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# On some matrices associated with interval valued fuzzy graphs

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**Abstract:** Interval Valued Fuzzy Node Arc Matrix (IVFNAM), Interval Valued Fuzzy Incidence Matrix (IVFIM) and Interval Valued Fuzzy Node Matrix (IVFNM) of an Interval Valued Fuzzy Graph (IVFG) are introduced here and some properties of these matrices are explained. The reachability matrix of an IVFG using IVFNAM is defined. It is shown that the strength of connectedness between any two pair of vertices in an IVFG can be found using this reachability matrix. We propose an algorithm to determine the nature of arcs in an IVFG using the reachability matrix. We also establish a relationship between the IVFIM of an IVFG and the IVFNAM of its corresponding line graph.

**Keywords:** Interval-valued Fuzzy node arc matrix, reachability matrix, strength of connectedness, interval-valued Fuzzy incidence matrix.

## 1 Introduction

Eventhough diagrams suitably specify graphs, they are not appropraiate for storing graphs in computers or for applying mathematical techniques to study their properties. As computers are more adept at manipulating numbers than at recognizing pictures, it is a standard practice to communicate the specification of a graph to a computer in matrix form. A plethora of study has been done on matrices associated with crisp graphs as well as fuzzy graphs. Fuzzy graphs are represented by fuzzy matrices. Extending this idea to the interval - valued fuzzy case in [9], we have shown that interval - valued fuzzy graphs can be represented by interval - valued fuzzy matrices. In [9], we defined interval - valued fuzzy adjacency matrix (IVFAM) and interval - valued fuzzy laplacian matrix (IVFLM) associated with an IVFG. In crisp graph theory, adjacency matrix completely determines the corresponding graph. But in the case of IVFGs, an IVFG is not completely determined by its IVFAM. We will not get any idea about the membership function of its nodes from the above defined IVFMs. Taking into consideration the membership degrees of nodes, we define a new IVFM called interval - valued fuzzy node arc matrix (IVFNAM) in this paper. Then we propose an algorithm to determine the nature of arcs using IVFNAM. Also we define interval - valued fuzzy incidence matrix. As in the case of IVFAM, we can see that an IVFG is not completely determined by its IVFIM.

Graph theoretic terms used in this work are either standard or are explained as and when they first appear. We consider only simple connected undirected graphs. That is, graphs with multiple edges and loops are not considered. Throughout the paper, we take $G = (A,B)$ as an IVFG on the crisp graph $G^* = (V,E)$ with $|V| = n$ and $|E| = m$. For the interval valued fuzzy graph theory preliminaries used in this work, refer [1,6,7,8]and for the interval - valued fuzzy matrix


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preliminaries, refer [5,9]. Studies on various types of fuzzy graphs such as bipolar fuzzy graphs are available in [3] and [4]. We see some of the basic defintions now.

**Definition 1.** [1] *Let $G^\star = (V,E)$ be a crisp graph. Then an interval - valued fuzzy graph (IVFG ) G on $G^\star$ is defined as a pair $G = (A,B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-(xy), \mu_B^+(xy)]$ is an interval - valued fuzzy set on E such that $\mu_B^-(xy) \le \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) \le \min\{\mu_A^+(x), \mu_A^+(y)\}$ for all $xy \in E$.*

**Definition 2.** [7] *The maximum of the $\mu^-$ strength (minimum of the $\mu_B^-$ values of the arcs in the path) of various paths connecting u and v is called the $\mu^-$ strength of connectedness between u and v and is denoted by $(\mu_{B^-})^\infty(u,v)$ or $NCONN_G(u,v)$.*

*The maximum of the $\mu^+$ strength (minimum of the $\mu_B^+$ values of the arcs in the path) of various paths connecting u and v is called the $\mu^+$ strength of connectedness between u and v and is denoted by $(\mu_{B^+})^\infty(u,v)$ or $PCONN_G(u,v)$.*

**Definition 3.** [5] *An interval - valued fuzzy matrix (IVFM) of order $m \times n$ is defined as $A = [a_{ij}]_{m\times n}$, where $A = [a_{ij}] = [a_{ij}^-, a_{ij}^+]$ , the $ij^{th}$ element of A represents the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval $[0,1]$.*

**Definition 4.** [5] *Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two IVFMs. Then the product $A \cdot B$ is the IVFM defined by $A \cdot B = [d_{ij}]_{m\times p} = [d_{ij}^-, d_{ij}^+]_{m\times p}$ where $d_{ij}^- = \vee_{k=1}^n a_{ik}^- \wedge b_{kj}^-$ and $d_{ij}^+ = \vee_{k=1}^n a_{ik}^+ \wedge b_{kj}^+$ where $i = 1,2,\ldots m$ and $j = 1,2,\ldots p$ i.e., here ordinary addition and multiplication are replaced by maximum and minimum respectively. This type of matrix multiplication is called minmax multiplication.*

**Definition 5.** *Let $A = [a_{ij}]$ be a square IVFM of order n. Then the powers of A, $A^n$ for $n \ge 2$ is defined by $A^n = A^{n-1} \cdot A$.*

