

Numerical solutions of fractional Boussinesq-Whitham-Broer-Kaup and diffusive Predator-Prey equations with conformable derivative

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Abstract: In this article, the residual power series method (RPSM) is used to obtain numerical solutions of the time-fractional Boussinesq-Whitham-Broer-Kaup (BBWK) and Diffusive Predator-Prey equations using the conformable fractional derivative definition. This definition is simple, effective and reliable in the solution procedure of the fractional differential equations that have complicated solutions with classical fractional derivative definitions like Caputo and Riemann-Liouville.

Keywords: The residual power series method, time-fractional Boussinesq-Whitham-Broer-Kaup equation, time-fractional Diffusive Predator-Prey equation, conformable fractional derivative, numerical Solution.

1 Introduction

Fractional analysis has been the subject of numerous applications in many areas of science and engineering branches such as fractional differential equations (FDEs) [7,24], mathematical physics, biology, image and signal processing, electrical network, fluid flow, viscoelasticity and control in recent years [8,9].

There are some common methods that are used to obtain approximate or analytical solutions of nonlinear fractional ordinary and partial differential equations in the literature [28,29,30,31,32,33,34,35,36,37]. These methods include, Laplace analysis method (LAM) [15] for the constant coefficient fractional differential equations, Adomian decomposition method (ADM) [13] for the dynamics of the fractional giving up smoking model of fractional order, homotopy perturbation method (HPM) [25] for the nonlinear fractional Schrödinger equation, homotopy analysis method (HAM) [18] for the conformable fractional Nizhnik–Novikov–Veselov system, differential transformation method (DTM) [19] for the convergence of fractional power series, Elzaki projected differential transform method [22] for system of linear and nonlinear fractional partial differential equations and perturbation-iteration algorithm (PIA) [20] for ordinary fractional differential equations.

In this article, the residual power series method [17,14,4,12,2] is used to obtain new approximate solutions of time-fractional Boussinesq-Whitham-Broer-Kaup equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial v}{\partial x} + u \frac{\partial u}{\partial x} = 0 \quad (1)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial(uv)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (2)$$

and diffusive predator-prey equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}} \right) u^2 - u^3 - uv \quad (3)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + \kappa uv - \beta v - \delta v^3 \quad (4)$$

where $u = u(x, t)$ and $v = v(x, t)$. In the RPSM, the coefficients of the power series are calculated by means of the concept of residual error with the help of one or more variable algebraic equation chains, and finally, a so-called truncated series solution is obtained [17].

The major improvement of the RPSM is that it can be implemented to the problem directly without linearization, perturbation or discretization and without any transformation by selecting appropriate initial conditions [14].

After giving a few preliminary definitions and brief description of the RPSM, we presented the solution procedure of two nonlinear fractional partial differential equations that shows the reliability and efficiency of the method. Also figures and tables are presented in order to compare numerical solutions of considered equations. Finally, we discussed the obtained results in a section as conclusion.

2 Preliminaries

There are a few definitions of fractional derivative of order $\alpha > 0$ in the literature. The most widely used are the Riemann-Liouville and Caputo fractional derivatives.

Definition 1. The Riemann-Liouville fractional derivative operator $D^\alpha f(x)$ for $\alpha > 0$ and $q - 1 < \alpha < q$ defined in [8, 9] as:

$$D^\alpha f(x) = \frac{d^q}{dx^q} \left[\frac{1}{\Gamma(q - \alpha)} \int_\alpha^x \frac{f(t)}{(x - t)^{\alpha + 1 - q}} dt \right]. \quad (5)$$

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ for $n \in \mathbb{N}$, $n - 1 < \alpha < n$, D_*^α , defined in [6] as:

$$D_*^\alpha f(x) = J^{n - \alpha} D^n f(x) = \frac{1}{\Gamma(n - \alpha)} \int_\alpha^x (x - t)^{n - \alpha - 1} \left(\frac{d}{dt} \right)^n f(t) dt. \quad (6)$$

Recently, a new definition of a fractional derivative called "conformable fractional derivative" has been proposed by R. Khalil et al. [16].

Definition 3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ is a function. α -th order "conformable fractional derivative" of function f defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1 - \alpha}) - (f)(t)}{\varepsilon} \quad (7)$$

for all $t > 0$, $\alpha \in (0, 1)$. The properties of this new definition are given in the following theorem [16]

Theorem 1. Let $\alpha \in (0, 1]$ and f, g functions are α -differentiable at $t > 0$, then

- (1) $T_\alpha(mf + ng) = mT_\alpha(f) + nT_\alpha(g)$ for all $m, n \in \mathbb{R}$
- (2) $T_\alpha(t^p) = pt^{p-\alpha}$ for all p
- (3) $T_\alpha(f \cdot g) = fT_\alpha(g) + gT_\alpha(f)$
- (4) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$
- (5) $T_\alpha(c) = 0$ for all constant functions $f(t) = c$
- (6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$.

Definition 4. Let f is a function with n variables x_1, \dots, x_n , and the conformable partial derivatives of f of order $\alpha \in (0, 1]$ in x_i is defined as follows [5]

$$\frac{d^\alpha}{dx_i^\alpha} f(x_1, \dots, x_n) = \lim_{\varepsilon \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, \dots, x_n) - f(x_1, \dots, x_n)}{\varepsilon} \tag{8}$$

Definition 5. The conformable integral of a function f for $a \geq 0$ is defined in [23] as

$$I_\alpha^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-\alpha}} dt. \tag{9}$$

3 Description of the residual power series method

In this section we are going to introduce some important definitions and theorems about residual power series

Theorem 2. Suppose that f is an infinitely α -differentiable function at a neighborhood of a point t_0 for some $0 < \alpha \leq 1$, then f has the fractional power series expansion of the form [1]:

$$f(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t - t_0)^{k\alpha}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{\frac{1}{\alpha}}, \quad R > 0. \tag{10}$$

Here $(T_\alpha^{t_0} f)^{(k)}(t_0)$ represents the application of the fractional derivative k -times.

