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Generalized Euler's $\Phi-$function of graph

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Abstract: Let G = (V, E) be a simple connected undirected graph. In this paper, we define generalized Euler's Φ-Function of a graph which is the summation of the Euler's $\varphi-$function of the degree of the vertices of a graph and it is denoted by Φ(G). It is determined the general form of Euler's Φ-function of some standard graphs. Finally, some important results and properties are studied.

Keywords: Euler's $\Phi-$function, Euler's $\varphi-$function.

1 Introduction

The Euler $\varphi-$function, which is also called the totient function or indicator function where Gauss introduced the symbol $\varphi(n)$, is counted as one of the typical area in number theory defined as the number of positive integers less than n which are relatively prime (or co-prime) to n [5]. For instance, there are 8 positive integers less than 20 which are relatively prime to 20. Also, there are 18 positive integers less than 19 which are relatively prime to 19. So, the above example is denoted by $\varphi(20) = 8$ and $\varphi(19) = 18$, respectively. Also, from the above example it is obtained that if $(n = p)$ is prime, then $\varphi(p) = p-1$ and in general, $\varphi\left(p^k\right) = p^k - p^{k-1}$ for any positive integer k. For any positive integer n we have that $\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_t}\right)$ where $n = {p_1}^{a_1}{p_2}^{a_2}\ldots{p_t}^{a_t}$ [1,4,6,8,9].

The Mobius function was defined in [1] and denoted by $\mu(n)$ which is defined to be

$$\mu(n) = \begin{cases} 1, & if\ n=1 \\ (-1)^t, & if n = p_1p_2\ldots p_t\ where\ the\ p_i\ are\ distinct\ primes; \\ 0, & otherwise \end{cases}$$

A function f is said to be multiplicative if for all positive integers m,n such that m,n are relatively prime, then $f(mn) = f(m)f(n)$. Both the Euler $\varphi-$ function and the Mobius function are multiplicative [8].

Providing an interesting connection between number theory and graph theory can be found and given in [2,3,7,10]. In this paper, we attempt to use a number theory function called the Euler's $\varphi-$function into graph theory. Let G be a simple connected undirected graph. We define generalized Euler Φ-function $\Phi(G)$ of any graph as the sum of the Euler's $\varphi-$ function for the degree of vertices of the graph G. It is shown that for all $v \in G$, $\deg(v) \geq 1$ such that $\deg(v)$ with at most 8 distinct prime factors, we have that $\Phi(G) > \frac{1}{6}\sum_{v\in V(G)}\deg(v)$. It is also shown that a relation the Euler's $\varphi-$ function and the Mobius function.

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2 Generalized Euler's $\Phi-$ function of some standard graphs

In this section, we define generalized Euler's $\Phi-$ Function of a graph and determine generalized Euler's $\Phi-$ function of some standard graphs in graph theory which are the path graph P_n, cycle graph C_n, complete graph K_n, complete bipartite graph $K_{m,n}$, complete $k-$ partite graph $K_{m_1,m_2,\ldots m_n}$, star graph S_n and wheel graph W_n.

Definition 1. *Let G be a simple connected graph and let φ $(deg(v))$ be defined as the Euler's $\varphi-$ Function for the degree of vertices v of any graph G which denoted by $\Phi(G)$. Then $\Phi(G) = \sum_{v\in V(G)} \varphi(deg(v))$.*

For example, generalized Euler's $\Phi-$function of a graph $G = K_4$ is given by

$$\Phi(K_4) = \sum_{v\in V(K_4)} \varphi(\deg(v)) = \varphi(\deg(v_1) + \varphi(\deg(v_2) + \varphi(\deg(v_3) + \varphi(\deg(v_4)$$
$$= 2+2+2+2 = 8.$$

Proposition 1. *For all $v \in V(G)$, $\deg(v) \geq 1$, and a graph G consists of n components $G_1,\ldots,G_n$, then*

$$\Phi(G) = \sum_{i=1}^{n} \Phi(G_i).$$

In the following proposition, we determine the general form of generalized Euler's $\Phi-$function of some standard graphs.

