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Convex functions in selected theory of functions

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Abstract: A valuable function $f(z)$ is univalent in a domain $D\subset\mathbb{C}$ if it is not get the same value twice; if $f(z_1)\neq f(z_2)$ points for all z_1 and z_2 in D with $z_1\neq z_2$. $f(z)$ locally said to be uniform at the point of function $z_0\in D$ if it is univalent in some neighborhood of z_0. For $f(z)$ is an analytic function, the condition $f'(z_0)\neq 0$ is equal to local unity in z_0. An analytical univalent function is called an appropriate mapping due to its angle protection feature. First of all, we will consider the class of phantomic and univalent S functions in the open unit disk $D=\{z:|z|<1\}$, normalized by the conditions $f(0)=0$ and $f'(0)=1$.

In this paper, it is known to be applicable to any real 2^{-r} value inequality , $4.2^{-r}\left(1-2^{-r}\right)=\frac{2^{r+2}-4}{2^{2r}}\leq 1$ **,** $\left(2^{-r}-2\geq 0,\ r\geq 1\right)$ our claim is proved that $|z|<1$ an arbitrary point z_0.

Keywords: Convex function, class K, local univalent.

1 Introduction

In the theory of univalent functions, two classes of functions are examined in more detail;

$$w=f(z)=\frac{z}{1-z}=z+z^2+z^3+z^4+z^5+\ldots \tag{1}$$

it is described as $|z|<1$ in unit disk D.

(a) Starlike fields with zero point as center (Class St),
(b) Konveks fields (Class K)

is represented. Löwner has proved two inequalities in class K functions.[2]

(i) $\frac{R}{1+R}\leq|f(z)|\leq\frac{R}{1-R},\quad |z|=R<1$
(ii) $\frac{1}{(1+R)^2}\leq\left|f'(z)\right|\leq\frac{R}{(1-R)^2},\quad |z|\leq R<1$. Bieberbach added the third inequality.[3]
(iii) $\left|arc f'(z)\right|\leq 2\arcsin R,\quad |z|\leq R<1.$

The aim of the present study is to concentrate these three inequalities, according to the boundary sets of the three processes.

$$\frac{f(z)}{z}=\frac{\frac{z}{1-z}}{z}=\frac{z+z^2+z^3+z^4+z^5+\ldots}{z}=1+z+z^2+z^3+z^4+z^5+\cdots;$$

$$\frac{zf'(z)}{f(z)}=\frac{z\frac{1}{(1-z)^2}}{\frac{z}{1-z}}=\frac{1}{1-z}=1+z+z^2+z^3+z^4+z^5+\cdots;$$

$$f'(z)=\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+5z^4+6z^5+\cdots.$$

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$f(z)$ is asked upon the assumption that class K has a function. Reaching the repetitive application of the Schwarz Lemma, the reel parts are in the $|z|<1$ grater then $\frac{1}{2}$ range in the form of $\frac{f(z)}{z}$ and $\frac{zf'(z)}{f(z)}$ [5].

The transformation of classes K and St into distortion and rotation sets; The main part is the result $\frac{f(z)}{z}$ and $\frac{zf'(z)}{f(z)}$ for selected function of class K.

Theorem 1. *$w=f(z)=\frac{z}{1-z}$ and $|z|<1$ is regular, we know that* $\operatorname{Re} f(z)\geq 0$ *is in unit disk $D=\{z\in\mathbb{C}:|z|<1\}$. Then $|z_0|\leq R<1$. and $f(z_0)$ located inside diameter of closed circular disk unit $\frac{1-R}{1+R}$ ... $\frac{1+R}{1-R}$ w-plane within the real axis.*

Proof. If we choose $f(z)=\frac{z}{1-z}=z+z^2+z^3+z^4+z^5+...$ then

$$w_1=f_1(z)=\frac{f(z)-1}{f(z)+1}$$

taking $|z|<1$ from the unit disk, we have $f_1(z)\leq 1$. Because $f_1(0)=0$ and $f_1(z)$ fulfills the requirements of Schwartz's Lemma. From here $f_1(z)$ is in $w_1\leq R$. Therefore

$$w_1=f_1(z)=\frac{\frac{z}{1-z}-1}{\frac{z}{1-z}+1}=2z-1.$$

Then we take

$$w=\frac{1+(2z-1)}{1-(2z-1)}=\frac{z}{1-z}$$

as shown in the proof shown on the circle. These functions are proved by return to the w-plane.

