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Soliton solutions for the Boussinesq equation with nonlinearity in the time-derivative term using polynomial function methods

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Abstract: In this study, solitary wave solutions were obtained by using polynomial function method for some cases in the 6th order nonlinear modified Boussinesq equation. The difference of this study from the other studies in the literature is that the Boussinesq equation and its analogues, which had been studied so far, had non-linearity in the terms having derivatives with respect to the space variables. However, in this study, the non-linearity is found in the time-derivative terms of the analogues of the Boussinesq equation.

Keywords: Nonlinear wave equation, solitary wave solutions, Boussinesq equation, polynomial function method.

1 Introduction

The Boussinesq equation was first described by Joseph Boussinesq in the 1870s as follows:

$$\lambda^2 \Delta u_{tt} - u_{tt} + \sigma^2 \Delta u = 0 \tag{1}$$

Here, Δ is the Laplace operator, λ and σ are real numbers.
The equation modeling the shallow water waves is the non-linear Boussinesq equation.
Christov and Christou [8] studied the following 2-dimensional Boussinesq equation:

$$u_{tt} = \Delta[u - u^2 - \Delta u] \tag{2}$$

Weakly time-dependent, nonlinear Boussinesq-type equations which include dispersion have become the most popular equation for predicting wave transformations in coastal regions. For the modeling of the waves in deeper waters and for the development of standard equations, several modified form Boussinesq equations have been studied.

Bingham et al. recreated the Boussinesq type models which were studied by Madsen et al. [7]. Liu et al. [14] developed a higher-order Boussinesq equation for the wave model of tsunami wave propagation. They proposed and applied a simplified model of a mobile base and simulated the effects of earthquakes of several kinds. The finite element method was used by Zhao et al. [26] to solve the generalized Boussinesq equations [27]. The travelling wave solutions for the Boussinesq equation with dimension (2+1) was examined by Senthilvelan [15] using the method of homogeneous balance. Similar type of equation was also examined by Chen et al. [9] using a new generalized transformation in the method of homogenous balance, where they obtained some new solutions for solitons and periodic waves. Again, the solutions for the Boussinesq equation of dimension (2+1) was studied by Abdel Rady et al. [4] using the method of

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repeated homogeneous balance and achieved new exact solutions for travelling waves.

A weak nonlinear Hamiltonian model had been developed by Zufiria for 2-dimensional water waves with a specific depth [28]. In his study, he obtained an equation of Boussinesq type by using various differential equation solution methods. Seadawy et al. found the solitary wave solutions of the Boussinesq equation, obtained by Zufiria, through a series expansion method [24]. This method was shown by Seadawy et al. [6,6,10,12,13,16,17,18,19,20,21,22,23,24] to give practical results for the solutions of several types of partial differential equations.

In this article, we study the analogue of the Boussinesq equation given below, which is used in the modelling of propagation of electrical signal in the communication lines, travelling waves in non-linear elastic bars [25], plasma waves [11] and heat transfer in spongy medium [11]:

$$\mu u_{xxxxxx} + u_{ttxx} + \alpha^2 u_{xx} - \beta^2 (q(u))_{tt} = f(x,t). \tag{3}$$

The main purpose of this article is to investigate solitary wave solutions of such equations. In the analogues of the Boussinesq equation, which has been researched and whose examples are given above, nonlinearity is found in the terms having derivatives with respect to the space variables. On the other hand, Boussinesq equations which contain nonlinearity in time-derivative terms, also appear. The following can be given as an example to these equations:

$$u_{ttxx} + \alpha^2 u_{xx} - \beta^2 (q(u))_{tt} = f(x,t). \tag{4}$$

Here, $q(\xi)$ is a continuous function. For such an equation, the solvability of different boundary-value problems has been proven by imposing certain conditions into the function $q(\xi)$ [1]. As a continuation of this study, the boundary-value problem was solved to find approximate solutions and the stability of the solution under certain conditions were investigated [2]. In addition, solitary wave solutions were obtained by using polynomial function and tanh function methods in the case of $q(\xi) = \xi^3$ for the same type of equation [3].