Definition 6. A multiple fractional power series about $t_0 = 0$ is defined by $\sum_{n=0}^{\infty} f_n(x)t^{n\alpha}$ for $0 \leq m - 1 < \alpha < m$, where t is a variable and $f_n(x)$ are functions called the coefficients of the series. [3, 12].

Theorem 3. Assume that $u(x, t)$ has a multiple fractional power series representation at $t_0 = 0$ of the form [3]

$$u(x, t) = \sum_{n=0}^{\infty} f_n(x)t^{n\alpha}, \quad 0 \leq m - 1 < \alpha < m, \quad x \in I, \quad 0 \leq t \leq R^{\frac{1}{\alpha}}. \tag{11}$$

If $u_t^{(n\alpha)}(x, t)$, $n = 0, 1, 2, \dots$ are continuous on $I \times (0, R^{\frac{1}{\alpha}})$, then $f_n(x) = \frac{u_t^{(n\alpha)}(x, 0)}{\alpha^n n!}$.

To clarify the basic concept of RPSM, let's take a nonlinear fractional differential equation of the form:

$$T_\alpha u(x, t) + N[x]u(x, t) + R[x]u(x, t) = c(x, t), \quad x \in \mathbb{R}, \quad n - 1 < n\alpha \leq n, \quad t > 0 \tag{12}$$

expressed by initial condition

$$f_0(x) = u(x, 0) = f(x) \tag{13}$$

where $R[x]$ is a linear, $N[x]$ is a non-linear operator and $c(x, t)$ are continuous functions.

The RPSM method made up of stating the solution of the equation (12) subject to (13) as a fractional power series expansion around $t = 0$.

$$f_{(n-1)}(x) = T_t^{(n-1)\alpha} u(x, 0) = h(x). \tag{14}$$

The expansion form of the solution is given by

$$u(x, t) = f(x) + \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \tag{15}$$

In the next step, the k -truncated series of $u(x, t)$, namely $u_k(x, t)$ can be written as:

$$u_k(x, t) = f(x) + \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \tag{16}$$

If the (1) RPS approximate solution $u_1(x, t)$ is

$$u_1(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha^n} \tag{17}$$

then $u_k(x, t)$ could be reformulated as

$$u_k(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha^n} + \sum_{n=2}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \tag{18}$$

for $0 < \alpha \leq 1, 0 \leq t < \mathbb{R}^{\frac{1}{\alpha}}, x \in I$ and $k = 2, 3, 4, \dots$ First we express the residual function as

$$Res(x, t) = T_\alpha u(x, t) + N[x]u(x, t) + R[x]u(x, t) - c(x, t) \tag{19}$$

and the k . residual function as

$$Res_k(x, t) = T_\alpha u_k(x, t) + N[x]u_k(x, t) + R[x]u_k(x, t) - g(x, t), k = 1, 2, 3, \dots \tag{20}$$

It is clear that $Res(x, t) = 0$ and $\lim_{k \rightarrow \infty} Res_k(x, t) = Res(x, t)$ for each $x \in I$ and $0 \leq t$. In fact this lead to $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Res_k(x, t)$ for $n = 1, 2, 3, \dots, k$ because in the conformable sense, the fractional derivative of a constant is zero [17, 14, 4]. Solving the equation $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Res_k(x, 0) = 0$ gives us the desired $f_n(x)$ coefficients. Thus the $u_n(x, t)$ approximate solutions can be obtained respectively.

We can express the following theorem for the convergence analysis for the method [21].

Theorem 4. *If there exists a fixed constant $0 < K < 1$ such that $\|u_{n+1}(x, y, t)\| \leq K \|u_n(x, y, t)\|$ for all $n \in \mathbb{N}$ and $0 < t < R < 1$, then the sequence of approximate solution converges to an exact solution.*

Proof. For all $0 < t < R < 1$, we have

$$\|u(x, y, t) - u_n(x, y, t)\| = \left\| \sum_{i=n+1}^{\infty} u_i(x, y, t) \right\| \leq \sum_{i=n+1}^{\infty} \|u_i(x, y, t)\| \leq \|f(x, y)\| \sum_{i=n+1}^{\infty} K^i = \frac{K^{n+1}}{1-K} \|f(x, y)\| \xrightarrow{n \rightarrow \infty} 0. \tag{21}$$

4 Application of residual power series method

4.1 Solution of time-Fractional Boussinesq-Whitham-Broer-Kaup equations

Consider the nonlinear time fractional BWBK equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial v}{\partial x} + u \frac{\partial u}{\partial x} = 0 \tag{22}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial(uv)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{23}$$

with the initial conditions obtained from the exact solutions as [27]

$$u(x, 0) = f(x) = -\frac{\omega \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx \right) \right)}{k} \tag{24}$$

$$v(x, 0) = g(x) = \frac{\omega^2 \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx \right) \right)}{k^2} - \frac{\omega^2 \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx \right) \right)^2}{2k^2}. \tag{25}$$

For residual power series

$$u(x, t) = f(x) + \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \tag{26}$$

and k -truncated series of $u(x, t)$

$$u_k(x, t) = f(x) + \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, \quad k = 1, 2, 3, \dots \tag{27}$$

The k_1 -th residual functions of time fractional BWBK equation is:

$$Resu_k(x, t) = \frac{\partial^\alpha u_k}{\partial t^\alpha} + \frac{\partial v_k}{\partial x} + u_k \frac{\partial u_k}{\partial x} \tag{28}$$

$$Resv_k(x, t) = \frac{\partial^\alpha v_k}{\partial t^\alpha} + \frac{\partial(u_k v_k)}{\partial x} + \frac{\partial^3 u_k}{\partial x^3}. \tag{29}$$

For determining the coefficients $f_1(x)$ and $g_1(x)$, in $u_k(x, t)$, we should subrogate the 1st truncated series $u_1(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha}$ and $v_1(x, t) = g(x) + g_1(x) \frac{t^\alpha}{\alpha}$ into the 1st truncated residual functions

$$Resu_1(x, t) = f_1(x) + g'(x) + \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} \right) \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} \right) + \frac{t^\alpha g_1'(x)}{\alpha} \tag{30}$$

$$Resv_1(x, t) = g_1(x) + f^{(3)}(x) + \frac{t^\alpha f_1^{(3)}(x)}{\alpha} + \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} \right) \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} \right) + \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} \right) \left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} \right). \tag{31}$$

