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Extension of some fixed point theorems type T-contraction in cone metric space

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Abstract: The aim of this paper is to prove fixed point theorem for an extended Kannan and Chatterjea type T-contraction mapping in a cone metric space. Our results generalize recent results existing in the literature of T-contraction mappings in cone metric space

Keywords: Fixed point theorem, cone metric space, T-contraction.

1 Introduction

In [10], Huang and Zhang introduced the concept of cone metric space as a generalization of metric space, in which they replace the set of real numbers with a real Banach space. After that, many others [1,2,4,5,6],[13] proved several fixed point theorems for contractive type mappings on a cone metric space.

In the other side, Morales and Rojas [8],[9] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Zamfirescu, T-weakly contraction mappings. The purpose of this paper is to prove fixed point theorem for an extended Kannan and Chatterjea type T-contraction mapping in a cone metric space. Our results extend and generalized fixed point theorems of [12].

2 Definitions and Preliminaries

First we define cone metric space and properties and other results that will be needed in the sequel

Definition 1. [11] *Let E be a real Banach space. A subset P of E is called a cone if and only if*

(1) *P is nonempty, closed and $P \neq \{0\}$;*
(2) *$\alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0$ and $x, y \in P \Rightarrow \alpha x + \beta y \in P$*
(3) *$x \in P$ and $-x \in P$ (i.e) $P \cap (-P) = \{0\}$.*

For a given cone $P \subseteq E$, a partial ordering is defined as $\leq$ on E with respect to P by $x \leq y$, if and only if $y - x \in P$. It is denoted as $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone $P \subset E$ is called normal, if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\|x\| \leq K \|y\| \tag{1}$$

The least positive number K satisfying (1) is called the normal constant of P.

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Definition 2. [10] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(d1) $0 < d(x,y)$ *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$*;*
(d2) $d(x,y) = d(y,x)$ *for all* $x,y \in X$*;*
(d3) $d(x,y) \leq d(x,z) + d(y,z)$ *for all* $x,y,z \in X$*.*

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Example 1. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x,y) = |x-y|(1,\frac{1}{2})$. Then (X,d) is a cone metric space

Lemma 1. *Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $n > N$, $d(x_n,x) \ll \frac{c}{2}$ and $d(x_n,y) \ll \frac{c}{2}$. We have

$$d(x,y) \leq d(x_n,x) + d(x_n,y) \leq c.$$

Hence $\|d(x,y)\| \leq K\|c\|$. Since c is arbitrary $d(x,y) = 0$; therefore $x = y$.

Definition 3. [3] *Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$.*

Definition 4. [10] *Let (X,d) be a cone metric space and let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ obeys the following.*

(1) *$\{x_n\}$ converges to x, if for every $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n,x) \ll c$, for all $n \geq N$.We denote this by $\lim_{n\to\infty} x_n = x$.*
(2) *$\{x_n\}$ is said to be Cauchy if for every $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n,x_m) \ll c$, for all $n,m \geq N$.*

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 5. [7] *Let (X,d) be a cone metric space, P be a normal cone with normal constant K and. Let $T : X \to X$. Then:*

(1) *T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X;*
(2) *T is said to be sequentially convergent, if we have, for every sequence $\{y_n\}$, if $T\{y_n\}$ is convergent, then $\{y_n\}$ also is convergent.*

Corollary 1. [14] *Let $a,b,c,u \in E$, the real Banach space.*

(1) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(3) *If $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.*

3 Mains results

In this section we shall prove some fixed point theorems of T-contractive mappings. The following theorems is extends and improves Theorem1 and Theorem 2 from [12]