In this study, the the analogue of the Boussinesq equation, of which we will find the solitary wave solutions in particular, is the following equation:

$$(q(u))_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxxxx} = 0. \tag{5}$$

Here, $q(\xi)$ is a continuous function. The solitary wave solutions of equation (5) corresponding to the $\mu = 0$ ve $\mu = 1$ cases for $q(\xi) = \xi^2$ will be investigated using the polynomial function method.

2 Method: polynomial function method

Step 1: x and t are independent variables. Let us look at the following nonlinear partial differential equation, where u is the dependent variable.

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \tag{6}$$

Here, $P(u)$ is a polynomial function. To obtain the solitary wave solutions of the equation (6) which includes partial derivatives, the following change of variables

$$\xi = x - ct \tag{7}$$

is performed such that the equation is transformed into an ordinary differenaital equation. Therefore, $u(x,t) = u(\xi)$.

Step 2: The partial differential equation in the form of (6) is converted into the below ordinary differential equation after performing the change of variables (7):

$$p(u, -cu', u', c^2u'', u'', \dots) = 0 \tag{8}$$

Step 3: Equation (8) is integrated in sufficient amount where the integration constants are taken as zero.

Step 4: For obtaining m (to be described in step 6), the degree of the highest order term in Equation (8) is taken equal to the highest degree of the nonlinear terms, such that the equation is homogenized. For this purpose, the degree of the nonlinear term will be calculated as follows:

$$deg\left[u^q\left(\frac{d^r u}{d\xi^r}\right)^s\right] = mq + s(m+r) \tag{9}$$

Step 5: The function ϕ is taken as the solution to the ordinary differential equation given below:

$$(\phi'(\xi))^2 = \alpha\phi^2(\xi) + \beta\phi^3(\xi) + \gamma\phi^4(\xi). \tag{10}$$

Here, α, β, γ are unknown constants.
Step 6: The solution of the equation (8) is searched in the following form:

$$u(x,t) = \sum_{i=0}^{m} a_i\phi^i \tag{11}$$

Here, a_i coefficients are unknowns in the beginning.
Step 7: By placing the function (11) in the equation (11), we have a polynomial in the powers of the function ϕ. After the polynomial coefficients are set as zero, an overdetermined algebraic equation system that is dependent on $a_i, \alpha, \beta, \gamma$ is obtained.
Step 8: This algebraic equation system is solved using Mathematica and unknown constants $a_i, \alpha, \beta, \gamma$ are obtained.
Step 9: The coefficients that we found are put into the equation (11). Changing the variables as given in (7), a solitan solution of the equation (6) is obtained.

After step 3 of the polynomial method, the equation (5) takes the form below.

$$c^2u^2 - u - c^2u_{\xi\xi} - \mu u_{\xi\xi\xi\xi} = 0 \tag{12}$$

3 Results and discussion

Case i. First, for $\mu = 0$, equation (12) becomes

$$c^2u^2 - u - c^2u_{\xi\xi} = 0. \tag{13}$$

After the homogenization of equation (13) as explained in step 4, the number m is obtained as follows:

$$2m = m + 2$$
$$m = 2$$

Then, according to step 6 of the polynomial method, we will search for the solution of equation (10) as follows:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi). \tag{14}$$

Here, $\phi(\xi)$ is the solution to the ordinary differential equation (10) given in step (5) for the case $\gamma = 0$.
By substituting the function (14) in the equation (13), we will obtain a polynomial in powers of $\phi(\xi)$, with coefficients depending on the constants $a_0, a_1, a_2, \alpha, \beta$ and c. If we equate those polynomial coefficients to zero, we obtain an overdetermined algebraic equation system with respect to the unknown constants $a_0, a_1, a_2, \alpha, \beta$ and c as given below:

$$\begin{aligned} -a_1 - (1+\alpha c^2) &= 0, \\ \frac{1}{2}\Big(-a_1\big(-2a_1+3\beta\big)c^2 - a_2\big(2+8\alpha c^2\big)\Big) &= 0, \\ a_2\big(2a_1-5\beta\big)c^2 &= 0, \\ a_2^2c^2 &= 0 \end{aligned} \tag{15}$$

Solving this equation system in Mathematica, we find the following:

$$a_0 = 0, \quad a_1 \neq 0, \quad a_2 = 0, \quad c \neq 0, \quad \alpha = -\frac{1}{c^2}, \quad \beta = \frac{2a_1}{3} \tag{16}$$