Now we take $t = 0$ through the equations $Resu_1(x, t)$ and $Resv_1(x, t)$ to obtain following equations

$$Resu_1(x, 0) = f_1(x) + f(x)f'(x) + g'(x) \tag{32}$$

$$Resv_1(x, 0) = g_1(x) + g(x)f'(x) + f(x)g'(x) + f^{(3)}(x). \quad (33)$$

Thus, for $Res_1(x, 0) = 0$, we get

$$f_1(x) = -f(x)f'(x) - g'(x) \quad (34)$$

$$g_1(x) = -g(x)f'(x) - f(x)g'(x) - f^{(3)}(x). \quad (35)$$

Therefore, we obtain the 1st RPS approximate solutions of the time fractional BWBK equation as

$$u_1(x, t) = f(x) + \frac{t^\alpha (-f(x)f'(x) - g'(x))}{\alpha} \quad (36)$$

$$v_1(x, t) = g(x) + \frac{t^\alpha (-g(x)f'(x) - f(x)g'(x) - f^{(3)}(x))}{\alpha}. \quad (37)$$

Again, to determine the second unknown coefficients $f_2(x)$ and $g_2(x)$, we subrogate the 2^{nd} truncated series solutions $u_2(x, t) = f(x) + f_1(x)\frac{t^\alpha}{\alpha} + f_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ and $v_2(x, t) = g(x) + g_1(x)\frac{t^\alpha}{\alpha} + g_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ into the 2^{nd} truncated residual functions to obtain following equations

$$Resu_2(x, t) = f_1(x) + g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^\alpha f_2(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2} + \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} \right) \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2} \right) \quad (38)$$

$$Resv_2(x, t) = g_1(x) + f^{(3)}(x) + \frac{t^\alpha f_1^{(3)}(x)}{\alpha} + \frac{t^{2\alpha} f_2^{(3)}(x)}{2\alpha^2} + \frac{t^\alpha g_2(x)}{\alpha} + \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} \right) \left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2} \right) + \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2} \right) \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2} \right). \quad (39)$$

Now, applying T_α on both sides of $Resu_2(x, t)$ and equating the equations above to zero for $t = 0$ gives:

$$f_2(x) = -f_1(x)f'(x) - f(x)f_1'(x) - g_1'(x) \quad (40)$$

$$g_2(x) = -g_1(x)f'(x) - f_1(x)g'(x) - g(x)f_1'(x) - f(x)g_1'(x) - f_1^{(3)}(x). \quad (41)$$

Therefore the 2nd RPS approximate solutions of time-fractional BWBK equation is obtained as:

$$u_2(x, t) = f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} (-f_1(x)f'(x) - f(x)f_1'(x) - g_1'(x))}{2\alpha^2} \quad (42)$$

$$v_2(x, t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} (-g_1(x)f'(x) - f_1(x)g'(x) - g(x)f_1'(x) - f(x)g_1'(x) - f_1^{(3)}(x))}{2\alpha^2}. \quad (43)$$

In the same manner, we apply the same procedure for $n = 3$ to obtain the following results.

$$f_3(x) = -f_2(x)f'(x) - 2f_1(x)f'_1(x) - f(x)f'_2(x) - g'_2(x) \tag{44}$$

$$u_3(x,t) = f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} + \frac{t^{3\alpha} (-f_2(x)f'(x) - 2f_1(x)f'_1(x) - f(x)f'_2(x) - g'_2(x))}{6\alpha^3} \tag{45}$$

$$g_3(x) = -g_2(x)f'(x) - f_2(x)g'(x) - 2g_1(x)f'_1(x) - g(x)f'_2(x) - 2f_1(x)g'_1(x) - f(x)g'_2(x) - f_2^{(3)}(x) \tag{46}$$

$$v_3(x,t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2} + \frac{t^{3\alpha} (-g_2(x)f'(x) - f_2(x)g'(x))}{6\alpha^3} + \frac{t^{3\alpha} (-2g_1(x)f'_1(x) - g(x)f'_2(x) - 2f_1(x)g'_1(x) - f(x)g'_2(x) - f_2^{(3)}(x))}{6\alpha^3} \tag{47}$$

In Table 1-2, the third order approximate RPSM solutions of time-fractional Boussinesq-Whitham-Broer-Kaup equations are compared numerically with exact solutions

$$u(x,t) = -\frac{\omega \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx + \frac{t^\alpha \omega}{\alpha} \right) \right)}{k} \tag{48}$$

$$v(x,t) = \frac{\omega^2 \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx + \frac{t^\alpha \omega}{\alpha} \right) \right)}{k^2} - \frac{\omega^2 \left(1 - \tanh \left(\frac{1}{2} \sqrt{\frac{\omega^2}{k^4}} kx + \frac{t^\alpha \omega}{\alpha} \right) \right)^2}{2k^2} \tag{49}$$

for $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$ and absolute errors are presented. The results indicate that as the x values increase the absolute errors increase. Besides, as the α values increase, the absolute errors decrease. Also the Table 1-2 show competitive solutions of the RPSM with highly approximate results. Moreover, in Figure 1-2, the surface plots of the approximate solutions are illustrated for $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75$ and $\alpha = 0.95$.