**Definition 6.** [8] *Let $G = (A,B)$ be an IVFG. Then the interval - valued fuzzy adjacency matrix (IVFAM) of G is the IVFM with rows and columns corresponding to $v_1, v_2, \ldots, v_n$. It is denoted by $A_G = [a_{ij}]$ where*

$$a_{ij} = \begin{cases} [a_{ij}^-, a_{ij}^+] = [0,0] & if \quad i = j, \\ [a_{ij}^-, a_{ij}^+] = [\mu_B^-(v_iv_j), \mu_B^+(v_iv_j)] & if \quad i \neq j \end{cases}$$

Some other concepts that are used in this work are provided in the following table.

| Notation | Concept |
|---|---|
| $O(G)$ | Order of $G$ [6] |
| S(G) | Size of $G$ [6] |
| $d(u)$ | Degree of a node $u$[6] |
| $td(u)$ | Total degree of a node $u$[6] |
| $P$ | Path in an IVFG [7] |

## 2 Interval valued Fuzzy node arc matrix

**Definition 7.***Let $G = (A,B)$ be an IVFG on $G^* = (V,E)$. Then the interval - valued fuzzy node - arc matrix (IVFNAM) of G is the IVFM with rows and columns corresponding to $v_1, v_2, \ldots v_n$. It is denoted by $N_G = [n_{ij}]$ where*

$$n_{ij} = \begin{cases} [n_{ij}^-, n_{ij}^+] = [\mu_A^-(v_i), \mu_A^+(v_i)] & if \quad i = j, \\ [n_{ij}^-, n_{ij}^+] = [\mu_B^-(v_iv_j), \mu_B^+(v_iv_j)] & if \quad i \neq j \end{cases}$$

*Remark.* Clearly the IVFG $G$ is completely determined by the corresponding IVFNAM $N_G$.

The definition of IVFNAM leads to the folowing obvious properties.

## 2.1 Properties of interval valued Fuzzy node arc matrix

(1) If $G^*$ has $n$ nodes, then $N_G$ is a square IVFM of order $n$.
(2) $N_G$ is a symmetric matrix.
(3) The sum of all entries in the row (or column) of $N_G$ gives the total degree of the corresponding node.
(4) Trace of $N_G$ gives $O(G)$.
(5) Sum of all entries of $N_G$ gives $2S(G)+O(G)$.
(6) The diagonal entries of $N_G$ are greater than or equal to all the entries in the corresponding row or column.

## 2.2 Connectedness strength

**Definition 8.** *Let $G=(A,B)$ be an IVFG on $G^*=(V,E)$. Let $v_i$ and $v_j$ be any two nodes of $G$. If there exists at least one path between $v_i$ and $v_j$ of length less than or equal to $k$, then the $\mu^-$ connectedness of strength $k$ between $v_i$ and $v_j$ is defined as the maximum of the $\mu^-$ strengths of all paths between them of length less than or equal to $k$. Similarly, $\mu^+$ connectedness of strength $k$ between $v_i$ and $v_j$ is defined as the maximum of the $\mu^+$ strengths of all paths between them of length less than or equal to $k$. If there is no such path, $\mu^-$ and $\mu^+$ connectedness of strength $k$ is defined to be zero.*

**Lemma 1.** *Let $G=(A,B)$ be an IVFG with $n$ nodes and $N_G$ be the corresponding IVFNAM. Let $N_G^2=P_G=[p_{ij}^-,p_{ij}^+]$. Then for $1\le i<j\le n$, $p_{ij}^-$ and $p_{ij}^+$ give the $\mu^-$ and $\mu^+$ connectedness of strength 2 between the nodes $v_i$ and $v_j$, respectively. Moreover, $p_{ii}^-=\mu_A^-(v_i)$ and $p_{ii}^+=\mu_A^+(v_i)$*

*Proof.* Let $G=(A,B)$ be an IVFG and $N_G$ be the corresponding IVFNAM. Then $N_G^2$ is also an IVFM. Let $N_G^2=P_G=[p_{ij}^-,p_{ij}^+]$. For $i=j, p_{ii}^-=\vee_{k=1}^n(n_{ik}^-\wedge n_{ki}^-)=n_{ii}^-=\mu_A^-(v_i)$ and $p_{ii}^+=\vee_{k=1}^n(n_{ik}^+\wedge n_{ki}^+)=n_{ii}^+=\mu_A^+(v_i)$. For $i\neq j$, $p_{ij}^-=\vee_{k=1}^n(n_{ik}^-\wedge n_{kj}^-)$. Here, $k$ takes all values from 1 to $n$ including $i$ and j. Hence the above equation can be rewritten as

$$\begin{aligned}p_{ij}^- &= \vee_{k=1,k\neq i,j}^n(n_{ik}^-\wedge n_{kj}^-)\vee(n_{ii}^-\wedge n_{ij}^-)\vee(n_{ij}^-\wedge n_{jj}^-)\\ &= \vee_{k=1,k\neq i,j}^n(n_{ik}^-\wedge n_{kj}^-)\vee n_{ij}^-\vee n_{ij}^-\\ &= \vee_{k=1,k\neq i,j}^n(n_{ik}^-\wedge n_{kj}^-)\vee n_{ij}^-\\ &= \vee_{k=1,k\neq i,j}^n(n_{ik}^-\wedge n_{kj}^-)\vee\mu_B^-(v_i,v_j)\end{aligned}$$

Thus, for $i\neq j$,

$$p_{ij}^-=\vee_{k=1,k\neq i,j}^n(n_{ik}^-\wedge n_{kj}^-)\vee\mu_B^-(v_i,v_j) \tag{1}$$

Clearly, $\mu_B^-(v_i,v_j)$ represents the $\mu^-$ strength of a path of length 1 between $v_i$ and $v_j$. Now we consider 2 cases.