Theorem 1. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T is a injective mapping and P is a normal cone with normal constant. If the mappings T and S satisfy*

$$d(TSx,TSy) \leq \alpha(d(Tx,TSx)+d(Ty,TSy))+\gamma d(Tx,Ty)$$

for all $x,y \in X$, where $\alpha > 0$ $\gamma \geq 0$, $2\alpha+\gamma < 1$ then S has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Sx_n$ for each $n = 0,1,2,...\infty$. We have

$$\begin{aligned} d(Tx_{n+1},Tx_n) &\leq \alpha(d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n))+\gamma d(Tx_{n-1},Tx_n) \\ &\leq \alpha d(Tx_n,Tx_{n+1})+(\alpha+\gamma)\, d(Tx_{n-1},Tx_n) \end{aligned}$$

then,

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\alpha+\gamma}{1-\alpha}\right) d(Tx_{n-1},Tx_n)$$

Proceeding further,

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\alpha+\gamma}{1-\alpha}\right)^n d(Tx_0,Tx_1)$$

Next, to claim that $\{Tx_n\}$ is a Cauchy sequence. Consider $m,n \in \mathbb{N}$ such that $m > n$,

$$\begin{aligned} d(Tx_n,Tx_m) &\leq d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})+....d(Tx_{m-1},Tx_m) \\ &\leq \left[\left(\frac{\alpha+\gamma}{1-\alpha}\right)^n+\left(\frac{\alpha+\gamma}{1-\alpha}\right)^{n+1}+...+\left(\frac{\alpha+\gamma}{1-\alpha}\right)^{m-1}\right] d(Tx_0,Tx_1) \end{aligned} \quad (2)$$

we take $\frac{\alpha+\gamma}{1-\alpha} = k$, The inequality (2) implies that for all $m,n \in \mathbb{N}, n > m$

$$d(Tx_n,Tx_m) \leq \frac{k^n}{1-k} d(Tx_0,Tx_1),$$

From the inequality(1), we get

$$\|d(Tx_n,Tx_m)\| \leq \frac{k^n}{1-k} \|d(Tx_0,Tx_1)\|,$$

Further, since $k \in (0,1)$,$k^n \to 0$ as $n \to \infty$. Therefore $\|d(Tx_m,Tx_n)\| \to 0$ as $m,n \to \infty$. Thus $\{Tx_n\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_m = u$. As T is continuous,

$$\lim_{n\to\infty} Tx_m = Tu \quad (3)$$

By the uniqueness of the limit, $z = Tu$. Since S is continuous, $\lim_{n\to\infty} Sx_m = Su$. Again as T is continuous, $\lim_{n\to\infty} TSx_m = TSu$. Therefore

$$\lim_{n\to\infty} Tx_{m+1} = TSu \quad (4)$$

Now consider,

$$\begin{aligned} d(TSu,Tu) &\leq d(TSu,Tx_m)+d(Tx_m,Tu) \\ &\leq \alpha d(Tu,TSu)+\alpha d(Tx_{m-1},Tx_m)+\gamma d(Tu,Tx_{m-1})+d(Tx_m,Tu) \end{aligned}$$

then

$$d(TSu,Tu) \leq \frac{\alpha}{1-\alpha}d(Tx_{m-1},Tx_m)+\frac{\gamma}{1-\alpha}d(Tu,Tx_{m-1})+\frac{1}{1-\alpha}d(Tx_m,Tu)$$
$$\leq \frac{\alpha}{1-\alpha}d(Tx_{m-1},Tx_m)+\frac{\gamma}{1-\alpha}\left(d(Tu,Tx_m)+d(Tx_m,Tx_{m-1})\right)+\frac{1}{1-\alpha}d(Tx_m,Tu).$$

So

$$d(TSu,Tu) \leq \frac{\alpha+\gamma}{1-\alpha}d(Tx_{m-1},Tx_m)+\frac{\gamma+1}{1-\alpha}d(Tx_m,Tu) \quad (5)$$

Let $0 \ll c$ be arbitrary. By (3) $d(Tu,Tx_m) \ll \frac{c(1-\alpha)}{2(1+\gamma)}$ and By (4) $d(Tx_{m-1},Tx_m) \ll \frac{c(1-\alpha)}{2(\gamma+\alpha)}$. Then, (5) becomes

$$d(TSu,Tu) \ll c, for\ each\ c \in intP$$

Now, Using Corollary(1) (iii), itfollows that $d(Tu,TSu)=0$ which implies that $Tu=TSu$. As T is injective, $u=Su$. Thus u is the fixed point of S.