Table 1: Comparison of the RPSM approximate ($u_3(x,t)$) and the exact solutions with absolute errors for $\omega = 0.25, k = 2$, and $t = 0.1$.

x	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error
0.0	-0.122804	-0.122804	2.7928E-11	-0.124382	-0.124382	4.9099E-14	-0.124768	-0.124768	3.7470E-16
0.1	-0.122023	-0.122023	7.7575E-11	-0.123601	-0.123601	3.5947E-13	-0.123987	-0.123987	6.5086E-15
0.2	-0.121242	-0.121242	1.2717E-10	-0.12282	-0.12282	6.6961E-13	-0.123206	-0.123206	1.2628E-14
0.3	-0.120462	-0.120462	1.7668E-10	-0.122039	-0.122039	9.7932E-13	-0.122425	-0.122425	1.8735E-14
0.4	-0.119682	-0.119682	2.2607E-10	-0.121258	-0.121258	1.2883E-12	-0.121644	-0.121644	2.4855E-14
0.5	-0.118902	-0.118902	2.7532E-10	-0.120478	-0.120478	1.5965E-12	-0.120864	-0.120864	3.0933E-14
0.6	-0.118123	-0.118123	3.2438E-10	-0.119698	-0.119698	1.9037E-12	-0.120083	-0.120083	3.7012E-14
0.7	-0.117344	-0.117344	3.7322E-10	-0.118918	-0.118918	2.2095E-12	-0.119304	-0.119304	4.3035E-14
0.8	-0.116566	-0.116566	4.2183E-10	-0.118139	-0.118139	2.5139E-12	-0.118524	-0.118524	4.9058E-14
0.9	-0.115789	-0.115789	4.7015E-10	-0.117361	-0.117361	2.8167E-12	-0.117745	-0.117745	5.5039E-14
1.0	-0.115012	-0.115012	5.1816E-10	-0.116583	-0.116583	3.1176E-12	-0.116967	-0.116967	6.1006E-14

Table 2: Comparison of the RPSM approximate ($v_3(x, t)$) and the exact solutions with absolute errors for $\omega = 0.25$, $k = 2$, and $t = 0.1$.

x	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error
0.0	0.007810	0.007810	4.9661E-10	0.007812	0.007812	3.1043E-12	0.007812	0.007812	6.1321E-14
0.1	0.007808	0.007808	4.9626E-10	0.007811	0.007811	3.1030E-12	0.007811	0.007811	6.1298E-14
0.2	0.007805	0.007805	4.9558E-10	0.007810	0.007810	3.0995E-12	0.007810	0.007810	6.1235E-14
0.3	0.007802	0.007802	4.9458E-10	0.007808	0.007808	3.0941E-12	0.007809	0.007809	6.1131E-14
0.4	0.007798	0.007798	4.9324E-10	0.007805	0.007805	3.0866E-12	0.007806	0.007806	6.0986E-14
0.5	0.007793	0.007793	4.9158E-10	0.007802	0.007802	3.0770E-12	0.007803	0.007803	6.0802E-14
0.6	0.007788	0.007788	4.8959E-10	0.007798	0.007798	3.0654E-12	0.007800	0.007800	6.0575E-14
0.7	0.007783	0.007783	4.8729E-10	0.007794	0.007794	3.0518E-12	0.007796	0.007796	6.0312E-14
0.8	0.007776	0.007776	4.8466E-10	0.007788	0.007788	3.0362E-12	0.007791	0.007791	6.0006E-14
0.9	0.007770	0.007770	4.8172E-10	0.007783	0.007783	3.0186E-12	0.007786	0.007786	5.9663E-14
1.0	0.007762	0.007762	4.7846E-10	0.007777	0.007777	2.9990E-12	0.007780	0.007780	5.9281E-14