Proposition 2. (1) *The Euler's $\Phi-$function of the path graph P_n, for $n \geq 2$ vertices, is $\Phi(P_n) = n$.*

(2) *The Euler's $\Phi-$function of the cycle graph C_n, for $n \geq 3$ vertices, is $\Phi(C_n) = n$.*

(3) *The Euler's Φ-function of the complete graph K_n, for $n \geq 3$ vertices, is*

$$\Phi(K_n) = n\varphi(n-1).$$

(4) *The Euler's $\Phi-$function of the complete bipartite graph $K_{m,n}$, for any positive integers m,n vertices, is $\Phi(K_{m,n}) = (m*\varphi(n)+(n*\varphi(m))$.*

(5) *The Euler's $\Phi-$function of the star graph S_n, for $n \geq 2$ vertices, is*

$$\Phi(S_n) = \varphi(n-1)+(n-1).$$

(6) *The Euler's $\Phi-$function of the complete k-partite graph $K_{m_1,m_2,\ldots,m_n}$, for any positive integers $m_1,m_2,\ldots,m_n$ vertices, is*

$$\Phi(K_{m_1,m_2,\ldots,m_n}) = (m_1*\varphi(m_2+m_3+\cdots+m_n)+(m_2*\varphi(m_1+m_3+\cdots+m_n)+\cdots+(m_n*\varphi(m_1+m_2+\cdots+m_n))$$
$$= \sum_{i=1}^{n}\left(m_i*\varphi\left(\sum_{j\neq i} m_j\right)\right).$$

(7) *The Euler's $\Phi-$function of the Wheel graph W_n, for $n \geq 4$ vertices, is*

$$\Phi(W_n) = \varphi(n-1)+2(n-1).$$

Proof.

(1) The path graph P_n of order n, if $n = 2$, we have two vertices of degree one, then we have $\Phi(P_2) = \sum_{v\in V(P_2)} \varphi(\deg(v)) = \varphi(\deg(v_1)) + \varphi(\deg(v_2)) = 1+1 = 2$. If $n \geq 3$, we have two vertices of degree one

and $n-2$ vertices of degree two, then we have

$$\Phi(P_n)=\sum_{v\in V(P_n)}\varphi(\deg(v))=1+1+n-2=n.$$

(2) In a cycle graph C_n of order n, we have n vertices of degree two, and then we have:

$$\begin{aligned}\Phi(C_n)=\sum_{v\in V(C_n)}\varphi(\deg(v))&=\varphi(\deg(v_1))+\varphi(\deg(v_2))+\cdots+\varphi(\deg(v_n))\\&=1+1+\cdots+1=n\end{aligned}$$

(3) In a complete graph K_n of order n, we have n vertices of degree $n-1$, and then we have:

$$\begin{aligned}\Phi(K_n)=\sum_{v\in V(K_n)}\varphi(\deg(v))&=\varphi(\deg(v_1))+\varphi(\deg(v_2))+\cdots+\varphi(\deg(v_n))\\&=\varphi(n-1)+\varphi(n-1)+\cdots+\varphi(n-1)\\&=n\varphi(n-1).\end{aligned}$$

(4) In a complete bipartite graph $K_{m,n}$ of order $m+n$, we have n vertices of degree m and we have m vertices of degree n, and then we have:

$$\begin{aligned}\Phi(K_{m,n})=&\sum_{v\in V(K_{m,n})}\varphi(\deg(v))=\varphi(\deg(v_1))+\varphi(\deg(v_2))+\cdots+\varphi(\deg(v_n))\\&+\varphi(\deg(v_1))+\varphi(\deg(v_2))+\cdots+\varphi(\deg(v_m))\\&=\varphi(n)+\varphi(n)+\cdots+\varphi(n)+\varphi(m)+\varphi(m)+\cdots+\varphi(m)\\&=m\varphi(n)+n\varphi(m).\end{aligned}$$