**Case - 1.** $\mu_B^-(v_i,v_j)\neq 0$. In otherwords, arc $(v_i,v_j)$ exists.

**Subcase-1.** If there is no arc between $v_i$ and $v_k$ or no arc between $v_k$ and $v_j$, then $n_{ik}^-\wedge n_{kj}^-=0$ and from equation (1), $p_{ij}^-$ gives the $\mu^-$ strength of a path of length 1 between $v_i$ and $v_j$ as there is no $v_i-v_j$ path of length 2.

**Subcase-2.** If there are arcs between $v_i$ and $v_k$ and between $v_k$ and $v_j$, then, $n_{ik}^-\wedge n_{kj}^-$ represent the $\mu^-$ strength of a path of length 2 between $v_i$ and $v_j$ with $v_k$ as an internal vertex. Then from $equation$(1), $p_{ij}^-$ gives the maximum of the $\mu^-$ strength of paths of length less than or equal to 2 between $v_i$ and $v_j$.
Hence from the above two subcases we can conclude that for $i\neq j$, $p_{ij}^-$ gives the maximum of the $\mu^-$ strength of paths of length less than or equal to 2 between $v_i$ and $v_j$.

**Case - 2.** $\mu_B^-(v_i,v_j)=0$. In otherwords, arc $(v_i,v_j)$ does not exist.

**Subcase-1.** If there is no arc between $v_i$ and $v_k$ or no arc between $v_k$ and $v_j$ for all $k = 1,2\ldots,n, k \neq i,j$ then $\vee_{k=1,k\neq i,j}^{n}(n_{ik}^{-} \wedge n_{kj}^{-}) = 0$. Thus from equation (1), $p_{ij}^{-} = 0$.
Here there is no $v_i - v_j$ path of length equal to 2 and arc $(v_i, v_j)$ does not exist. Hence we cannot find a $v_i - v_j$ path of length less than or equal to 2. Hence from definition 8, $\mu^-$ connectedness of strength 2 between the nodes $v_i$ and $v_j$ equals zero and thus the lemma follows.

**Subcase-2.** If there are arcs between $v_i$ and $v_k$ and between $v_k$ and $v_j$, then, $n_{ik}^{-} \wedge n_{kj}^{-}$ represent the $\mu^-$ strength of a path of length 2 between $v_i$ and $v_j$ with $v_k$ as an internal vertex. Then from equation (1), $p_{ij}^{-}$ gives the $\mu^-$ strength of paths of length equal to 2 between $v_i$ and $v_j$.

Thus from the above two subcases also we can conclude that for $i \neq j$, $p_{ij}^{-}$ gives the maximum of the $\mu^-$ strength of paths of length less than or equal to 2 between $v_i$ and $v_j$. In otherwords, for $i \neq j$, $p_{ij}^{-}$ gives the $\mu^-$ connectedness of strength 2 between the nodes $v_i$ and $v_j$.

Similarly, for $i \neq j$, we can prove that $p_{ij}^{+}$ gives the $\mu^+$ connectedness of strength 2 between the nodes $v_i$ and $v_j$.

**Lemma 2.** *Let $G = (A, B)$ be an IVFG with n nodes and $N_G$ be the corresponding IVFNAM. Let $N_G^k = Q_G = [q_{ij}^{-}, q_{ij}^{+}]$. Then for $1 \leq i < j \leq n$, $q_{ij}^{-}$ and $q_{ij}^{+}$ give the $\mu^-$ and $\mu^+$ connectedness of strength k between the nodes $v_i$ and $v_j$, respectively. Further, $q_{ii}^{-} = \mu_A^{-}(v_i)$ and $q_{ii}^{+} = \mu_A^{+}(v_i)$ $\forall\ i = 1,2,\ldots,n$.*

*Proof.* Let $G = (A, B)$ be an IVFG and $N_G$ be the corresponding IVFNAM. We prove the lemma by the method of mathematical induction on the power $k$ of $N_G$. By lemma 1, the statement is true for $k = 2$. Now, suppose that it is true for k = m. Let $N_G^m = T_G = [t_{ij}^{-}, t_{ij}^{+}]$. Then by our assumption, for $i = j$, $t_{ii}^{-} = \mu_A^{-}(v_i)$ and $t_{ii}^{+} = \mu_A^{+}(v_i)$ and for $i \neq j$, $t_{ij}^{-}$ and $t_{ij}^{+}$ give respectively the $\mu^-$ and $\mu^+$ connectedness of strength $m$ between the nodes $v_i$ and $v_j$.

Let $N_G^{m+1} = Q_G = [q_{ij}^{-}, q_{ij}^{+}]$. For $i = j, q_{ii}^{-} = \vee_{k=1}^{n}(t_{ik}^{-} \wedge n_{ki}^{-}) = n_{ii}^{-} = \mu_A^{-}(v_i)$ and $q_{ii}^{+} = \vee_{k=1}^{n}(t_{ik}^{+} \wedge n_{ki}^{+}) = n_{ii}^{+} = \mu_A^{+}(v_i)$ for all $i = 1,2,\ldots,n$.