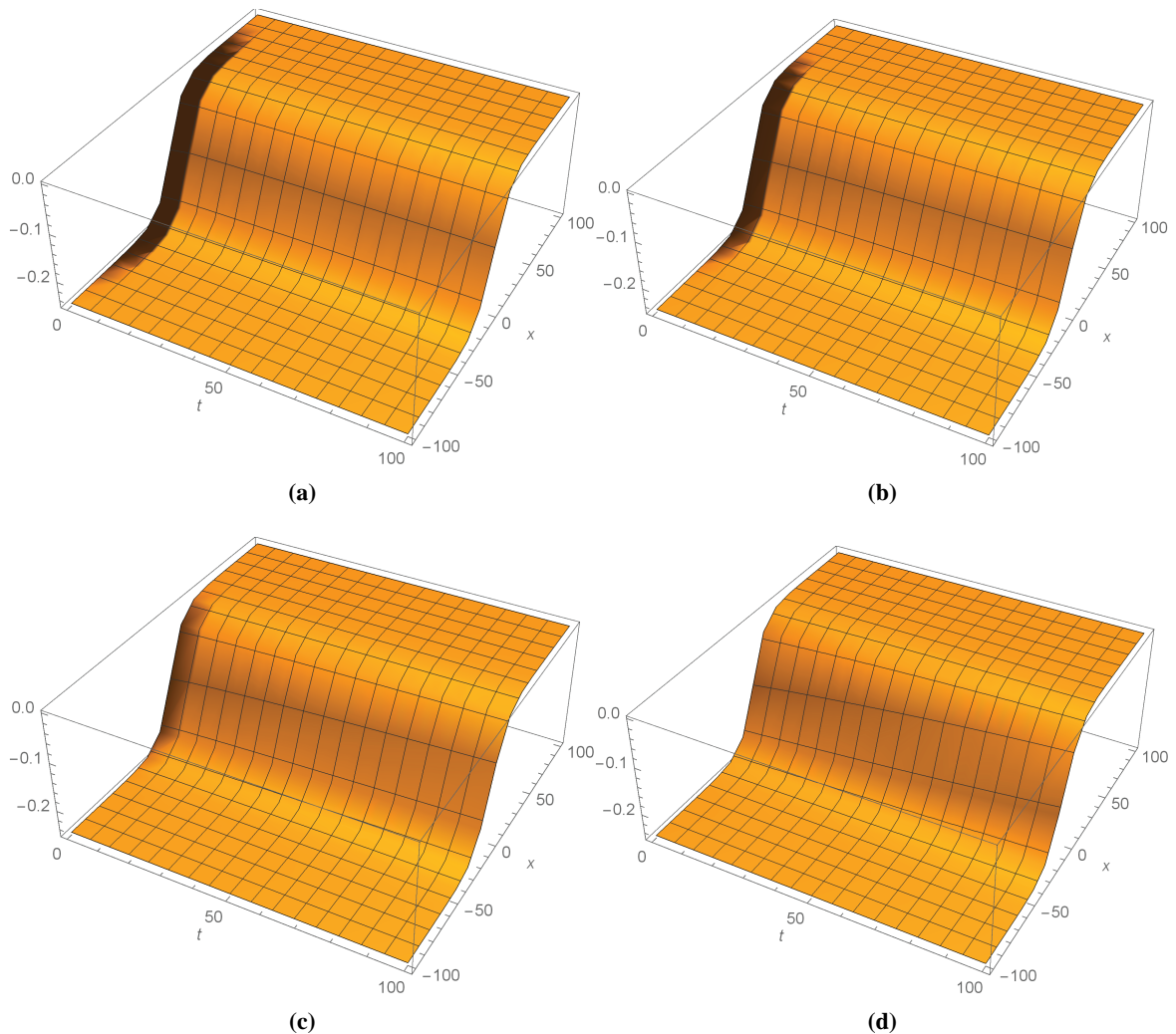


Fig. 1: The surface plots of $u_3(x, t)$ for $\omega = 0.25$, $k = 2$, $t = 0.1$ and for (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 0.95$.

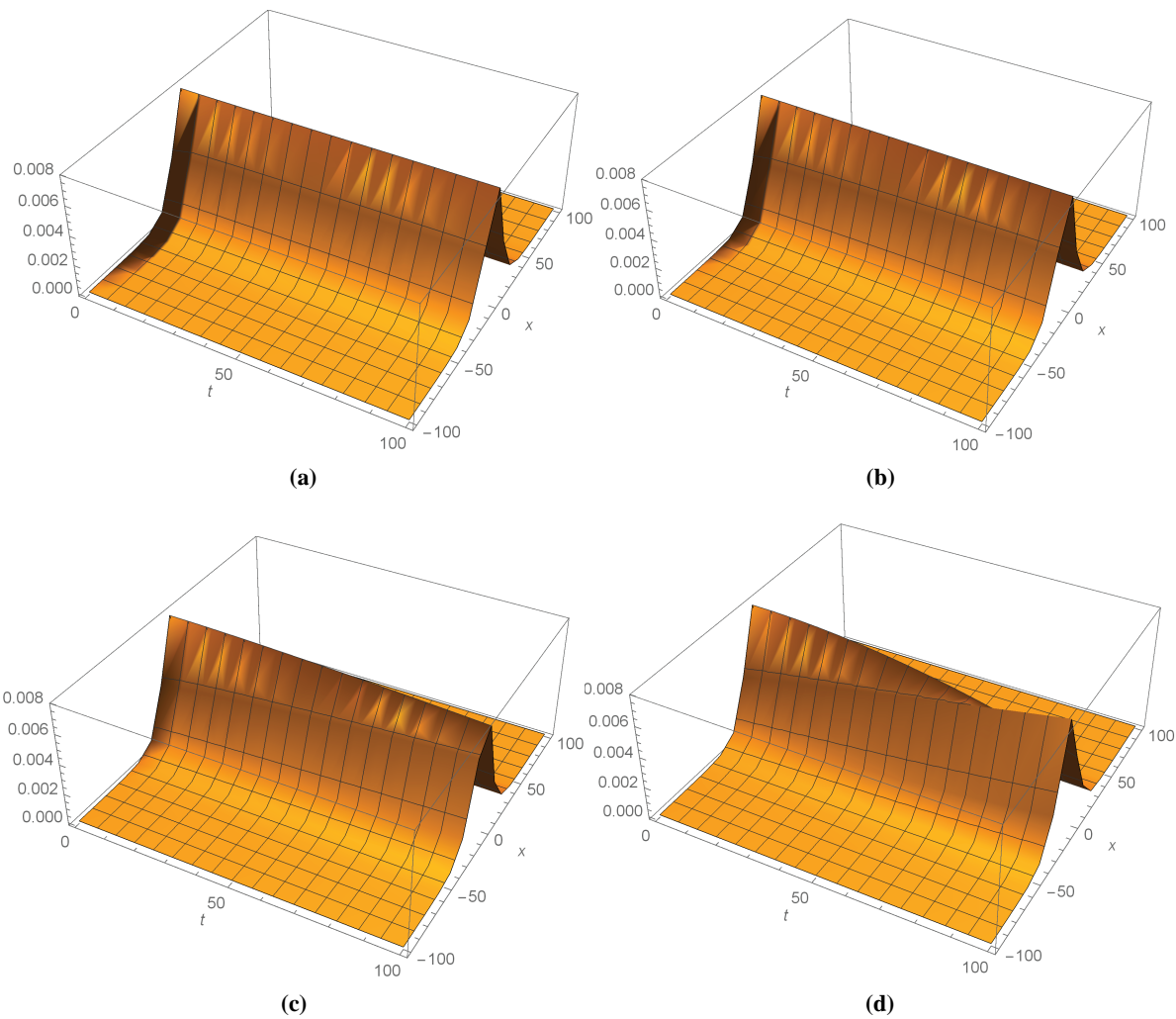


Fig. 2: The surface plots of $v_3(x, t)$ for $\omega = 0.25, k = 2, t = 0.1$ and for (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 0.95$.

4.2 Solution of time-Fractional Predator-Prey Equations

Now consider the time-fractional diffusive predator-prey equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}} \right) u^2 - u^3 - uv \tag{50}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + \kappa uv - \beta v - \delta v^3 \tag{51}$$

with the initial conditions

$$u(x, 0) = f(x) = \frac{v}{\sqrt{2}} + \sqrt{2} \tanh x \tag{52}$$

$$v(x, 0) = g(x) = \frac{1}{\sqrt{\delta}} \left(\frac{v}{\sqrt{2}} + \sqrt{2} \tanh x \right). \tag{53}$$

Now repeating the above residual power series procedure we obtained the following results.

