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On time-fractional Lotka-Volterra diffusion model

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Abstract: The interest of this paper to find the solution of time-fractional Lotka-Volterra diffusion problem by implementing the residual power series method (RPSM). The fractional derivative is described in the Caputo sense. The desired solution of the nonlinear equations are established in the form of rapidly convergent series whose components are computed by Matlab Software Package. The obtained results and graphical consequences show that the suggested method in this study is very efficient, effective and reliable for the solution of the time-fractional Lotka-Volterra equation.

Keywords: Fractional differential equation, Lotka-Volterra equation, Caputo derivative, residual power series method.

1 Introduction

The Lotka-Volterra diffusion equations which has many applications in ecology , chemistry, genetics, etc. have been widely analyzed in recent years. In homogeneous model there could be different cases which is tough to guess. In one case, the more diffusive species could ignore its competitor for a long time and in another case they eliminate the competitor gradually when they invade the whole region. The unexpected result could happen in competitive reaction-diffusion system. The general form of Lotka-Volterra diffusion equations for two species can be given in the following form:

$$D_t^{\alpha} u = d_1 \Delta_x u + u(r_1 - a_1 u - k_1 v),\ \Omega \times (0,+\infty) \tag{1}$$

$$D_t^{\alpha} v = d_2 \Delta_x v + v(r_2 - a_2 v - k_2 u),\ \Omega \times (0,+\infty) \tag{2}$$

where Ω represents some spatial domain and $d_1, d_2, r_1, r_2, a_1, a_2, k_1, k_2$ are some positive constants. Without diffusion , this system can be written as an ODE system. Then, the system has to steady state $(u,v) = (0,1)$ and $(u,v) = (1,0)$ which are stable when $\frac{k_1 r_2}{r_1 a_2} > 1$ and $\frac{k_2 r_1}{r_2 a_1} > 1$, respectively [1-4].

In recent years, fractional calculus which has been considerable interest are used in bioengineering, thermodynamics, viscoelasticity, control theory, aerodynamics, electromagnetics, signal processing, chemistry, finance [5-11]. Various numerical methods have been applied and analyzed for differential equations with fractional order derivative of Riemann-Liouville or Caputo sense [10-16]. The RPSM was established as an powerful method for fuzzy differential equations [17]. It has been successfully applied in various fields [18-29]. The solution of problems by RPSM are obtained in the form of Maclaurin series.

2 Preliminaries

In this section, the fundamental definitions and various features for fractional calculus are shown [10,30-32].

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Definition 1. *The Riemann-Liouville fractional integral of order* α $(\alpha \geq 0)$ *is given as [19,22]*

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \ \alpha > 0, \ x > 0 \tag{3}$$

$$J^0 f(x) = f(x) \tag{4}$$

Definition 2. *The Caputo fractional derivative with order* α *is given as [19,22]*

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \int_0^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} f(t)dt, \ m-1 < \alpha < m, \ x > 0 \tag{5}$$

where D^m *represents the differential operator with order m.*
Taking the Caputo derivative, we have

$$D^{\alpha} x^{\beta} = 0, \quad \beta < \alpha \tag{6}$$

$$D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \quad \beta \geq \alpha \tag{7}$$

Definition 3. *The Caputo's time fractional derivative of order* α *of* $u(x,t)$ *is defined as [19,22]*

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial t^m} d\xi, \ m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m} \qquad ,\alpha = m \in N \end{cases} \tag{8}$$

Theorem 1. *Suppose that* $u(x,t)$ *has a multiple fractional power series representation at* $t = t_0$ *of the form*

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, \quad x \in I, \quad t_0 \leq t \leq t_0 + R \tag{9}$$

If $D_t^{m\alpha} u(x,t), \ m = 0,1,2,...$ *are continuous on* $I \times (t_0, t_0 + R)$*, then* $f_m(x) = \frac{D_t^{m\alpha} u(x,t_0)}{\Gamma(m\alpha+1)}$.