Now, for $i \neq j$,
$$q_{ij}^{-} = \vee_{k=1}^{n}(t_{ik}^{-} \wedge n_{kj}^{-}) \tag{2}$$
where $t_{ik}^{-}$ is the $\mu^-$ connectedness of strength $m$ between the nodes $v_i$ and $v_k$. That is, $t_{ik}^{-}$ is the maximum of the $\mu^-$ strength of all paths of length less than or equal to $m$ between the nodes $v_i$ and $v_k$. Since $G$ is connected, there exists paths between every two nodes $v_i$ and $v_j$. Now, we have three cases.

(1) There exists at least one $v_i - v_k$ path of length less than or equal to $m$ and arc $(v_k, v_j)$ exists
In this case, the above said $v_i - v_k$ paths together with arc $(v_k, v_j)$ form $v_i - v_j$ paths of length less than or equal to $m+1$. Since $t_{ik}^{-}$ is the maximum of the $\mu^-$ strength of all paths of length less than or equal to $m$ between the nodes $v_i$ and $v_k$, from equation 2,$q_{ij}^{-}$ gives the maximum of the $\mu^-$ strength of all paths of length less than or equal to $m+1$ between the nodes $v_i$ and $v_j$ and hence the lemma.

(2) There does not exist any $v_i - v_k$ path of length less than or equal to $m$ for all k.
In this case there will not exist any $v_i - v_j$ path of length less than or equal to $m+1$ whether arc $(v_k, v_j)$ exists or not. Hence from equation 2, $q_{ij}^{-} = 0$, which by definition 8 is same as the $\mu^-$ connectedness of strength $m+1$ between $v_i$ and $v_j$ and hence the lemma.

(3) There exists at least one $v_i - v_k$ path of length less than or equal to $m$ and arc $(v_k, v_j)$ does not exit for all k. This is not possible since $G$ is connected.

Similarly, for $i \neq j$, we can prove that $q_{ij}^{+}$ gives the $\mu^+$ connectedness of strength $k$ between the nodes $v_i$ and $v_j$.

## 2.3 Reachability matrix

The reachability matrix of a fuzzy graph was introduced by Yeh and Bang in [10]. Analogous to this definition we define the reachability matrix of an IVFG.

**Definition 9.** *Let $G = (A,B)$ be an IVFG on $G^* = (V,E)$ and let $N_G$ be the corresponding IVFNAM. Then the IVFM $N_G^k$ is called the reachability matrix of $G$ if there exists a positive integer $k$ such that $N_G^k = N_G^{k+1}$. The reachability matrix of $G$ is usually denoted by $R_G = [r_{ij}]$*

The $\mu^-$ and $\mu^+$ strength of connectedness between any pair of nodes can be easily obtained using the reachability matrix and is given by the following theorem.

**Theorem 1.** *Let $G = (A,B)$ be an IVFG and let $N_G$ be the corresponding IVFNAM. Let $R_G = r_{ij} = [r_{ij}^-, r_{ij}^+]$ be the reachability matrix of $G$. Then $NCONN_G(v_i, v_j) = r_{ij}^-$ and $PCONN_G(v_i, v_j) = r_{ij}^+$*

*Proof.* Let $G = (A,B)$ be an IVFG and let $N_G$ be the corresponding IVFNAM. Let $R_G = r_{ij} = [r_{ij}^-, r_{ij}^+]$ be the reachability matrix of $G$. By the definition of the reachability matrix, $R_G$ is the reachability matrix of $G$ if there exists a +ve integer $k$ such that $R_G = N_G^k = N_G^{k+1}$. Let $N_G^k = [(n_{ij}^-)^{(k)}, (n_{ij}^+)^{(k)}]$ and $N_G^{k+1} = [(n_{ij}^-)^{(k+1)}, (n_{ij}^+)^{(k+1)}]$. Then we have $r_{ij}^- = (n_{ij}^-)^{(k)} = (n_{ij}^-)^{(k+1)}$. By lemma 2, $(n_{ij}^-)^{(k)}$ represents the maximum of the $\mu^-$ strength of all paths of length less than or equal to $k$ between the nodes $v_i$ and $v_j$. Since $(n_{ij}^-)^{(k)} = (n_{ij}^-)^{(k+1)}$ the maximum of the $\mu^-$ strength of all paths of length less than or equal to $k$ between the nodes $v_i$ and $v_j$ is same as the maximum of the $\mu^-$ strength of all paths of length less than or equal to $k+1$ between the nodes $v_i$ and $v_j$. Again we have $N_G^k = N_G^{k+1} = N_G^{k+2} = N_G^{k+3} = \ldots$ Hence using the above argument, we can conclude that $r_{ij}^- = (n_{ij}^-)^{(k)}$ represents the maximum of the $\mu^-$ strength of all paths between the nodes $v_i$ and $v_j$. Therefore, $NCONN_G(v_i, v_j) = r_{ij}^-$.

Similarly, we can prove that $PCONN_G(v_i, v_j) = r_{ij}^+$.

Now we see an illustration.