$$f_1(x) = -\beta f(x) + \left(\frac{1}{\sqrt{\delta}} + \kappa\right) f^2(x) - f^3(x) - f(x)g(x) + f''(x) \quad (54)$$

$$u_1(x,t) = f(x) + \frac{t^\alpha}{\alpha} \left(-\beta f(x) + \left(\frac{1}{\sqrt{\delta}} + \kappa\right) f^2(x) - f^3(x) - f(x)g(x) + f''(x) \right) \quad (55)$$

$$g_1(x) = -\beta g(x) + \kappa f(x)g(x) - \delta g^3(x) + g''(x) \quad (56)$$

$$v_1(x,t) = g(x) + \frac{t^\alpha}{\alpha} \left(-\beta g(x) + \kappa f(x)g(x) - \delta g^3(x) + g''(x) \right) \quad (57)$$

$$f_2(x) = -\beta f_1(x) + 2\left(\frac{1}{\sqrt{\delta}} + \kappa\right) f(x)f_1(x) - 3f^2(x)f_1(x) - f_1(x)g(x) - f(x)g_1(x) + f''(x) \quad (58)$$

$$u_2(x,t) = f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha}}{2\alpha^2} \left(-\beta f_1(x) + 2\left(\frac{1}{\sqrt{\delta}} + \kappa\right) f(x)f_1(x) \right) \\ + \frac{t^{2\alpha}}{2\alpha^2} \left(-3f^2(x)f_1(x) - f_1(x)g(x) - f(x)g_1(x) + f''(x) \right) \quad (59)$$

$$g_2(x) = \kappa f_1(x)g(x) - \beta g_1(x) + \kappa f(x)g_1(x) - 3\delta g^2(x)g_1(x) + g_1''(x) \quad (60)$$

$$v_2(x,t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha}}{2\alpha^2} \left(\kappa f_1(x)g(x) - \beta g_1(x) + \kappa f(x)g_1(x) - 3\delta g^2(x)g_1(x) + g_1''(x) \right) \quad (61)$$

$$f_3(x) = \frac{1}{\sqrt{\delta}} \left(2f_1^2(x) + 2\sqrt{\delta}\kappa f_1^2(x) - 6\sqrt{\delta}f(x)f_1^2(x) - \beta\sqrt{\delta}f_2(x) + 2f(x)f_2(x) + 2\sqrt{\delta}\kappa f(x)f_2(x) \right) \\ - \frac{1}{\sqrt{\delta}} \left(3\sqrt{\delta}f^2(x)f_2(x) + \sqrt{\delta}g(x)f_2(x) - 2\sqrt{\delta}f_1(x)g_1(x) - \sqrt{\delta}f(x)g_2(x) + \sqrt{\delta}f_2''(x) \right) \quad (62)$$

$$u_3(x,t) = f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} + \frac{1}{\sqrt{\delta}} \frac{t^{3\alpha}}{6\alpha^3} \left(2f_1^2(x) + 2\sqrt{\delta}\kappa f_1^2(x) - 6\sqrt{\delta}f(x)f_1^2(x) \right) \\ - \frac{1}{\sqrt{\delta}} \frac{t^{3\alpha}}{6\alpha^3} \left(-\beta\sqrt{\delta}f_2(x) + 2f(x)f_2(x) + 2\sqrt{\delta}\kappa f(x)f_2(x) + 3\sqrt{\delta}f^2(x)f_2(x) \right) \\ + \frac{1}{\sqrt{\delta}} \frac{t^{3\alpha}}{6\alpha^3} \left(\sqrt{\delta}g(x)f_2(x) - 2\sqrt{\delta}f_1(x)g_1(x) - \sqrt{\delta}f(x)g_2(x) + \sqrt{\delta}f_2''(x) \right) \quad (63)$$

$$g_3(x) = \kappa g(x)f_2(x) + 2\kappa f_1(x)g_1(x) - 6\delta g(x)g_1^2(x) - \beta g_2(x) + \kappa f(x)g_2(x) - 3\delta g^2(x)g_2(x) + g_2''(x) \quad (64)$$

$$v_3(x,t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2} + \frac{t^{3\alpha}}{2\alpha^2} \left(\kappa g(x)f_2(x) + 2\kappa f_1(x)g_1(x) \right) \\ + \frac{t^{3\alpha}}{2\alpha^2} \left(-6\delta g(x)g_1^2(x) - \beta g_2(x) + \kappa f(x)g_2(x) - 3\delta g^2(x)g_2(x) + g_2''(x) \right). \quad (65)$$

In Table 3-4, the third order approximate RPSM solutions of time-fractional Predator-Prey equations are compared numerically with the exact solutions [26]

$$u(x, t) = \frac{v}{\sqrt{2}} + \sqrt{2} \tanh\left(x + \frac{t^\alpha v}{\alpha}\right) \tag{66}$$

$$v(x, t) = \frac{1}{\sqrt{\delta}} \left(\frac{v}{\sqrt{2}} + \sqrt{2} \tanh\left(x + \frac{t^\alpha v}{\alpha}\right) \right) \tag{67}$$

for $\alpha = 0.75$, $\alpha = 0.85$ and $\alpha = 0.95$ and absolute errors are presented. The results show that as the absolute value of the x points increase the absolute errors decrease. Besides, as the α values increase, the absolute errors decrease. Also in Figure 3 and 4, the surface plots of the approximate solutions are illustrated for $\alpha = 0.65$, $\alpha = 0.75$, $\alpha = 0.85$ and $\alpha = 0.95$.

Table 3: Comparison of the RPSM approximate ($u_3(x, t)$) and the exact solutions with absolute errors for $\delta = 0.25$, $\kappa = 2$, $\beta = -1$, $v = \sqrt{2}$ and $t = 0.1$.