To Prove Uniqueness: If w is another fixed point of S, then $w=Sw$.

$$d(Tu,Tw)=d(TSu,TSw) \leq \alpha(d(Tu,TSu)+d(Tw,TSw))+\gamma d(Tu,Tw)$$
$$\leq \gamma d(Tu,Tw)$$

a contradiction. Hence $d(Tu,Tw)=0$ which implies $Tu=Tw$. As T is injective, $u=w$. Therefore the fixed point of S is unique. This completes the proof of the Theorem

Corollary 2. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T be injective and P be a normal cone with normal constant. If the mappings T and S satisfy*

$$d(Sx,Sy) \leq \alpha\left(d(x,Sx)+d(y,Sy)\right)$$

for all $x,y \in X$, for some $\alpha \in \left(0,\frac{1}{2}\right)$, then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $\gamma=0$ and $T=I$, the identity mapping in Theorem (1)

Corollary 3. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T be injective and P be a normal cone with normal constant. If the mappings T and S satisfy*

$$d(TSx,TSy) \leq \zeta\left(d(Tx,TSx).d(Ty,TSy).d(Tx,Ty)\right)^{\frac{1}{3}}$$

for all $x,y \in X$, for some $\zeta \in (0,1)$, then S has a unique fixed point in X.

Proof. The arithmetic mean-geometric mean inequality implies that

$$d(TSx,TSy) \leq \frac{\zeta}{3}\left(d(Tx,TSx)+d(Ty,TSy)+d(Tx,Ty)\right)$$

then, The proof of this Corollary follows by taking $\alpha=\gamma=\frac{\zeta}{3}$ in Theorem (1)

Theorem 2. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T is a injective mapping and P is a normal cone with normal constant. If the mappings T and S satisfy*

$$d(TSx,TSy) \leq \alpha(d(Ty,TSx)+d(Tx,TSy))+\gamma d(Tx,Ty)$$

for all $x,y \in X$, where $\alpha > 0$ $\gamma \geq 0$, $2\alpha+\gamma < 1$ then S has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Sx_n$ for each $n = 0,1,2,...\infty$. Consider

$$\begin{aligned} d(Tx_{n+1},Tx_n) &\leq \alpha(d(Tx_{n-1},Tx_{n+1})+d(Tx_n,Tx_n))+\gamma d(Tx_{n-1},Tx_n) \\ &\leq \alpha d(Tx_{n-1},Tx_{n+1})+\gamma d(Tx_{n-1},Tx_n) \\ &\leq \alpha\left(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\right)+\gamma d(Tx_{n-1},Tx_n) \end{aligned}$$

then,

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\alpha+\gamma}{1-\alpha}\right) d(Tx_{n-1},Tx_n)$$

Proceeding further,

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\alpha+\gamma}{1-\alpha}\right)^n d(Tx_0,Tx_1)$$

Next, to claim that $\{Tx_n\}$ is a Cauchy sequence. Consider $m,n \in \mathbb{N}$ such that $m > n$,

$$\begin{aligned} d(Tx_n,Tx_m) &\leq d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})+....d(Tx_{m-1},Tx_m) \\ &\leq \left[\left(\frac{\alpha+\gamma}{1-\alpha}\right)^n+\left(\frac{\alpha+\gamma}{1-\alpha}\right)^{n+1}+...+\left(\frac{\alpha+\gamma}{1-\alpha}\right)^{m-1}\right] d(Tx_0,Tx_1) \end{aligned} \tag{6}$$