**Example 1.** Consider the IVFG $G = (A,B)$ given in Figure 1.

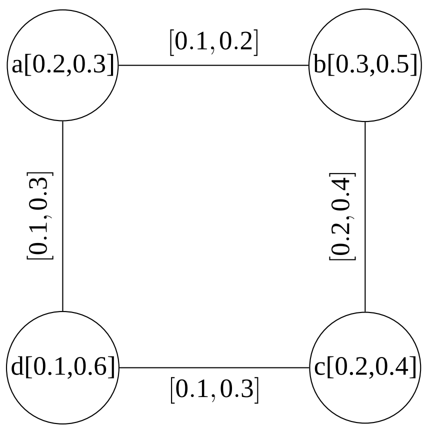


**Fig. 1:** Example to illustrate theorem 1

By definition 2, $CONN_G(a,b) = [NCONN_G(a,b),\ PCONN_G(a,b)] = [0.1,0.3]$, $CONN_G(a,c) = [0.1,0.3]$, $CONN_G(a,d) = [0.1,0.3]$, $CONN_G(b,c) = [0.2,0.4]$, $CONN_G(b,d) = [0.1,0.3]$ and $CONN_G(c,d) = [0.1,0.3]$.

This can also be obtained from the corresponding IVFNAM by applying theorem 1. Using definition 7,

$$N_G = \begin{bmatrix} [0.2,0.3] & [0.1,0.2] & [0.0,0.0] & [0.1,0.3] \\ [0.1,0.2] & [0.3,0.5] & [0.2,0.4] & [0.0,0.0] \\ [0.0,0.0] & [0.2,0.4] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.0,0.0] & [0.1,0.3] & [0.1,0.6] \end{bmatrix}. \quad \text{Then,} \quad N_G^2 = \begin{bmatrix} [0.2,0.3] & [0.1,0.2] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.2] & [0.3,0.5] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.6] \end{bmatrix},$$

$$N_G^3 = \begin{bmatrix} [0.2,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.5] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.6] \end{bmatrix}, \quad N_G^4 = \begin{bmatrix} [0.2,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.5] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.2,0.4] & [0.2,0.4] & [0.1,0.3] \\ [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.6] \end{bmatrix},$$

Here, since $N_G^3 = N_G^4$, the reachability matrix $R_G = N_G^3$. Hence by theorem 1, $CONN_G(a,b) = [NCONN_G(a,b), PCONN_G(a,b)] = [0.1,0.3]$, $CONN_G(a,c) = [0.1,0.3]$, $CONN_G(a,d) = [0.1,0.3]$, $CONN_G(b,c) = [0.2,0.4]$, $CONN_G(b,d) = [0.1,0.3], CONN_G(c,d) = [0.1,0.3]$.

**Theorem 2.** *The reachability matrix of a Complete Interval Valued Fuzzy Graph (CIVFG) G is G itself.*

*Proof.* Let $G$ be a CIVFG on $G^* = (V,E)$. Then by definition of a CIVFG, $\mu_B^-(xy) = min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y))$ for all $x, y \in V$. Let $N_G = [n_{ij}^-, n_{ij}^+]$ be the IVFNAM of $G$. Then for $i = j$, $n_{ii}^- = \mu_A^-(v_i)$ and $n_{ii}^+ = \mu_A^+(v_i)$ and for $i \neq j$, $n_{ij}^- = \mu_B^-(v_i, v_j) = min(\mu_A^-(v_i), \mu_A^-(v_j))$ and $n_{ij}^+ = \mu_B^+(v_i, v_j) = min(\mu_A^+(v_i), \mu_A^+(v_j))$.
Let $N_G^2 = P_G = [p_{ij}^-, p_{ij}^+]$. Then by lemma 1 for $i = j$, $p_{ii}^- = \mu_A^-(v_i)$ and $p_{ii}^+ = \mu_A^+(v_i)$. And for $i \neq j$, $p_{ij}^-$ and $p_{ij}^+$ gives respectively the $\mu^-$ and $\mu^+$ connectedness of strength 2 between the nodes $v_i$ and $v_j$. From equation 1, we have $p_{ij}^- = \vee_{k=1,k\neq i,j}^n (n_{ik}^- \wedge n_{kj}^-) \vee \mu_B^-(v_i, v_j)$. Since $G^*$ is complete, for any $k$, $v_i v_k v_j$ is a path in $G^*$. Then there are 2 cases.

**Case 1.** $\mu_A^-(v_k) \leq min(\mu_A^-(v_i), \mu_A^-(v_j))$ for every $k$. In this case,

$$\begin{aligned} p_{ij}^- &= \vee_{k=1,k\neq i,j}^n (n_{ik}^- \wedge n_{kj}^-) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (\mu_B^-(v_i, v_k) \wedge \mu_B^-(v_k, v_j)) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_i), \mu_A^-(v_k)) \wedge min(\mu_A^-(v_k), \mu_A^-(v_j))) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_i), \mu_A^-(v_k), \mu_A^-(v_j))) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_k), min(\mu_A^-(v_i), \mu_A^-(v_j)))) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n \mu_A^-(v_k) \vee min(\mu_A^-(v_i), \mu_A^-(v_j)) \\ &= min(\mu_A^-(v_i), \mu_A^-(v_j)) \end{aligned}$$