x	$\alpha = 0.75$			$\alpha = 0.85$			$\alpha = 0.95$		
	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error
-10	-0.414213	-0.414213	5.6537E-11	-0.414214	-0.414213	1.3064E-11	-0.414213	-0.414213	3.2387E-12
-8	-0.414212	-0.414212	3.0868E-9	-0.414213	-0.414213	7.1328E-10	-0.414213	-0.414213	1.7684E-10
-6	-0.414179	-0.414179	1.6851E-7	-0.414186	-0.414185	3.8940E-8	-0.414189	-0.414189	9.6541E-9
-4	-0.412368	-0.412359	9.1442E-6	-0.412698	0.412696	2.1136E-6	-0.412889	-0.412889	5.2412E-7
-2	-0.316762	-0.316414	3.4829E-4	-0.333766	-0.333684	8.202E-5	-0.343687	-0.343666	2.0573E-5
0	1.456435	1.457199	7.6453E-4	1.326242	1.326374	1.3223E-4	1.234017	1.234041	2.4240E-5
2	2.388259	2.387967	2.9187E-4	2.382275	2.382203	7.2407E-5	2.377619	2.377601	1.8822E-5
4	2.413735	2.413728	6.9999E-6	2.413622	2.413620	1.7529E-6	2.413534	2.413534	4.5887E-7
6	2.414204	2.414204	1.2881E-7	2.4142027	2.414202	3.2261E-8	2.414201	2.414201	8.4462E-9
8	2.414213	2.414213	2.3594E-9	2.414213	2.414213	5.9094E-10	2.414213	2.414213	1.5471E-10
10	2.414213	2.414213	4.3215E-11	2.414213	2.414213	1.0823E-11	2.414213	2.414213	2.8337E-12

Table 4: Comparison of the RPSM approximate ($v_3(x, t)$) and the exact solutions with absolute errors for $\delta = 0.25$, $\kappa = 2$, $\beta = -1$, $v = \sqrt{2}$ and $t = 0.1$.

x	$\alpha = 0.75$			$\alpha = 0.85$			$\alpha = 0.95$		
	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error	RPSM	Exact	Abs. Error
-10	-0.828427	-0.828427	1.1307E-10	-0.828427	-0.828427	2.6128E-11	-0.828427	-0.828427	6.4776E-12
-8	-0.828425	-0.828425	6.1736E-9	-0.828426	-0.828426	1.4265E-9	-0.828426	-0.828426	3.5368E-10
-6	-0.828359	-0.828359	3.3703E-7	-0.828371	-0.828371	7.7880E-8	-0.828378	-0.828378	1.9308E-8
-4	-0.824737	-0.824718	1.8288E-5	-0.825396	-0.825392	4.2273E-6	-0.825779	-0.825778	1.0482E-6
-2	-0.633525	-0.632828	6.9659E-4	-0.667532	-0.667368	16405E-4	-0.687374	-0.687333	4.1147E-5
0	2.912870	2.914399	1.5290E-3	2.652485	2.652749	26446E-4	2.468035	2.468083	4.8480E-5
2	4.776518	4.775934	5.8375E-4	4.764550	4.764406	14481E-4	4.755239	4.755202	3.7644E-5
4	4.827470	4.827456	1.3999E-5	4.827244	4.827241	3.5058E-6	4.827069	4.827068	9.1774E-7
6	4.828409	4.828409	2.5762E-7	4.828405	4.828405	6.4522E-8	4.828402	4.828402	1.6892E-8
8	4.828426	4.828426	4.7189E-9	4.828426	4.828426	1.1818E-9	4.828426	4.828426	3.0942E-10
10	4.828427	4.828427	8.6430E-11	4.828427	4.828427	2.1646E-11	4.828427	4.828427	5.6665E-12

5 Conclusion

In this paper, approximate solutions of the nonlinear time-fractional Boussinesq-Whitham-Broer-Kaup and Diffusive Predator-Prey equations are obtained by the residual power series method (RPSM) . With conformable fractional derivative definition, we can easily transform fractional differential equations to the nonlocal classical differential equations. By the presented method and the conformable fractional derivative definition, it is shown that there is no need

