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Contra P_p-continuous functions

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Abstract: In this paper, we apply the notion of P_p-open sets in topological spaces to present and study a new class of functions called contra P_p-continuous functions which lies between classes of contra θ-continuous functions and contra-precontinuous functions. It is shown that contra P_p-continuous is weaker than contra θ-continuous, but it is stronger than contra-precontinuous and weakly P_p-continuous. Furthermore, we obtain basic properties and preservation theorems of contra P_p-continuity.

Keywords: *P_p-open, preopen, contra-continuous; contra P_p-continuous, Contra-precontinuous.*

1 Introduction

In 1996, Dontchev [3] introduced and investigated a new notion of continuity called contra-continuity. Following this, many authors introduced many types of new generalizations of contra-continuity called as contra θ-continuous [2], perfectly continuous [14] and contra-precontinuous [6]. Long and Herrington [9] have introduced a new class of functions called strongly θ-continuous function. Noiri and Popa [15] have introduced and studied quasi θ-continuous function. In this direction, we will introduce and investigate the concept of contra P_p-continuous function via the notion of P_p-open set and study some properties of contra P_p-continuous.

2 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1. *A subset A of a space X is said to be*

(1) *preopen* [10] *if $A \subset Int(Cl(A))$.*
(2) *α-open* [13] *if $A \subset Int(Cl(Int(A)))$.*
(3) *regular open* [19] *if A =Int(Cl(A)).*

The complement of a preopen (resp., α-open and regular open) set is preclosed (resp., α-closed and regular closed). The family of all preopen of X is denoted by $PO(X)$. In 1968, Velicko [20] defined the concept of θ-open set in X which is denoted by $\theta O(X)$. A subset A of a space X is called θ-open set if for each $x \in A$, there exists an open set G such that $x \in G \subset Cl(G) \subset A$. The complement of θ-open set is said to be θ-closed set.

Definition 2. [8] *A subset A of a space X is called P_p-open, if for each $x \in A \in PO(X)$, there exists a preclosed set F such that $x \in F \subseteq A$. The complement of a P_p-open set is P_p-closed. The family of all P_p-open subsets of a topological space (X,τ) is denoted by $P_pO(X,\tau)$ or $P_pO(X)$. The intersection of all P_p-closed sets of X containing A is called the P_p-closure*

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of A and is denoted by $P_pCl(A)$. The union of all P_p-open sets of X contained in A is called the P_p-interior of A and is denoted by $P_pInt(A)$.

Definition 3. *A function $f : X \to Y$ is called*

(1) *contra-continuous* [3] *if $f^{-1}(V)$ is closed in X for each open set V of Y.*
(2) *contra θ-continuous* [2] *if $f^{-1}(V)$ is θ-closed in X for each open set V of Y.*
(3) *contra-precontinuous* [6] *if $f^{-1}(V)$ is preclosed in X for each open set V of Y.*
(4) *perfectly continuous* [14] *if $f^{-1}(V)$ is clopen in X for each open set V of Y.*
(5) *strongly θ-continuous* [9] *if $f^{-1}(V)$ is θ-open in X for each open set V of Y*
(6) *quasi θ-continuous* [15] *at a point $x \in X$ if for each θ-open V of Y containing $f(x)$, there exists a θ-open U of X containing x such that $f(U) \subset Cl(V)$.*
(7) *weakly P_p-continuous* [11] *at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a P_p-open U of X containing x such that $f(U) \subset Cl(V)$.*

Theorem 1. [11] *Let $f : X \to Y$ be a function. If the inverse image of each regular open set of Y is P_p-closed in X, then f is weakly P_p-continuous.*

Definition 4. [7] *A subset A of a space X is called preclopen, if A is both preopen and peclosed.*

Definition 5. [12] *Let $A \subseteq X$. The set $\cap\{U \in \tau : A \subset U\}$ is called the kernel of A and is denoted by $ker(A)$.*

Lemma 1. [5] *The following properties hold for subsets A and B of a space X:*

(1) *$x \in ker(A)$ if and only if $A \cap F \neq \phi$ for any closed subset F of X containing x.*
(2) *$A \subset ker(A)$ and $A = ker(A)$ if A is open in X.*
(3) *If $A \subset B$, then $ker(A) \subset ker(B)$.*