we take $\frac{\alpha+\gamma}{1-\alpha} = k$, The inequality (6) implies that for all $m,n \in \mathbb{N}, n > m$

$$d(Tx_n,Tx_m) \leq \frac{k^n}{1-k} d(Tx_0,Tx_1),$$

From (1), it follows that

$$\|d(Tx_n,Tx_m)\| \leq \frac{k^n}{1-k} d(Tx_0,Tx_1),$$

Since $k \in (0,1)$,$k^n \to 0$ as $n \to \infty$. Therefore $\|d(Tx_m,Tx_n)\| \to 0$ as $m,n \to \infty$. Consequently $\{Tx_n\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$.

Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_m = u$. As T is continuous,

$$\lim_{n\to\infty} Tx_m = Tu \tag{7}$$

By the uniqueness of the limit, $z = Tu$. Since S is continuous, $\lim_{n\to\infty} Sx_m = Su$. Again as T is continuous, $\lim_{n\to\infty} TSx_m = TSu$. Therefore

$$\lim_{n\to\infty} Tx_{m+1} = TSu \tag{8}$$

Now consider,

$$\begin{aligned} d(TSu,Tu) &\leq d(TSu,Tx_m)+d(Tx_m,Tu) \\ &\leq \alpha(d(Tx_{m-1},TSu)+d(Tu,Tx_m))+\gamma d(Tu,Tx_{m-1})+d(Tx_m,Tu) \end{aligned}$$

then

$$\begin{aligned} d(TSu,Tu) &\leq \alpha(d(Tx_{m-1},Tu)+d(Tu,TSu)+d(Tu,Tx_m))+\gamma d(Tu,Tx_{m-1})+d(Tx_m,Tu) \\ &\leq \frac{\alpha+\gamma}{1-\alpha}(d(Tx_{m-1},Tx_m)+d(Tu,Tx_m)+\frac{\alpha+1}{1-\alpha}d(Tu,Tx_m). \end{aligned}$$

Therefore,

$$d(TSu,Tu) \leq \frac{\alpha+\gamma}{1-\alpha}d(Tx_{m-1},Tu)+\frac{2\alpha+\gamma+1}{1-\alpha}d(Tu,Tx_m). \tag{9}$$

Let $0 \ll c$ be arbitrary. By (7) $d(Tu,Tx_m) \ll \frac{c(1-\alpha)}{2(2\alpha+\gamma+1)}$ and By (8) $d(Tx_{m-1},Tx_m) \ll \frac{c(1-\alpha)}{2(\gamma+\alpha)}$. Then, (9) becomes

$$d(TSu,Tu) \ll c, for\ each\ c \in intP$$

Now, Using Corollary(1) (iii), it follows that $d(Tu,TSu) = 0$ which implies that $Tu = TSu$. As T is injective, $u = Su$. Thus u is the fixed point of S. To Prove Uniqueness: If w is another fixed point of S, then $w = Sw$.

$$\begin{aligned} d(Tu,Tw) = d(TSu,TSw) &\leq \alpha(d(Tw,TSu)+d(Tu,TSw))+\gamma d(Tu,Tw) \\ &\leq (2\alpha+\gamma)d(Tu,Tw) \end{aligned}$$

a contradiction. Hence $d(Tu,Tw) = 0$ which implies $Tu = Tw$. As T is injective, $u = w$ is the unique fixed point of S.

Corollary 4. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T be injective and P be a normal cone with normal constant. If the mappings T and S satisfy*

$$d(Sx,Sy) \leq \alpha\,(d(x,Sy)+d(y,Sx))$$

for all $x,y \in X$, for some $\alpha \in \left(0,\frac{1}{2}\right)$, then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $\gamma = 0$ and $T = I$, the identity mapping in Theorem (2)

4 Conclusion

In this paper, we study a fixed point theorems for self-mapping satisfying T-contractive condition in cone metric spaces which generalized and extend the result of [12].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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