts to $T(M)$, with the metric ${}^H g$, of a vector field X in M be a harmonic vector field in $T(M)$ are that, respectively, (a) X is a harmonic vector field in M ; (b) X is a parallel with vanishing second covariant derivative in M and ∇ is the affine connection of M such that $\nabla g = 0$; (c) X is a parallel in M .*

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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