As seen from this solution the coefficients c and a_1 are arbitrary constants different than zero. For simplicity, if we equate $c = 1$, $a_1 = 1$, the constants in (16) takes the values below:

$$a_0 = 0, \quad a_1 = 1, \quad c = 1, \quad \alpha = -1, \quad \beta = \frac{2}{3}$$

According to these values, the ordinary differential equation (10) has the following solution:

$$\phi(\xi) = \frac{3}{2}\left(1 + \tan\left[\frac{1}{2}(\xi + \sqrt{3})\right]^2\right) \tag{17}$$

In this regard, the solution of the partial differential equation for $\mu = 0$ is found as given below after considering the change of variables $\xi = x - ct$:

$$u(x,t) = \frac{3}{2}\left(1 + \tan\left[\frac{1}{2}(x - t + \sqrt{3})\right]^2\right) \tag{18}$$

The following graph of the solution can be plotted using Mathematica:

Case ii. Now, let us consider equation (12) for $\mu = 1$ which becomes:

$$c^2u^2 - u - c^2u_{\xi\xi} - u_{\xi\xi\xi\xi} = 0 \tag{19}$$

Homogenizing the equation (19) as described in step 4, we find the value of m as follows:

$$2m = m + 4$$
$$m = 4$$

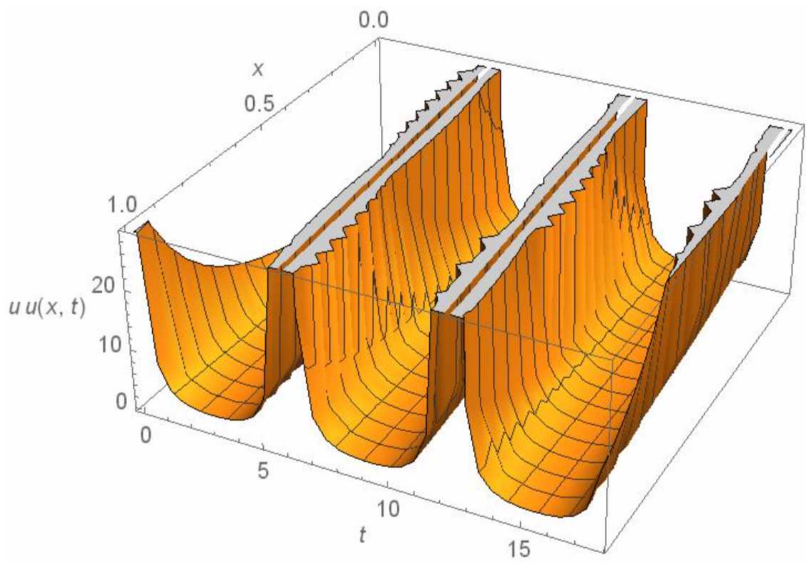

Fig. 1: The behavior of the solution of equation (5) for $\mu = 0$

In this case, we search for the solution of the equation as given below according to the step 6:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + a_3\phi^3(\xi) + a_4\phi^4(\xi) \tag{20}$$

Here, $\phi(\xi)$ is the solution of the ordinary differential equation (10) given in step (5) for the case β. Substituting the function (20) in the equation (19), we will have a polynomial in powers of $\phi(\xi)$, with coefficients depending on the constants $a_0, a_1, a_2, a_3, a_4, \alpha, \gamma$ and c. Similarly, when we set those polynomial coefficients to zero, an overdetermined algebraic equation system is obtained with respect to the unknown constants $a_0, a_1, a_2, a_3, a_4, \alpha, \gamma$ and c as given below:

$$\begin{aligned} a_0(-1+a_0c^2) &= 0, \\ -a_1(1+\alpha^2-2a_0c^2+ac^2) &= 0, \\ a_1^2c^2 - a_2(1+16\alpha^2-2a_0c^2+4\alpha c^2) &= 0, \\ -a_3(1+81\alpha^2-2a_0c^2+9\alpha c^2) - 2a_1[-a_2c^2+(10\alpha+c^2)\gamma] &= 0, \\ a_2^2c^2+2a_1a_3c^2 - a_4(1+256\alpha^2-2a_0c^2+16\alpha c^2) - 6a_2(20\alpha+c^2)\gamma &= 0, \\ 2[a_2a_3c^2 - 6a_3(34\alpha+c^2)\gamma + a_1(a_4c^2-12\gamma^2)] &= 0, \\ a_3^2c^2 + 2[-10a_4(52\alpha+c^2)\gamma + a_2(a_4c^2-60\gamma^2)] &= 0, \\ 2a_3(a_4c^2-180\gamma^2) &= 0, \\ a_4(a_4c^2-840\gamma^2) &= 0. \end{aligned} \tag{21}$$