**Case 2.** $\mu_A^-(v_k) > min(\mu_A^-(v_i), \mu_A^-(v_j))$ for every $k$. Here,

$$\begin{aligned} p_{ij}^- &= \vee_{k=1,k\neq i,j}^n (n_{ik}^- \wedge n_{kj}^-) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (\mu_B^-(v_i, v_k) \wedge \mu_B^-(v_k, v_j)) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_i), \mu_A^-(v_k)) \wedge min(\mu_A^-(v_k), \mu_A^-(v_j))) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_i), \mu_A^-(v_k), \mu_A^-(v_j))) \vee \mu_B^-(v_i, v_j) \\ &= \vee_{k=1,k\neq i,j}^n (min(\mu_A^-(v_k), min(\mu_A^-(v_i), \mu_A^-(v_j)))) \vee \mu_B^-(v_i, v_j) \\ &= min(\mu_A^-(v_i), \mu_A^-(v_j)) \vee min(\mu_A^-(v_i), \mu_A^-(v_j)) \\ &= min(\mu_A^-(v_i), \mu_A^-(v_j)). \end{aligned}$$

**Table 1:** Types of arcs.

| Name | Requirement |
|---|---|
| $\alpha^-$ strong | $\mu_{B^-}(u,v) > NCONN_{G-(u,v)}(u,v)$ |
| $\alpha^+$ strong | $\mu_{B^+}(u,v) > PCONN_{G-(u,v)}(u,v)$ |
| $\alpha$ strong | $\alpha^-$ strong and $\alpha^+$ strong |
| $\beta^-$ strong | $\mu_{B^-}(u,v) = NCONN_{G-(u,v)}(u,v)$ |
| $\beta^+$ strong | $\mu_{B^+}(u,v) = PCONN_{G-(u,v)}(u,v)$ |
| $\beta$ strong | $\beta^-$ strong and$\beta^+$ strong |
| $\alpha\beta$ strong | $\alpha^-$ strong and $\beta^+$ strong |
| $\beta\alpha$ strong | $\beta^-$ strong and $\alpha^+$ strong |
| $\delta^- arc$ | $\mu_{B^-}(u,v) < NCONN_{G-(u,v)}(u,v)$ |
| $\delta^+ arc$ | $\mu_{B^+}(u,v) < PCONN_{G-(u,v)}(u,v)$ |
| $\delta$ arc | $\delta^- arc$ and $\delta^+ arc$ |
| $\alpha\delta$ | $\alpha^- strong$ and $\delta^+$ |
| $\beta\delta$ | $\beta^- strong$ and $\delta^+$ |
| $\delta\alpha$ | $\delta^-$ and $\alpha^+ strong$ |
| $\delta\beta$ | $\delta^-$ and $\beta^+ strong$ |

Similarly, we can prove that $p^+_{ij} = min(\mu^+_A(v_i), \mu^+_A(v_j))$ Hence from the above two cases, we can see that $N_G = N^2_G$. Therefore, the reachability matrix of $G$ is $G$ itself.

**Corollary 1.** *Let $G = (A,B)$ be a CIVFG. Then $\mu_{B^-}(u,v) = NCONN_G(u,v)$ and $\mu_{B^+}(u,v) = PCONN_G(u,v)$ for all arcs $(u,v) \in G$.*

*Proof.* Follows from theorem 1 and theorem 2.

Next we propose an algorithm to determine the nature of arcs in an IVFG using IVFNAM. See the table 1 for various types of arcs $(u,v)$ in an IVFG, $G$ and their requirements to be of that type[8].

### *2.4 Algorithm for determining the nature of $(v_i, v_j)(i \neq j)$ arc of an IVFG*

Let $G$ be an IVFG on $n$ nodes. A non zero non diagonal entry $n_{ij}$ of the corresponding IVFNAM $N_G$ indicates the existence of a $(v_i, v_j)$ arc. The nature of such an arc can be determined using the following steps.

(1) Write the IVFNAM $N_G = n_{ij}$ corresponding to $G$.
(2) Form the new matrix $N_{G-(v_i,v_j)}$ by replacing the entries $n_{ij}$ and $n_{ji}$ by $[0,0]$. Let it be $S_G$.
(3) Let $R_i = s_{i1}, s_{i2}, \ldots, s_{ij}, \ldots, s_{in}$ denotes the $i^{th}$ row of $S_G$.
Obtain the new $s_{ij}$ by the following procedure.
$s_{ij}(new) = \vee_{k=1}^{n}(s_{ik}(old) \wedge s_{kj}(old))$
(4) Do step 3. for every pair $i, j = 1, 2, \ldots n$.
(5) Form the matrix $N^2_{G-(v_i,v_j)} = [s_{ij}(new)]$
(6) Repeat steps 3. and 4. with $s_{ik}s$ replaced by those obtained in step 3.
(7) Form the matrix $N^3_{G-(v_i,v_j)} = [s_{ij}(new)]$
(8) Again repeat steps 3. and 4. with $s_{ik}s$ replaced by those obtained in step 6.
(9) Continue this process until $N^k_{G-(v_i,v_j)} = N^{k+1}_{G-(v_i,v_j)}$
(10) Form $R_{G-(v_i,v_j)} = [r^-_{ij}, r_{ij}+] = N^k_{G-(v_i,v_j)}$, the reachability matrix of $G-(v_i,v_j)$
(11) Find $d^-_{ij} = n^-_{ij} - r^-_{ij}$ and $d^+_{ij} = n^+_{ij} - r^+_{ij}$
(12) Depending on the values of $d^-_{ij}$ and $d^+_{ij}$ make the following conclusions:
(i) $d^-_{ij} > 0, d^+_{ij} > 0 \Rightarrow (v_i, v_j)$ *is* $\alpha$ *strong*