3 Application RPSM to the time-fractional Lotka-Volterra diffusion problem

We first consider the following one-dimensional Lotka-Volterra competition-diffusion problem:

$$D_t^{\alpha} u = d_1 D_{xx} u + u(r_1 - a_1 u - k_1 v), \ R \times (0, +\infty) \tag{10}$$

$$D_t^{\alpha} v = d_2 D_{xx} v + v(r_2 - a_2 v - k_2 u), \ R \times (0, +\infty) \tag{11}$$

subject to initial condition

$$u(x,0) = A_1(x), \tag{12}$$

$$v(x,0) = B_1(x) \tag{13}$$

where $d_1, d_2, r_1, r_2, a_1, a_2, k_1, k_2$ are positive constants with ecological meaning. The RPSM is applied to find out series solution for these equations with given initial conditions by replacing its fractional power series expansion with its truncated residual function. From each equation, a repetition formula for the determination of coefficients is supplied, while coefficients in fractional power series expansion can be calculated by repeatedly fractional differentiation of the truncated residual function [17-24]. The RPSM propose the solutions for Eq. (10)-(13) as a fractional power series at $t = 0$ [17]

$$u(x,t) = \sum_{k=0}^{\infty} A_{k+1}(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{14}$$

$$v(x,t) = \sum_{k=0}^{\infty} B_{k+1}(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{15}$$

where $x \in I, 0 \leq t < R$. To obtain the numerical values from this series, let $u_m(x,t)$ denotes the m-th truncated series of $u(x,t)$. That is

$$u_m(x,t) = \sum_{k=0}^{m} A_{k+1}(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{16}$$

$$v_m(x,t) = \sum_{k=0}^{m} B_{k+1}(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{17}$$

By the condition at $t = 0$, we have

$$u_0(x,t) = f_0(x) = u(x,0) = A_1(x), \tag{18}$$

$$v_0(x,t) = g_0(x) = v(x,0) = B_1(x) \tag{19}$$

From Eqs.(14)-(15)

$$u_m(x,t) = A_1(x) + \sum_{k=2}^{m} A_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \; 0 < \alpha \leq 1, \tag{20}$$

$$v_m(x,t) = B_1(x) + \sum_{k=2}^{m} B_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \; 0 < \alpha \leq 1. \tag{21}$$

Define the residual function as for Eqs.(10)-(11)[35]

$$Resu_0 = D_t^{\alpha} u_0 - d_1 D_{xx} u_0 + u_0(r_1 - a_1 u_0 - k_1 v_0) \tag{22}$$

$$Resv_0 = D_t^{\alpha} v_0 - d_2 D_{xx} v_0 + v_0(r_2 - a_2 v_0 - k_2 u_0) \tag{23}$$

$$Resu_m = D_t^{\alpha} u_m - d_1 D_{xx} u_m + u_m(r_1 - a_1 u_m - k_1 v_m) \tag{24}$$

$$Resv_m = D_t^{\alpha} v_m - d_2 D_{xx} v_m + v_m(r_2 - a_2 v_m - k_2 u_m) \tag{25}$$

From [17-24], by making use of some results such as $Res(x,t) = 0$ and $D_t^{k\alpha} Res_m(x,0) = 0, \; k = 0,1,2,\ldots,m, m = 1,2,3,\ldots$ are used to obtain the solution.

Substituting $u_m(x,t), v_m(x,t)$ into Eqs. (24)-(25), calculating the fractional derivative $D_t^{(m-1)\alpha}$ of $Res(x,t)$ at $t = 0$ and solving the following obtained algebraic system

$$D_t^{(m-1)\alpha} Res_m(x,0) = 0, 0 < \alpha \leq 1, m = 1,2,3,\ldots \tag{26}$$

the required coefficients $A_k(x), k = 2,3,\ldots,m$ in Eq. (20) are determined.