(5) In a star graph S_n of order n, we have $n-1$ vertices of degree one and we have one vertex of degree $n-1$, say v_1, and then we have:

$$\begin{aligned}\Phi(S_n)=\sum_{v\in V(S_n)}\varphi(\deg(v))&=\varphi(\deg(v_1))+\varphi(\deg(v_2))+\cdots+\varphi(\deg(v_n))\\&=\varphi(n-1)+1+\cdots+1\\&=\varphi(n-1)+(n-1).\end{aligned}$$

(6) In a complete k-partite graph $K_{m_1,m_2,\ldots,m_n}$ of order $m_1+m_2+\cdots+m_n$, we have m_i vertices of degree $\sum_{j\neq i}m_j$ where $j,i=1,2,\ldots,n$, and then we have:

$$\begin{aligned}\Phi(K_{m_1,m_2,\ldots,m_n})=&\sum_{v\in V(K_{m_1,m_2,\ldots,m_n})}\varphi(\deg(v))\\&=(m_1*\varphi(m_2+m_3+\cdots+m_n))+(m_2*\varphi(m_1+m_3+\cdots+m_n))+\cdots+(m_n*\varphi(m_1+m_2+\cdots+m_{n-1}))\\&=\sum_{i=1}^{n}\left(m_i*\varphi\left(\sum_{j\neq i}m_j\right)\right)\end{aligned}$$

(7) In a wheel graph W_n of order n, we have $n-1$ vertices of degree three and we have one vertex of degree $n-1$, say v_1, and then we have:

$$\begin{aligned}\Phi(W_n) = \sum_{v\in V(W_n)} \varphi(\deg(v)) &= \varphi(\deg(v_1)) + \varphi(\deg(v_2)) + \cdots + \varphi(\deg(v_n)) \\ &= \varphi(n-1) + \varphi(3) + \cdots + \varphi(3) \\ &= \varphi(n-1) + 2(n-1).\end{aligned}$$

In the following proposition, we introduce generalized Euler's $\Phi-$Function of the corona product of some special graphs. The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1 and $\mid V(G_1) \mid$ copies of G_2; and by joining each vertex of the $i-$th copies of G_2 to the i-th vertices of G_1, where $1 \le i \le\mid V(G_1) \mid$.

Proposition 3.

(1) $\Phi(P_n \circ K_1) = 2n$.
(2) $\Phi(C_n \circ K_1) = 3n$.
(3) $\Phi(K_n \circ K_1) = n(\varphi(n)+1)$.
(4) $\Phi(K_{m,n} \circ K_1) = m\varphi(n+1) + n\varphi(m+1) + m + n$.
(5) $\Phi(S_n \circ K_1) = \varphi(n) + 2n + 1$.
(6) $\Phi(K_{m_1,m_2,\ldots,m_n} \circ K_1) = \sum_{i=1}^{n} \left(\left(m_i * \varphi\left(\sum_{j\neq i}(m_j+1)\right)\right) + m_i\right)$.
(7) $\Phi(W_n \circ K_1) = \varphi(n) + 3n - 2$.

*Proof.*The proof is similar to the proof of Proposition 2.

3 Generalized Euler's $\Phi-$function of graphs

In this section, it is shown that for all $v \in G$, $\deg(v) \ge 1$ such that $\deg(v)$ with at most 8 distinct prime factors, we have that $\Phi(G) > \frac{1}{6}\sum_{v\in V(G)} \deg(v)$. It is also shown a relation between the Euler's $\varphi-$function and the Mobius function.

The following lemma is the most useful properties of Generalized Euler's $\Phi-$function for the divisor sum of the degree of the vertices in graph.