$$\begin{array}{rl}
\text{(ii)} & d^-_{ij} = 0, d^+_{ij} = 0 \Rightarrow (v_i, v_j) \quad is \quad \beta \quad strong \\
\text{(iii)} & d^-_{ij} > 0, d^+_{ij} = 0 \Rightarrow (v_i, v_j) \quad is \quad \alpha\beta \quad strong \\
\text{(iv)} & d^-_{ij} = 0, d^+_{ij} > 0 \Rightarrow (v_i, v_j) \quad is \quad \beta\alpha \quad strong \\
\text{(v)} & d^-_{ij} < 0, d^+_{ij} < 0 \Rightarrow (v_i, v_j) \quad is \quad a \quad \delta \quad arc \\
\text{(vi)} & d^-_{ij} > 0, d^+_{ij} < 0 \Rightarrow (v_i, v_j) \quad is \quad a \quad \alpha\delta \quad arc \\
\text{(vii)} & d^-_{ij} = 0, d^+_{ij} < 0 \Rightarrow (v_i, v_j) \quad is \quad a \quad \beta\delta \quad arc \\
\text{(viii)} & d^-_{ij} < 0, d^+_{ij} > 0 \Rightarrow (v_i, v_j) \quad is \quad a \quad \delta\alpha \quad arc \\
\text{(ix)} & d^-_{ij} < 0, d^+_{ij} = 0 \Rightarrow (v_i, v_j) \quad is \quad a \quad \delta\beta \quad arc
\end{array}$$

## 3 Interval valued Fuzzy incidence matrix

**Definition 10.** *Let $G = (A,B)$ be an IVFG. Then the interval - valued fuzzy incidence matrix (IVFIM) of $G$ is the IVFM with rows corresponding to $v_1, v_2, \ldots v_n$ and columns corresponding to $e_1, e_2, \ldots e_m$. It is denoted by $E_G = [e_{ij}]$ where*

$$e_{ij} = \begin{cases} [e^-_{ij}, e^+_{ij}] = [\mu^-_B(e_j), \mu^+_B(e_j)] \quad if \quad the \quad j^{th} \quad arc \quad has \quad one \quad end \quad v_i, \\ [0,0] \quad otherwise. \end{cases}$$

### *3.1 Properties of Interval Valued Fuzzy Incidence Matrix*

(1) $E_G$ is a $n \times m$ matrix
(2) Each column consists of exactly two non - zero equal entries as each arc is incident with exactly two nodes.
(3) The sum of all entries in the row corresponding to $v_i$ gives the degree of $v_i$.
(4) The sum of all entries of $E_G$ gives twice $S_G$ .

The next theorem relates the IVFIM of an IVFG $G$ to the IVFNAM of the line graph of $G$. We denote by $E^T_G$, the transpose of the matrix $E_G$

**Theorem 3.** *Let $G = (A,B)$ be an IVFG with IVFIM $E_G$. Then $E^T_G E_G = N_{L(G)}$ where $L(G)$ denotes the line graph of $G$.*

*Proof.* Let $G = (A,B)$ be an IVFG with $n$ nodes and $m$ arcs.Also let $E_G$ be the corresponding IVFIM. Clearly, $E_G$ is a $n \times m$ symmetric IVFM. Hence $E^T_G$ is a $m \times n$ symmetric IVFM. Therefore $E^T_G E_G$ is a $m \times m$ IVFM. Let $E^T_G E_G = [d_{ij}] = [d_{ij^-}, d_{ij^+}]$ . Let $L(G)$ be the line graph of $G$. We have to show that $E^T_G E_G = N_{L(G)}$.
Let

$$E_G = \begin{bmatrix} [a^-_{11}, a^+_{11}] & [a^-_{12}, a^+_{12}] & \ldots [a^-_{1j}, a^+_{1j}] & \ldots [a^-_{1m}, a^+_{1m}] \\ [a^-_{21}, a^+_{21}] & [a^-_{22}, a^+_{22}] & \ldots [a^-_{2j}, a^+_{2j}] & \ldots [a^-_{2m}, a^+_{2m}] \\ \ldots & \ldots & \ldots & \ldots \\ [a^-_{i1}, a^+_{i1}] & [a^-_{i2}, a^+_{i2}] & \ldots [a^-_{ij}, a^+_{ij}] & \ldots [a^-_{im}, a^+_{im}] \\ \ldots & \ldots & \ldots & \ldots \\ [a^-_{n1}, a^+_{n1}] & [a^-_{n2}, a^+_{n2}] & \ldots [a^-_{nj}, a^+_{nj}] & \ldots [a^-_{nm}, a^+_{nm}] \end{bmatrix}$$