In order to determine $A_2(x)$ and $B_2(x)$, the 1 *st* residual function in Eqs. (24)-(25) can be written as follows:

$$Resu_0 = D_t^\alpha u_0 - d_1 D_{xx} u_0 + u_0(r_1 - a_1 u_0 - k_1 v_0) \tag{27}$$

$$Resv_0 = D_t^\alpha v_0 - d_2 D_{xx} v_0 + v_0(r_2 - a_2 v_0 - k_2 u_0) \tag{28}$$

where $u_0(x,t) = A_1(x) + A_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$ and $v_0(x,t) = B_1(x) + B_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$. Therefore,

$$\begin{aligned} Resu_0(x,t) = A_2 - d_1(A_1'' + A_2''\frac{t^\alpha}{\Gamma(1+\alpha)}) - r_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)}) + a_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)})^2 \\ + k_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)}) \end{aligned} \tag{29}$$

$$\begin{aligned} Resv_0(x,t) = B_2 - d_2(B_1'' + B_2''\frac{t^\alpha}{\Gamma(1+\alpha)}) - r_2(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)}) + a_2(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)})^2 \\ + k_2(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)}) \end{aligned} \tag{30}$$

From Eq.(26), it is concluded that $Resu_0(x,0) = 0$ and $Resv_0(x,0) = 0$, which leads to

$$A_2(x) = d_1 A_1'' + r_1 A_1 - a_1 A_1^2 - k_1 A_1 B_1 \tag{31}$$

$$B_2(x) = d_2 B_1'' + r_2 B_1 - a_2 B_1^2 - k_2 A_1 B_1 \tag{32}$$

Similarly, to obtain $A_3(x)$ and $B_3(x)$, the 2*nd* residual function in Eqs. (24)-(25) becomes

$$Resu_1(x,t) = D_t^\alpha u_1 - d_1 D_{xx} u_1 + u_1(r_1 - a_1 u_1 - k_1 v_1) \tag{33}$$

$$Resv_1(x,t) = D_t^\alpha v_1 - d_2 D_{xx} v_1 + v_1(r_2 - a_2 v_1 - k_2 u_1) \tag{34}$$

where $u_1(x,t) = A_1(x) + A_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ and $v_1(x,t) = B_1(x) + B_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$. Therefore,

$$\begin{aligned} Resu_1(x,t) =& (A_2 + A_3\frac{t^\alpha}{\Gamma(1+\alpha)}) - d_1(A_1'' + A_2''\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3''\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ & - r_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) + a_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})^2 \\ & + k_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \end{aligned} \tag{35}$$

$$\begin{aligned} Resv_1(x,t) =& (B_2 + B_3\frac{t^\alpha}{\Gamma(1+\alpha)}) - d_2(B_1'' + B_2''\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3''\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ & - r_2(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) + a_2(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})^2 \\ & + k_2(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \end{aligned} \tag{36}$$

The operator D_t^α is applied on both sides of Eqs.(34)-(35) as follows:

$$\begin{aligned} D_t^\alpha Resu_1(x,t) =& A_3 - d_1(A_2'' + A_3''\frac{t^\alpha}{\Gamma(1+\alpha)}) - r_1(A_2 + A_3\frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &+ 2a_1(A_2 + A_3\frac{t^\alpha}{\Gamma(1+\alpha)})(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ &+ k_1(A_2 + A_3\frac{t^\alpha}{\Gamma(1+\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ &+ k_1(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})(B_2 + B_3\frac{t^\alpha}{\Gamma(1+\alpha)}) \end{aligned} \tag{37}$$

$$\begin{aligned} D_t^\alpha Resv_1(x,t) =& B_3 - d_2(B_2'' + B_3''\frac{t^\alpha}{\Gamma(1+\alpha)}) - r_2(B_2 + B_3\frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &+ 2a_2(B_2 + B_3\frac{t^\alpha}{\Gamma(1+\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ &+ k_2(A_2 + A_3\frac{t^\alpha}{\Gamma(1+\alpha)})(B_1 + B_2\frac{t^\alpha}{\Gamma(1+\alpha)} + B_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) \\ &+ k_2(A_1 + A_2\frac{t^\alpha}{\Gamma(1+\alpha)} + A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)})(B_2 + B_3\frac{t^\alpha}{\Gamma(1+\alpha)}) \end{aligned} \tag{38}$$