Lemma 1.*For all $v \in V(G)$, $\deg(v) \ge 1$, we have*

$$\sum_{d|\deg(v) \text{ for all } v\in V(G)} \Phi_d(G) = \sum_{v\in V(G)} \sum_{d|\deg(v)} \varphi(d) = \sum_{v\in V(G)} \deg(v),$$

where d is the divisor of the degree of vertices in a graph G.

Proof. Let, for $v \in V(G)$, $F(\deg(v)) = \varphi(d_1) + \varphi(d_2) + \cdots + \varphi(d_t)$, where d_t is any divisor of $(\deg(v))$ and $\deg(v) = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$. Then, by using the telescoping series, we have that $F(p^k) = 1 + (p-1) + (p^2-p) + (p^3-p^2) + \cdots + (p^k - p^{k-1}) = p^k$, where $(1, p, p^2, p^3, \ldots)$ are the divisors of p^k. Since F is multiplicative, so

$$\begin{aligned}\sum_{v\in V(G)} F(\deg(v)) &= \sum_{d|\deg(v)} F\left(p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}\right) = \sum_{d|\deg(v)} \left(F\left(p_1^{a_1}\right) * F\left(p_2^{a_2}\right) * \cdots * F\left(p_t^{a_t}\right)\right) \\ &= \sum_{d|\deg(v)} \left(p_1^{a_1} * p_2^{a_2} * \cdots * p_t^{a_t}\right) = \sum_{d|\deg(v)} \deg(v)\end{aligned}$$

Lemma 2. *For all* $v \in V(G)$, $\deg(v) \geq 1$, *we have*

$$\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) = \sum_{v\in V(G)} \left(\deg(v) * \prod_{p|\deg(v)} \left(1 - \frac{1}{p}\right)\right),$$

where $(\deg(v)) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$.

Proof. Let $v \in V(G)$, $\left(\deg(v) = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}\right) \geq 1$ and since φ is multiplicative, thus

$$\begin{aligned}
\Phi(G) &= \sum_{v\in V(G)} \varphi(\deg(v)) = \sum_{v\in V(G)} \varphi\left(p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}\right) = \sum_{v\in V(G)} \left(\varphi\left(p_1^{a_1}\right) * \varphi\left(p_2^{a_2}\right) \dots \varphi\left(p_t^{a_t}\right)\right) \\
&= \sum_{v\in V(G)} \left(\left(p_1^{a_1} - p_1^{a_1-1}\right) * \left(p_2^{a_2} - p_2^{a_2-1}\right) * \dots * \left(p_t^{a_t} - p_t^{a_t-1}\right)\right) \\
&= \sum_{v\in V(G)} \left(p_1^{a_1}\left(1 - \frac{1}{p_1}\right) * p_2^{a_2}\left(1 - \frac{1}{p_2}\right) * \dots * p_t^{a_t}\left(1 - \frac{1}{p_t}\right)\right) \\
&= \sum_{v\in V(G)} \left(\left(p_1^{a_1} * p_2^{a_2} * \dots * p_t^{a_t}\right)\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_t}\right)\right) \\
&= \sum_{v\in V(G)} \left(\deg(v) * \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_t}\right)\right) \\
&= \sum_{v\in V(G)} \left(\deg(v) * \prod_{p|\deg(v)} \left(1 - \frac{1}{p}\right)\right).
\end{aligned}$$

From the definition of Generalized Euler's $\Phi-$function of graph, a relation between the Euler's $\varphi-$function and the Mobius function is given in the following lemma:

Lemma 3. *For all* $v \in V(G)$, $\deg(v) \geq 1$, *we have*

$$\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) = \sum_{v\in V(G)} \left(\sum_{d|\deg(v)} \left(\mu(d) * \frac{\deg(v)}{d}\right)\right),$$

where μ *is the Mobius function.*

Proof. Let $v \in V(G)$, $\deg(v) \geq 1$. Then by lemma 2, we have

$$\begin{aligned}
\Phi(G) &= \sum_{v\in V(G)} \varphi(\deg(v)) = \sum_{v\in V(G)} \left(\deg(v) * \prod_{p|\deg(v)} \left(1 - \frac{1}{p}\right)\right) \\
&= \sum_{v\in V(G)} \left(\deg(v) * \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_t}\right)\right) \\
&= \sum_{v\in V(G)} \left(\deg(v) * \left(1 - \sum_i \frac{1}{p_i} + \sum_{i\neq j} \frac{1}{p_i p_j} + \dots + (-1)^t \frac{1}{p_1 p_2 \dots p_t}\right)\right)
\end{aligned}$$

If we denote $\left(1 - \sum_i \frac{1}{p_i} + \sum_{i\neq j} \frac{1}{p_i p_j} + \dots + (-1)^t \frac{1}{p_1 p_2 \dots p_t}\right) = A$, so each term in A is $\frac{\pm 1}{d}$, where d is 1 in the first term or a product of distinct primes. The signs in front of each term are alternated by $(-1)^k$ according to the number of primes p's which is exactly what is done by the Mobius function. So

$$\begin{aligned}\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) &= \sum_{v\in V(G)} \left((\deg(v)) \sum_{d|\deg(v)} \frac{\mu(d)}{d} \right) \\ &= \sum_{v\in V(G)} \left(\sum_{d|\deg(v)} \mu(d) * \frac{\deg(v)}{d} \right).\end{aligned}$$

Theorem 1. *For all* $v \in V(G)$, $\deg(v) \geq 1$ *such that* $\deg(v)$ *with at most* 8 *distinct prime factors, we have that*

$$\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) > \frac{1}{6} \sum_{v\in V(G)} \deg(v).$$

Proof. Firstly, the proposition for a $\deg(v)$ with 8 distinct prime factors will be proved. Let $\deg(v) = {p_1}^{a_1} {p_2}^{a_2} \dots {p_t}^{a_t}$, then, for generalized Euler $\Phi-$function, we use Lemma 2 and we realize that

$$\begin{aligned}\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) &= \sum_{v\in V(G)} \left(\deg(v) * \prod_{p|\deg(v)} \left(1 - \frac{1}{p}\right) \right) \\ &= \left({p_1}^{a_1} {p_2}^{a_2} \dots {p_t}^{a_t} * \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_t}\right) \right).\end{aligned}$$

Because of the 8 distinct prime factors of $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$, we realize that $p_1 \geq 2, p_2 \geq 3, p_3 \geq 5, p_4 \geq 7, p_5 \geq 11, p_6 \geq 13, p_7 \geq 17, p_8 \geq 19$, since the first 8 primes are distinct.
Thus,

$$\frac{1}{p_1} \leq \frac{1}{2}, \frac{1}{p_2} \leq \frac{1}{3}, \frac{1}{p_3} \leq \frac{1}{5}, \frac{1}{p_4} \leq \frac{1}{7}, \frac{1}{p_5} \leq \frac{1}{11}, \frac{1}{p_6} \leq \frac{1}{13}, \frac{1}{p_7} \leq \frac{1}{17}, \frac{1}{p_8} \leq \frac{1}{19} \tag{1}$$

and then $\left(1 - \frac{1}{2}\right) \geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{19}\right) \geq \left(1 - \frac{1}{19}\right)$. We are now able to substitute (1) in the equation for $\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v))$, we will obtain that

$$\begin{aligned}\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) &\geq \sum_{v\in V(G)} (\deg(v)) * \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{19}\right) \\ &= \sum_{v\in V(G)} (\deg(v)) * \left(\frac{1}{2} * \frac{2}{3} * \frac{4}{5} * \frac{6}{7} * \frac{10}{11} * \frac{12}{13} * \frac{16}{17} * \frac{18}{19} \right) \\ &= \frac{1658880}{9699690} \sum_{v\in V(G)} (\deg(v)) = 0.171 * \sum_{v\in V(G)} (\deg(v)).\end{aligned}$$