Then

$$E^T_G = \begin{bmatrix} [a^-_{11}, a^+_{11}] & [a^-_{21}, a^+_{22}] & \ldots [a^-_{i1}, a^+_{i1}] & \ldots [a^-_{n1}, a^+_{n1}] \\ [a^-_{12}, a^+_{12}] & [a^-_{22}, a^+_{22}] & \ldots [a^-_{i2}, a^+_{i2}] & \ldots [a^-_{n2}, a^+_{n2}] \\ \ldots & \ldots & \ldots & \ldots \\ [a^-_{1j}, a^+_{1j}] & [a^-_{2j}, a^+_{2j}] & \ldots [a^-_{ij}, a^+_{ij}] & \ldots [a^-_{nj}, a^+_{nj}] \\ \ldots & \ldots & \ldots & \ldots \\ [a^-_{1m}, a^+_{1m}] & [a^-_{2m}, a^+_{2m}] & \ldots [a^-_{im}, a^+_{im}] & \ldots [a^-_{nm}, a^+_{nm}] \end{bmatrix}$$

Let $E_G^T E_G = [d_{ij}] = [d_{ij}^-, d_{ij}^+]$. Clearly, from the above two IVFMs, when $i = j$

$$\begin{aligned} d_{ij}^- &= [a_{1i}^- \wedge a_{1i}^-] \vee [a_{2i}^- \wedge a_{2i}^-] \vee \cdots \vee [a_{ii}^- \wedge a_{ii}^-] \vee \cdots \vee [a_{ni}^- \wedge a_{ni}^-] \\ &= a_{1i}^- \vee a_{2i}^- \vee \cdots \vee a_{ii}^- \vee \cdots \vee a_{ni}^- \\ &= \mu_B^-(e_i) \end{aligned}$$

Similarly, $d_{ij}^+ = \mu_B^+(e_i)$. When $i \neq j$,

$$\begin{aligned} d_{ij}^- &= [a_{1i}^- \wedge a_{1j}^-] \vee [a_{2i}^- \wedge a_{2j}^-] \vee \cdots \vee [a_{ii}^- \wedge a_{ij}^-] \vee \cdots \vee [a_{ni}^- \wedge a_{nj}^-] \\ &= \min(\mu_B^-(e_i), \mu_B^-(e_j)) \quad if \quad e_i \quad and \quad e_j \quad are \quad incident \\ &= 0 \quad otherwise \end{aligned}$$

Similarly,

$$d_{ij}^+ = \begin{cases} \min(\mu_B^+(e_i), \mu_B^+(e_j)) \quad if \quad e_i \quad and \quad e_j \quad are \quad incident, \\ 0 \quad otherwise. \end{cases}$$

Thus the diagonal entries $(d_{ii}, i = 1, 2, \ldots m)$ of $E_G^T E_G$ are the membership degrees of the corresponding arcs($e_i, i = 1, 2, \ldots m$). Also $d_{ij}$ is non zero if and only if $e_i$ and $e_j$ are incident, the non zero value being the minimum of membership degrees of $e_i$ and $e_j$. Now, clearly by the definition of the line graph of $G$ and IVFNAM, $E_G^T E_G$ is same as $N_{L(G)}$.

**Definition 11.** *Let $G = (A, B)$ be an IVFG. Then the interval - valued fuzzy node matrix (IVFNM) of $G$ is the IVFM with rows and columns corresponding to $v_1, v_2, \ldots, v_n$. It is denoted by $M_G = [m_{ij}]$ where*

$$m_{ij} = \begin{cases} [m_{ij}^-, m_{ij}^+] = [\mu_A^-(v_i), \mu_A^+(v_i)] \quad if \quad i = j, \\ [0, 0] \quad otherwise. \end{cases}$$

Obviously, $M_G$ is a diagonal matrix and trace of $M_G = O(G)$ .

**Theorem 4.** *Let $G = (A, B)$ be an IVFG with IVFAM $A_G$, IVFNAM $N_G$ and IVFNM $M_G$. Then $A_G = N_G - M_G$ where '-' denotes ordinary subtraction.*

*Proof.* Proof follows from the definitions of $A_G$, $N_G$ and $M_G$.

In crisp graph theory, adjacency matrix of the line graph of a graph $G$ with $n$ nodes and $m$ arcs is related to the incidence matrix of $G$ by the formula $A(L(G)) = B^T B - 2I_m$ where $B$ is the incidence matrix and $I_m$ denotes the identity matrix of order $m$ [2]. But in IVFGs, IVFAM of the line graph of an IVFG $G$ with $n$ nodes and $m$ arcs is related to the IVFIM of $G$ by the following theorem.

**Theorem 5.** *Let $G = (A, B)$ be an IVFG with IVFIM $E_G$ and IVFNM $M_G$. Then $A_{L(G)} = E_G^T E_G - M_{L(G)}$ where $L(G)$ denotes the line graph of $G$.*

*Proof.* Follows directly from *theorem 3 and theorem 4*.

## 4 Conclusion

In this paper, we have shown that Inteval-Valued Fuzzy Graphs (IVFG) can be completely represented using a special type of Inteval Valued Fuzzy Matrix called Inteval Valued Fuzzy Node Arc Matrix (IVFNAM). Then we defined reachability matrix of an IVFG using IVFNAM and proved that the strength of connectedness between any two pair of vertices in an IVFG can be found using the corresponding reachability matrix. We proposed an algorithm to determine the nature of arcs in an IVFG using the reachability matrix. We also defined Inteval Valued Fuzzy Incidene Matrix and Inteval Valued Fuzzy Node Matrix of an IVFG, studied their properties and established relationships between those matrices. We have also found a relationship between IVFIM and the IVFNAM of the corresponding line graph.

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## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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