From Eq. (26),

$$A_3(x) = d_1A_2'' + r_1A_2 - 2a_1A_1A_2 - k_1A_2B_1 - k_1A_1B_2 \tag{39}$$

$$B_3(x) = d_2B_2'' + r_2B_2 - 2a_2B_1B_2 - k_2A_2B_1 - k_2A_1B_2 \tag{40}$$

The same manner is repeated as above, the following recurrence results is obtained

$$A_4(x) = d_1A_3'' + r_1A_3 - 2a_1A_2^2 - 2a_1A_1A_3 - k_1A_3B_1 - 2k_1A_2B_2 - k_1A_1B_3 \tag{41}$$

$$B_4(x) = d_2B_3'' + r_2B_3 - 2a_2B_2^2 - 2a_2B_1B_3 - k_2A_3B_1 - 2k_2A_2B_2 - k_2A_1B_3 \tag{42}$$

and so on.

4 Numerical results

Example 1. We take $d_1 = d_2 = 0.1, r_1 = r_2 = 0.1, a_1 = a_2 = 0.1, k_1 = k_2 = 0.2$ and initial conditions $A_1(x) = 1, B_1(x) = 1 - x^2$ in Eqs.(3)-(4).

Based on the obtained results, we conclude that RPS approximate solution is getting closer to the exact solution of time-fractional Lotka-Volterra diffusion problem as the order of fractional derivative α increases to one.It is clear from Figs. 1-2 that convergence of the approximate solution depend on the order of the fractional derivative.

In Tables 1-2, the approximate solutions $u_k(x,t), v_k(x,t), k = 0,1,2,3$ are given for $\alpha = 0.25, 0.5, 1$. These tables show that as the fractional derivative α is getting closer to 1, approximate solution getting closer to the exact solution of time-fractional Lotka-Volterra diffusion problem.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 1$
0	1.00000	1.00000	1.00000
0.2	0.92888	0.94624	0.97274
0.4	0.91550	0.93116	0.95042
0.6	0.90480	0.91970	0.93223
0.8	0.89527	0.90905	0.91735
1	0.88641	0.89823	0.90498

Table 1: The u solution for Ex. 1 for several values α and $x = 0.5$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 1$
0	0.75000	0.75000	0.75000
0.2	0.52964	0.60076	0.68627
0.4	0.48152	0.54185	0.62687
0.6	0.44577	0.49413	0.57071
0.8	0.41580	0.45075	0.51670
1	0.38929	0.40926	0.46375

Table 2: The v solution for Ex. 1 for several values α and $x = 0.5$.

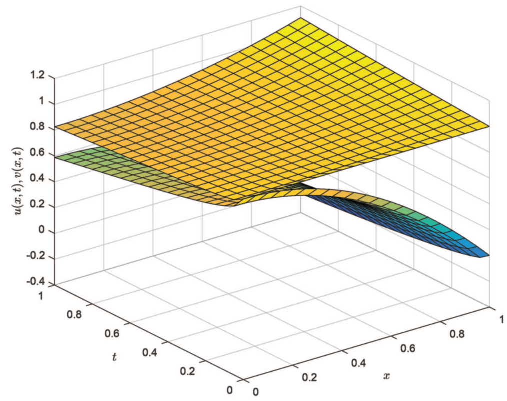

Fig. 1: The RPS solution for Ex.1 for $\alpha = 0.5$.