However,

$$\frac{1}{6} \sum_{v\in V(G)} \deg(v) \approx 0.167 * \sum_{v\in V(G)} \deg(v).$$

This means that

$$\Phi(G) = \sum_{v\in V(G)} \varphi(\deg(v)) > \frac{1}{6} \sum_{v\in V(G)} \deg(v).$$

Each of the following factors $\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_8}\right)$ is less than one because two is the smallest possible prime, so each of the term in the bracket will be less than one, that is to say that once the product is multiplied by it, its value will be decreased. Therefore, if $v \in V(G)$, $\deg(v)$ has less than 8 distinct prime factors, the value for its generalized Euler $\Phi-$function will be greater than the value of generalized Euler $\Phi-$function of $\deg(v)$, for all $v \in V(G)$, with at most 8 distinct prime factors. Thus, for all $v \in G$, $\deg(v) \geq 1$ such that $\deg(v)$ with at most 8 distinct prime factors

$$\Phi(G) = \sum_{v \in V(G)} \varphi(\deg(v)) > \frac{1}{6} \sum_{v \in V(G)} \deg(v).$$

Theorem 2. *For all $v \in V(G)$, we have that*

$$\sum_{v \in V(G)} \frac{\deg(v)}{\varphi(\deg(v))} = \sum_{v \in V(G)} \sum_{d|\deg(v)} \frac{\mu^2(d)}{\varphi(d)}.$$

Proof. By lemma 2, we have that for each $v \in V(G)$ $\deg(v) = {p_1}^{a_1} {p_2}^{a_2} \ldots {p_t}^{a_t}$ we have

$$\begin{aligned}
\varphi(\deg(v)) &= \deg(v) * \prod_{p|\deg(v)} \left(1 - \frac{1}{p}\right) \\
&= \deg(v) * \left(1 - \frac{1}{p_1}\right) * \left(1 - \frac{1}{p_2}\right) * \cdots * \left(1 - \frac{1}{p_t}\right) \\
&= \frac{\deg(v) * (p_1 - 1) * (p_2 - 1) * \cdots * (p_t - 1)}{p_1 * p_2 * \cdots * p_t}.
\end{aligned}$$

Hence, for each $v \in V(G)$, we have that

$$\begin{aligned}
\frac{\deg(v)}{\varphi(\deg(v))} &= \frac{\deg(v)}{\frac{\deg(v)*(p_1-1)*(p_2-1)*\cdots*(p_t-1)}{p_1*p_2*\cdots*p_t}} \\
&= \frac{p_1 * p_2 * \cdots * p_t}{(p_1 - 1) * (p_2 - 1) * \cdots * (p_t - 1)}.
\end{aligned}$$

By taking summation over the vertices $v \in V(G)$, then

$$\sum_{v \in V(G)} \frac{\deg(v)}{\varphi(\deg(v))} = \sum_{p_i|\deg(v)} \frac{p_1 * p_2 * \cdots * p_t}{(p_1 - 1) * (p_2 - 1) * \cdots * (p_t - 1)}.$$

This is what we have obtained from the left hand side of our identity above. From now on in this prove LHS will be represented to $\sum_{v \in V(G)} \frac{\deg(v)}{\varphi(\deg(v))}$.