Theorem 2. *$w=f(z)=\frac{z}{1-z}=z+z^2+z^3+z^4+z^5+...$ function is in class K. Then*

$$w=h(z)=\frac{f(z_0)-f\left(\frac{z_0-z}{1-\overline{z_0}z}\right)}{f'(z_0)\left(1-|z_0|^2\right)} \tag{2}$$

in class K. Here is the point z_0 where an arbitrary point $|z|<1$.

Proof. Linear function $\xi=\frac{z_0-z}{1-\overline{z_0}z}$, converts $|z|<1$ to $|\xi|<1$, this obtained $z_0=z$ and $\xi=0$. $z=0$ and we take that $\xi=z_0$. This function

$$w_1=f(\xi)=f\left(\frac{z_0-z}{1-\overline{z_0}z}\right),$$

$|z|<1$, $w=f(z)$ is based on the area that appears with G, which is caused by the letter $|z|<1$. Hence we obtained that $z=0$ and $w_1=f(z_0)$. For the function $w=h(z)$ it will be $h(0)=0$. If we take derivative

$$h'(z)=\frac{f'\left(\frac{z_0-z}{1-\overline{z_0}z}\right)}{f'(z_0)\left(1-|z_0|^2\right)}, \tag{3}$$

thus $h'(0)=0$. Finally, $w=h(z)$ the resulting $|z|<1$ pictorial field G in $w=h(z)$ also appears with an extension of twist and rotation, so that G is univalent and convex. The result is $h(z)$ in class K.

Theorem 3. *(Main Theorem) Let's choose $w=f(z)=\frac{z}{1-z}=z+z^2+z^3+z^4+z^5+...$ in class K;*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\geq 2^{-r} \text{ and } |z|<1, \left(0\leq 2^{-r}\leq\frac{1}{2}\right).$$

It is obtained in

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)\geq\frac{1}{4(1-2^{-r})}\geq 2^{-r},$$

form for class K functions.

Proof. Let $w=f(z)=\frac{z}{1-z}=z+z^2+z^3+z^4+z^5+...$, function is in class **K**; let z_0 have an arbitrary but fixed point in $|z|<1$ and $|z_0|<R<1$. Consider the $w=h(z)$ function in (2) next to f(z). For $z=z_0$ in (2) and (3)

$$h(z_0)=\frac{f(z_0)}{f'(z_0)\left(1-|z_0|^2\right)}. \tag{4}$$

$$h'(z_0)=\frac{1}{f'(z_0)\left(1-|z_0|^2\right)^2}. \tag{5}$$

From Theorem 2, Since $w=h(z)$ is a function of class K , it must be $|z|<1$

$$\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right)\geq 2^{-r}. \tag{6}$$

As a result, the process performs the following statements

$$\begin{aligned}g(z)&=\left(\frac{1}{1-2^{-r}}\right)\left(\frac{zh'(z)}{h(z)}-2^{-r}\right)=\left(\frac{1}{1-2^{-r}}\right)\left(\frac{1-z}{1-|z|^2}-2^{-r}\right)\\&=\left(\frac{2^r}{2^r-1}.\frac{1-z}{1-|z|^2}\right)-\left(\frac{2^r}{2^r-1}.\frac{1}{2^r}\right)=\left(\frac{2^r}{2^r-1}.\frac{1-z}{1-|z|^2}\right)-\frac{1}{2^r-1},\end{aligned}$$