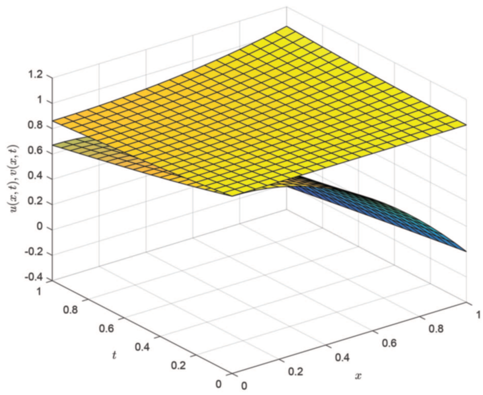

Fig. 2: The RPS solution for Ex.1 for $\alpha = 1$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 1$
0	1.00000	1.00000	1.00000
0.2	0.03685	0.01989	0.00573
0.4	0.04799	0.03241	0.01290
0.6	0.05621	0.04372	0.02154
0.8	0.06299	0.05442	0.03163
1	0.06886	0.06472	0.04318

Table 3: The u solution for Ex. 2 for several values α and $x = 0.5$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 1$
0	0.75000	0.75000	0.75000
0.2	0.60314	0.65555	0.71365
0.4	0.56927	0.61268	0.67689
0.6	0.54485	0.57731	0.63945
0.8	0.52492	0.54545	0.60103
1	0.50770	051559	0.56135

Table 4: The v solution for Ex. 2 for several values α and $x = 0.5$.

Figs. 1-2, the approximate solutions $u(x,t), v(x,t)$ for $\alpha = 0.5, 1$ are plotted. It is clear from these figures that as the amount of α enlarges to one, the approximate solution getting closer to exact solution.

Example 2. Let $d_1 = d_2 = 0.1, r_1 = r_2 = 0.2, a_1 = a_2 = 0.1, k_1 = k_2 = 0.1$ and initial conditions $A_1(x) = 1, B_1(x) = 1 - x^2$ in Eqs.(3)-(4).

Based on the obtained results, we conclude that RPS approximate solution is getting closer to the exact solution of time-fractional Lotka-Volterra diffusion problem as the order of fractional derivative α increases to one.It is clear from Figs. 1-2 that convergence of the approximate solution depend on the order of the fractional derivative.

In Tables 3-4, the approximate solutions $u_k(x,t), v_k(x,t), k = 0,1,2,3$ are presented for $\alpha = 0.25, 0.5, 1$. These tables show that as the fractional derivative α is getting closer to 1, approximate solution getting closer to the exact solution of

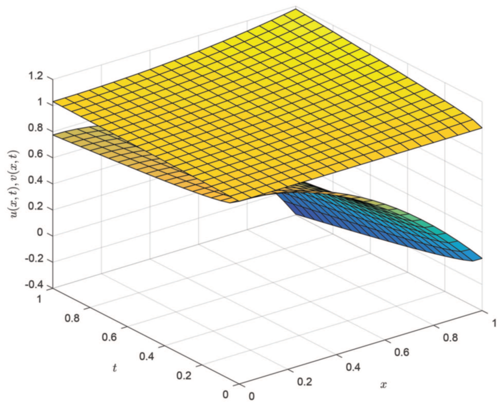

Fig. 3: The RPS solution for Ex.2 for $\alpha = 0.5$.

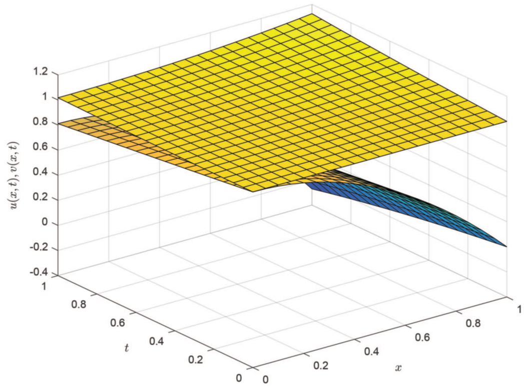

Fig. 4: The RPS solution for Ex.2 for $\alpha = 1$.

time-fractional Lotka-Volterra diffusion problem.

Figs. 3-4, the approximate solution $u(x,t), v(x,t)$ are drawn for $\alpha = 0.5, 1$. It is clear from these figures that as the amount of α enlarges to one, the approximate solution getting closer to exact solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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