For the right hand side (RHS) of our identity above. It is known from the definition of the Mobius function, where $\deg(v) = p_1 p_2 \ldots p_t$, $\mu(\deg(v)) = (-1)^t$ and $\mu(\deg(v)) = 0$ if $\deg(v)$ has a square term. Hence, $\mu^2(d) = 1$ if d does not have a square term and $\mu^2(d) = 0$ if d has a square term. Thus, we only have $\deg(v)$ which can be factorised as the product of distinct primes or we also call the product of distinct primes as square-free. From now on in this proof d_s will be represented as a square-free. Thus,

$$\sum_{d_s|\deg(v)} \frac{\mu^2(d_s)}{\varphi(d_s)} = \sum_{d_s|\deg(v)} \frac{1}{\varphi(d_s)}.$$

Since d_s is square-free, so each prime in the factorisation of $\deg(v)$ is used once. Therefore, the above expression means that each of the value 1, $(p_1, p_2, \ldots, p_t), (p_1 * p_2, p_2 * p_3, \ldots, p_{t-1} * p_t), (p_1 * p_2 * \cdots * p_t)$ is taken by d_s. Therefore, the

RHS is becoming

$$\begin{aligned}
\sum_{d_s|\deg(v)} \frac{1}{\varphi(d_s)} &= \frac{1}{\varphi(1)} + \frac{1}{\varphi(p_1)} + \cdots + \frac{1}{\varphi(p_1 * p_2)} + \cdots + \frac{1}{\varphi(p_1 * p_2 * \ldots p_t)} \\
&= 1 + \frac{1}{p_1 - 1} + \cdots + \frac{1}{(p_1-1)*(p_2-1)} + \cdots + \frac{1}{(p_1-1)*(p_2-1)*\cdots*(p_t-1)} \\
&= \frac{((p_1-1)*\cdots*(p_t-1)) + ((p_2-1)*\cdots*(p_t-1)) + \cdots + ((p_t-1)+\cdots+1)}{\prod_{i=1}^{t}(p_i-1)} \\
&= \frac{1+(p_1-1)+\cdots+(p_t-1)+(p_1-1)*(p_2-1)+\cdots+(p_1-1)*(p_2-1)*\cdots*(p_t-1)}{\prod_{i=1}^{t}(p_i-1)} \\
&= \frac{\varphi(1)+\varphi(p_1)+\cdots+\varphi(p_t)+\varphi(p_1)*\varphi(p_2)+\cdots+\varphi(p_1)*\varphi(p_2)*\cdots*\varphi(p_t)}{\prod_{i=1}^{t}(p_i-1)}.
\end{aligned}$$

Since φ is multiplicative as all primes are relatively primes, so the RHS will become

$$\begin{aligned}
\sum_{d_s|\deg(v)} \frac{1}{\varphi(d_s)} &= \frac{\varphi(1)+\varphi(p_1)+\cdots+\varphi(p_t)+\varphi(p_1*p_2)+\cdots+\varphi(p_1*p_2*\cdots*p_t)}{\prod_{i=1}^{t}(p_i-1)} \\
&= \frac{\sum_{d_N|(p_1*p_2*\cdots*p_t)} \varphi(d_N)}{\prod_{i=1}^{t}(p_i-1)}.
\end{aligned}$$

Therefore, for each $v \in V(G)$ we have, from lemma 2, that

$$\frac{\sum_{d_N|(p_1*p_2*\cdots*p_t)} \varphi(d_N)}{\prod_{i=1}^{t}(p_i-1)} = \frac{\deg(v)}{\prod_{i=1}^{t}(p_i-1)}$$

By taking summation over all vertices $v \in V(G)$, we have

$$\sum_{v\in V(G)} \sum_{d_s|\deg(v)} \frac{1}{\varphi(d_s)} = \sum_{v\in V(G),\ p_i|\deg(v)} \frac{\deg(v)}{\prod_{i=1}^{t}(p_i-1)} = \sum_{p_i|\deg(v)} \frac{(p_1*p_2*\cdots*p_t)}{\prod_{i=1}^{t}(p_i-1)},$$

when $\deg(v) = (p_1 * p_2 * \cdots * p_t)$, which is equal to the LHS. Hence,

$$\sum_{v\in V(G)} \frac{\deg(v)}{\varphi(\deg(v))} = \sum_{v\in V(G)} \sum_{d|\deg(v)} \frac{\mu^2(d)}{\varphi(d)}.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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