So that we obtained

$$\frac{1}{2^r-1}\left(\frac{2^r(1-z)}{1-|z|^2}-1\right),$$

$g(0)=1$ and $\operatorname{Re}g(z)\geq 0$ with $|z|<1$ we took Theorem 1. Accordingly, the real line $g(z_0)$, $\frac{1-R}{1+R}...\frac{1+R}{1-R}$ located on a circular disk as a diameter above the distance. So that $\frac{z_0h'(z_0)}{h(z_0)}$. The diameter of the axis of the real axis

$$\left(1-2^{-r}\right)\frac{1-R}{1+R}+2^{-r}...\left(1-2^{-r}\right)\frac{1+R}{1-R}+2^{-r}$$

is located on the disc. Where

$$Re\left(.\frac{1-|z|^2}{1-z}\right)\geq\frac{1}{(1-2^{-r}).\left(\frac{1+R}{1-R}\right)+2^{-r}}\geq\frac{2^r(1-R)}{(2^r-2)(1+R)}=\left(1-\frac{2}{2^r-2}\right)\left(\frac{1-R}{1+R}\right), \tag{7}$$

is located. Now we use (4) and (5)

$$\frac{f(z_0)}{z_0}=\frac{1}{1-|z_0|}.\left(\frac{1}{z_0\frac{h'(z_0)}{h(z_0)}}\right) \tag{8}$$

from (7) and (8) we obtained from the following

$$
\begin{aligned}
Re\left(\frac{f(z_0)}{z_0}\right) &= \frac{1}{1-R^2}\,\mathrm{Re}\left(\frac{1}{z_0\frac{h'(z_0)}{h(z_0)}}\right)\\
&\geq \frac{1}{(1-R^2)}\,\frac{1}{(1-2^{-r}).\left(\frac{1+R}{1-R}\right)+2^{-r}}\\
&= \frac{1}{(1-2^{-r})(1+R)^2+2^{-r}(1-R^2)}\\
&= \frac{2^r}{2^r(1+R)^2-2R(1+R)}\\
&\geq \frac{1}{(1+R)(2-2.2^{-r})+R^2(1-2.2^{-r})}\\
&= \frac{1}{\frac{2^r+2^r.2R-2R+2^rR^2-2R^2}{2^r}}\\
&= \frac{1}{\frac{2^r(1+R)^2-2R(1+R)}{2^r}}\\
&= \frac{2^r}{(R+1)\left[2^r(R+1)-2R\right]}.
\end{aligned}
\tag{9}
$$

We put it on the following theorem

$$
F(R) = 1+R(2-2.2^{-r})+R^2(1-2.2^{-r}) = \frac{(R+1)\left[2^r(R+1)-2R\right]}{2^r}
$$

so,

$$
F'(R) = (2-2.2^{-r})+2R(1-2.2^{-r}) = 2+2R-\frac{1+2R}{2^{r-1}} > 0 \text{ for } 0 \leq R \leq 1 \text{ and } 0 \leq 2^{-r} \leq \frac{1}{2}.
$$

Therefore $F(R) \leq F(1)$ in $0 \leq R \leq 1$. If so, from (9) we obtained that

$$
Re\left(\frac{f(z_0)}{z_0}\right) > \frac{1}{(1+R)(2-2.2^{-r})+R^2(1-2.2^{-r})} = \frac{1}{4(1-2^{-r})} = \frac{1}{\frac{2^{r+2}-4}{2^r}} = \frac{2^r}{2^{r+2}-4} \geq \frac{1}{2^r}.
$$

Finally, it is known to be applicable to any real 2^{-r} value inequality

$$
4.2^{-r}\left(1-2^{-r}\right) = \frac{2^{r+2}-4}{2^{2r}} \leq 1, \quad (2^r-2 \geq 0, \quad r \geq 1).
$$

Therefore our claim is proved that $|z| < 1$ an arbitrary point z_0. The proof is completed.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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[5] Wie ich nachtraglich erfuhr, ist dieser Satz und Satz V auch in einer kürzlich erschienenen Arbeit von A. Marx enthalten. A. Marx, Untersunchungen über schlichte Abbildungen, Math. Annalen 107, 1932, S. 40-67. Die Beweise sind aber dort andere als hier.