Proposition 1. [8] *For any subset A of a space (X, τ). The following statements are equivalent:*

(1) *A is clopen.*
(2) *A is P_p-open and closed .*
(3) *A is preopen and closed.*

Definition 6. *A space X is said to be:*

(1) *Locally indiscrete* [4] *if every open subset of X is closed.*
(2) *Pre-R_0* [1] *if U is a preopen and $x \in U$, then $PCl(\{x\}) \subseteq U$.*
(3) *Pre-T_1* [7] *if for each pair of distinct points x,y of X, there exist two preopen sets one containing x but not y and the other containing y but not x.*
(4) *P_p-T_1* [11] *if for each pair of distinct points x,y of X, there exist two disjoint P_p-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*
(5) *P_p-T_2* [11] *if for each pair of distinct points x,y of X, there exist two disjoint P_p-open sets U and V containing x and y respectively.*

Definition 7. [16] *A space X is said to be pre-regular if for each preclosed F and each point $x \notin F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$.*

Proposition 2. *The following statements are true:*

(1) *If a space X is pre-T_1, then $PO(X) = P_pO(X)$* [8].
(2) *If a space X is X is pre-R_0, then $PO(X) = P_pO(X)$* [11].
(3) *If a space X is pre-regular, then $\tau \subseteq P_pO(X)$* [8].

(4) *If a space* (X,τ) *is locally indiscrete, then* $\tau \subseteq P_pO(X)$ [8].
(5) *If a space* (X,τ) *is locally indiscrete, then* $PO(X) = P_pO(X)$ [8].

Corollary 1. [8] *Let* A *and* B *be any subsets of a space* X*. If* $A \in P_pO(X)$ *and* B *is both* α*-open and preclosed subset of* X*, then* $A \cap B \in P_pO(B)$.

Proposition 3. [8] *Let*(Y,τ_Y) *be a subspace of a space* (X,τ) *and* $A \subset Y$*. If* $A \in P_pO(Y,\tau_Y)$ *and* Y *is preclopen, then* $A \in P_pO(X,\tau)$.

Definition 8. *A topological space* (X,τ) *is said to be*

(1) *Ultra Hausdorrf* [18] *if for each pair of distinct points* x, y *of* X*, there exist two clopen sets* U *and* V *such that* $x \in U$*,* $y \in V$ *and* $U \cap V = \phi$.
(2) *Ultra normal* [18] *if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.*
(3) *Weakly Hausdorff* [17] *if each element of* X *is an intersection of regular closed sets.*

Proposition 4. [8] *Let* X *and* Y *be two topological spaces and* $X \times Y$ *be the product topology. If* $A \in P_pO(X)$ *and* $B \in P_pO(Y)$*, then* $A \times B \in P_pO(X \times Y)$.

Theorem 2. [8] *For a function* $f : X \to Y$*, the following statements are equivalent:*

(1) f *is* P_p*-continuous.*
(2) $f^{-1}(V)$ *is* P_p*-open set in* X*, for each open set* V *of* Y.
(3) $f^{-1}(F)$ *is* P_p*-closed set in* X*, for each closed set* F *of* Y.

Corollary 2. [8] *Every quasi* θ*-continuous function is a* P_p*-continuous function.*

3 Contra P_p-continuous functions

Definition 9. *A function* $f : X \to Y$ *is called contra* P_p*-continuous if* $f^{-1}(V)$ *is* P_p*-closed in* X *for each open set* V *of* Y.

Lemma 2. *Every contra* θ*-continuous function is contra* P_p*-continuous and every contra* P_p*-continuous function is contra-precontinuous.*

Proof. Follows directly from their definitions.

Theorem 3. *If a function* $f : X \to Y$ *is contra* P_p*-continuous, then* f *is weakly* P_p*-continuous.*

Proof. Let V be any regular open of Y, then V is open. Since f is contra P_p-continuous, then $f^{-1}(V)$ is P_p-closed of X. Therefore, by Theorem 1, f is weakly P_p-continuous.