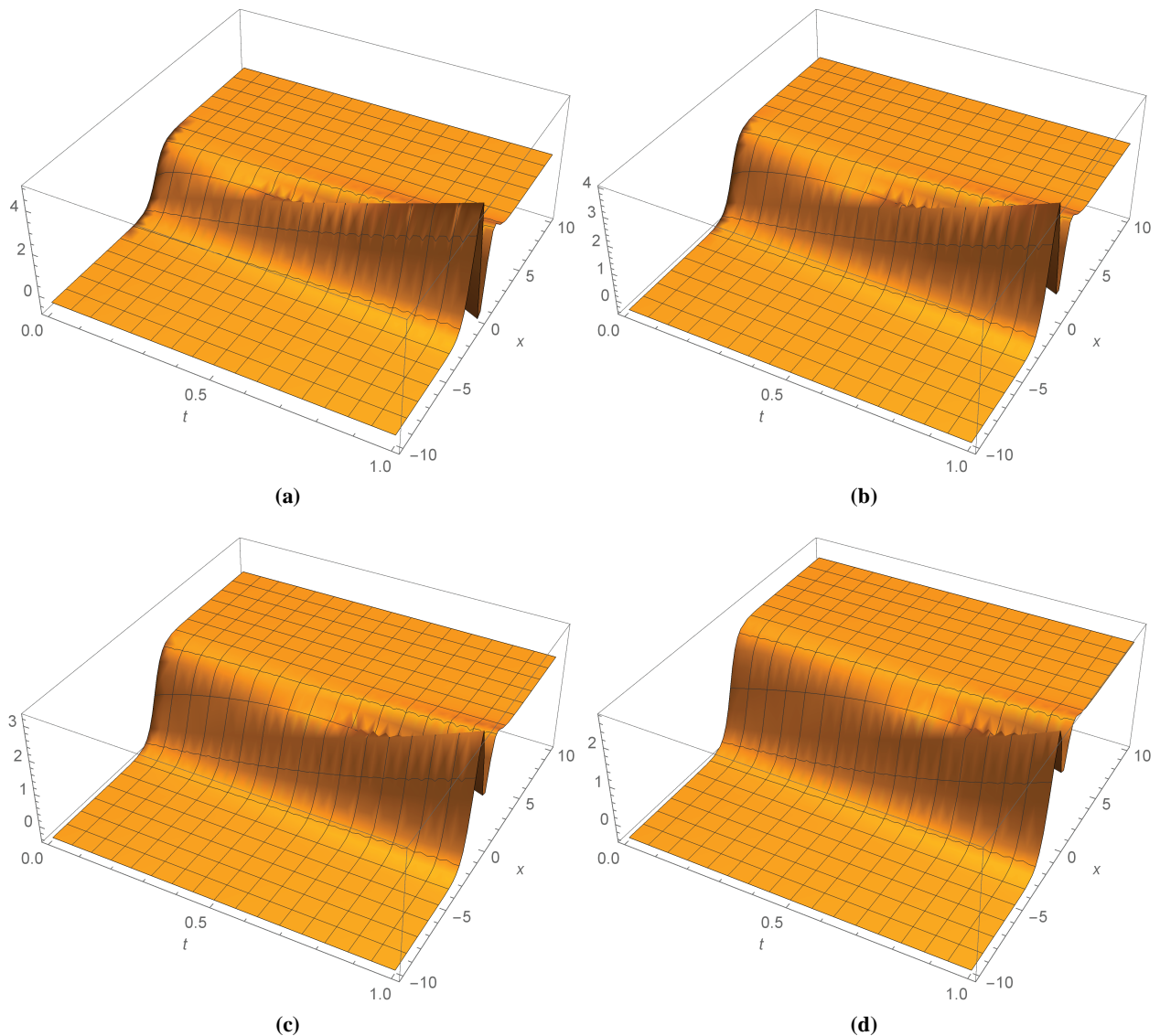


Fig. 3: The surface plots of $u_3(x, t)$ for $\delta = 0.25$, $\kappa = 2$, $\beta = -1$, $v = \sqrt{2}$, $t = 0.1$ and for (a) $\alpha = 0.65$, (b) $\alpha = 0.75$, (c) $\alpha = 0.85$, (d) $\alpha = 0.95$.

to another complex method or definition. Approximate and exact solutions of time-fractional nonlinear differential equations are compared to show the reliability of the method. Absolute errors were given with approximate and exact solutions with the help of figures and tables. Also our work reported here is a first step towards understanding structural and physical behaviour of mathematical models in biology. We hope that our work will be very useful in better understanding the events occurring at mathematical biology.

Competing interests

The authors declare that they have no competing interests.

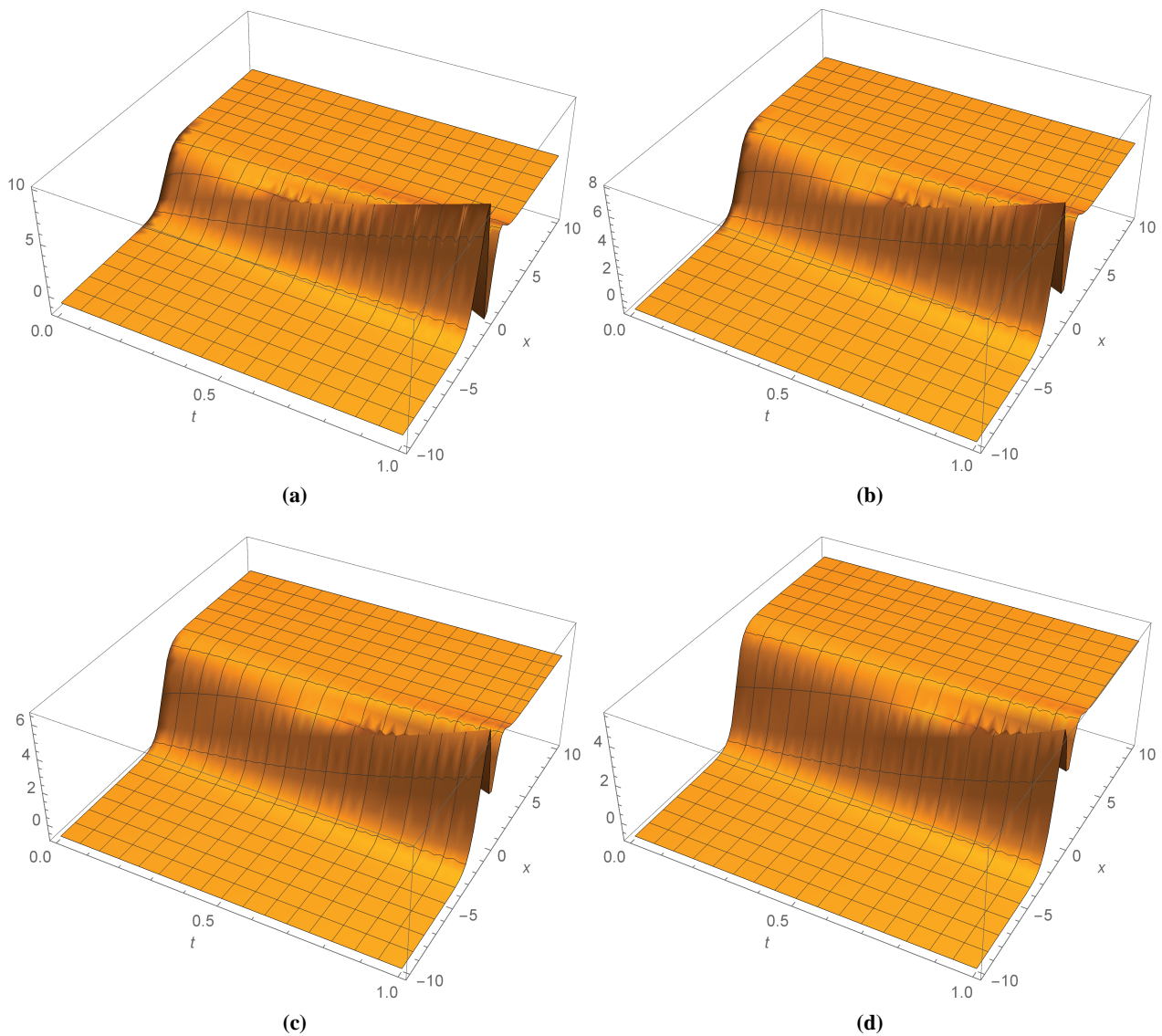


Fig. 4: The surface plots of $v_3(x, t)$ for $\delta = 0.25$, $\kappa = 2$, $\beta = -1$, $v = \sqrt{2}$, $t = 0.1$ and for (a) $\alpha = 0.65$, (b) $\alpha = 0.75$, (c) $\alpha = 0.85$, (d) $\alpha = 0.95$.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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