Solving this equation system in Mathematica, we find the following:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 \neq 0, \quad c = \sqrt{\frac{13}{6}}, \quad \alpha = -\frac{c^2}{52}, \quad \gamma = \frac{\sqrt{a_4}c}{2\sqrt{210}} \tag{22}$$

Since the coefficients c and a_4 are arbitrary numbers different than zero, we can take $a_4 = 1$, $c = \sqrt{\frac{13}{6}}$. Then, the constants in (22) becomes as seen below:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 1, \quad c = \sqrt{\frac{13}{6}}, \quad \alpha = -\frac{1}{24}, \quad \gamma = \frac{\sqrt{13}}{12\sqrt{35}}$$

According to these values, the ordinary differential equation (10) has the following solution:

$$\phi(\xi) = \frac{\cot\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}\xi\right)\right]\sqrt{\sqrt{455} + \sqrt{455}\tan\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}\xi\right)\right]^2}}{\sqrt{26}} \tag{23}$$

Accordingly, the solution of the partial differential equation (5) for can be written as below:

$$u(\xi) = \frac{1}{676}\cot\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}\xi\right)\right]^4\left(\sqrt{455} + \sqrt{455}\tan\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}\xi\right)\right]^2\right)^2 \tag{24}$$

The solution of the partial differential equation for $\mu = 1$ is found as given below after considering the change of variables $\xi = x - ct$:

$$u(x,t) = \frac{1}{676}\cot\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}x - \sqrt{\frac{13}{6}t}\right)\right]^4\left(\sqrt{455} + \sqrt{455}\tan\left[\frac{1}{12}\left(-2\sqrt{2730} - \sqrt{6}x - \sqrt{\frac{13}{6}t}\right)\right]^2\right)^2 \tag{25}$$

We obtain the following graph of the solution by the help of Mathematica:

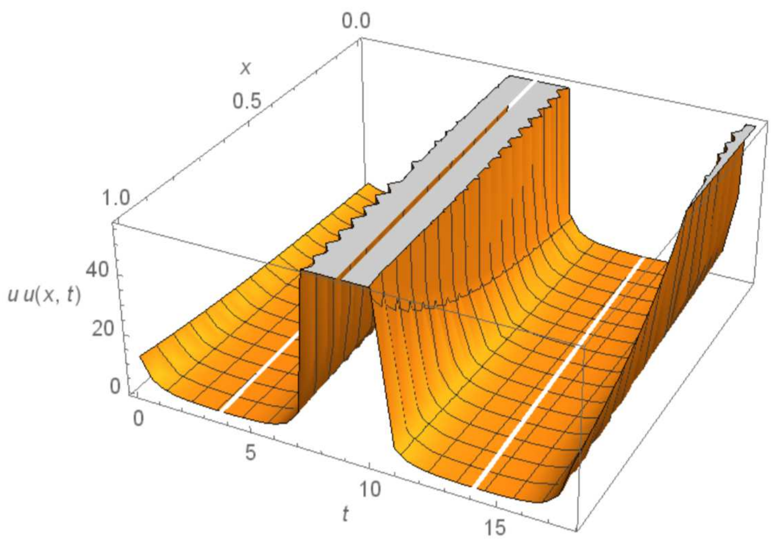

Fig. 2: The behavior of the solution of equation (5) for $\mu = 1$

4 Conclusion

In this paper, the soliton solutions of the nonlinear analogues of the Boussinesq Equation (5) were found using the polynomial function method at Mathematica software for the cases $\mu = 0$ and $\mu = 1$ by. The behaviors of the solitary wave solutions are shown in the plotted graphs.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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