By Lemma 2 and Theorem 3, the following diagram is obtained:

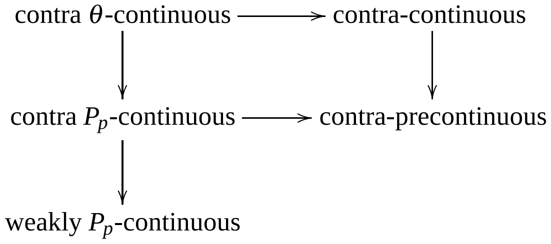

Diagram 1

In the sequel, we shall show that none of the implications that concerning contra P_p-continuity in Diagram 1 is reversible.

Example 31 *Let* $X = \{a,b,c,d\}$ *with the two topologies* $\tau = \{\phi,\{a\},\{b,c\},\{a,b,c\},X\}$ *and* $\sigma = \{\phi,\{b\},\{a,d\},\{a,b,d\},X\}$. *Let* $f:(X,\tau)\to(X,\sigma)$ *be the identity function. Then* f *is contra* P_p*-continuous but not contra* θ*-continuous, since* $\{b\}\in\sigma$ *but* $f^{-1}(\{b\}) = \{b\}$ *is not* θ*-closed in* (X,τ).

Example 32 *Let* $X = \{a,b,c,d\}$ *with the two topologies* $\tau = \{\phi,\{a\},\{a,b\},\{c,d\},\{a,,c,d\},X\}$ *and* $\sigma = \{\phi,\{b\},\{b,c\},\{b,c,d\},X\}$. *Let* $f:(X,\tau)\to(X,\sigma)$ *be the identity function. Then* f *is contra-precontinuous but not contra* P_p*-continuous, since* $\{b\}\in\sigma$ *but* $f^{-1}(\{b\}) = \{b\}$ *is not* P_p*-closed in* (X,τ).

Example 33 *Let* $X = \{a,b,c,d\}$ *with the two topologies* $\tau = \{\phi,\{b\},\{a,d\},\{a,b,d\},X\}$ *and* $\sigma = \{\phi,\{a\},\{d\},\{a,d\},\{a,b,d\},X\}$. *Let* $f:(X,\tau)\to(X,\sigma)$ *be the identity function. Then* f *is weakly* P_p*-continuous but not contra* P_p*-continuous, since* $\{a,b,d\}\in\sigma$ *but* $f^{-1}(\{a,b,d\}) = \{a,b,d\}$ *is not* P_p*-closed in* (X,τ).

Theorem 4. *For a function* $f : X \to Y$*, the following statements are equivalent:*

(1) f *is contra* P_p*-continuous.*
(2) *for every closed subset* F *of* Y, $f^{-1}(F) \in P_pO(X)$.
(3) *For each* $x\in X$ *and each closed set* F *of* Y *containing* $f(x)$*, there exists a* P_p*-open* U *of* X *containing* x *such that* $f(U)\subset F$.
(4) $f(P_pCl(A)) \subset ker(f(A))$ *for each* $A\subset X$.
(5) $P_pCl(f^{-1}(B)) \subset f^{-1}(ker(B))$ *for each* $B\subset Y$.

Proof. The implications $(1)\Leftrightarrow(2)$ and $(2)\Rightarrow(3)$ are obvious.
$(3)\Rightarrow(2)$ Let F be any closed set of Y and $x\in f^{-1}(F)$. Then $f(x)\in F$ and by (3) there exists $U\in P_pO(X)$ containing x such that $f(U)\subset F$. Therefore, we obtain that $f^{-1}(F) = \cup\{U_x : x\in f^{-1}(F)\} \in P_pO(X)$.
$(2)\Rightarrow(4)$ Let A be any subset of X. Suppose that $y\notin ker(f(A))$. Then by Lemma 1(1), there exists a closed set F of Y containing y such that $f(A)\cap F=\phi$. Thus, we have $A\cap f^{-1}(F) = \phi$ and $P_pCl(A)\cap P_pInt(f^{-1}(F)) = \phi$. Since $f^{-1}(F)$ is P_p-open in X. Hence, $P_pCl(A)\cap f^{-1}(F)=\phi$ which implies that $f(P_pCl(A))\cap F=\phi$ and hence $y\notin f(P_pCl(A))$. Therefore, we obtain that $f(P_pCl(A))\subset ker(f(A))$.
$(4)\Rightarrow(5)$ Let B be any subset of Y. By (4) and Lemma 1, we have $f(P_pCl(f^{-1}(B)))\subset ker(f(f^{-1}(B)))\subset ker(B)$ and $P_pCl(f^{-1}(B))\subset f^{-1}(ker(B))$.
$(5)\Rightarrow(1)$ Let V be any open set of Y. By (5) and Lemma 1(2), we have $P_pCl(f^{-1}(V))\subset f^{-1}(ker(V)) = f^{-1}(V)$ and $P_pCl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is P_p-closed in X. Therefore, f is contra P_p-continuous.

Theorem 5. *A function* $f : X\to Y$ *is contra* P_p*-continuous if and only if* f *is contra-precontinuous and for each* $x\in X$ *and each closed set* F *of* Y *containing* $f(x)$*, there exists a preclosed* E *in* X *containing* x *such that* $f(E)\subset F$.

Proof. **Necessity**. Let $x\in X$ and F be any closed set of Y containing $f(x)$. Since f is contra P_p-continuous, then by Theorem 4, there exists a P_p-open set U of X containing x such that $f(U)\subset F$. Since U is P_p-open set. Then for each $x\in U$, there exists a preclosed E of X such that $x\in E\subset U$. Therefore, we have $f(E)\subset F$. Hence, contra P_p-continuous always implies contra-precontinuous.
Sufficiency. Let F be any closed set of Y. We have to show that $f^{-1}(F)$ is P_p-open set in X. Since f is contra-precontinuous, then $f^{-1}(F)$ is preopen in X. Let $x\in f^{-1}(F)$, then $f(x)\in F$. By hypothesis, there exists a preclosed E of X containing x such that $f(E)\subset F$, which implies that $x\in E\subset f^{-1}(F)$. Therefore, $f^{-1}(F)$ is P_p-open set in X. Hence, by Theorem 4, f is contra P_p-continuous.

Theorem 6. *If a function* $f : X\to Y$ *is contra* P_p*-continuous and* Y *is regular, then* f *is* P_p*-continuous.*

Proof. Let x be any arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set G in Y containing $f(x)$ such that $Cl(G)\subset V$. Since f is contra P_p-continuous, so by Theorem 4, there exists a P_p-open U of X containing x such that $f(U)\subset Cl(G)$. Then $f(U)\subset Cl(G)\subset V$. Hence, f is P_p-continuous.

Corollary 3. *If a function $f : X \to Y$ is contra P_p-continuous and Y is regular, then f is quasi θ-continuous.*

Proof. Follows from Corollary 2.

Theorem 7. *The following statements are equivalent for a function $f : X \to Y$:*

(1) *f is perfectly continuous.*
(2) *f is contra P_p-continuous and continuous.*
(3) *f is contra-precontinuous and continuous.*

Proof. This is an immediate consequence of Proposition 1.

Corollary 4. *Let $f : X \to Y$ be a function and X be a pre-T_1 space. f is contra P_p-continuous if and only if f is contra-precontinuous.*

Proof. Follows from Proposition 2(1).

Corollary 5. *Let $f : X \to Y$ be a function and X be a pre-R_0 space. f is contra P_p-continuous if and only if f is contra-precontinuous.*

Proof. Follows from Proposition 2(2).

Corollary 6. *Let $f : X \to Y$ be a function and X be a pre-regular space. If f is contra-continuous, then f is contra P_p-continuous.*

Proof. Follows from Proposition 2(3).

Corollary 7. *Let $f : X \to Y$ be a function and X be a locally indiscrete space. If f is contra-continuous, then f is contra P_p-continuous.*

Proof. Follows from Proposition 2(4).

Corollary 8. *Let $f : X \to Y$ be a function and X be a locally indiscrete space. f is contra-precontinuous if and only if f is contra P_p-continuous.*

Proof. Follows from Proposition 2(5).

Corollary 9. *If X is both pre-T_1 and X locally indiscrete space, the following statements are equivalent for a function $f : X \to Y$:*

(1) *f is contra P_p-continuous.*
(2) *f is contra-precontinuous.*

Proof. Follows from Corollary 4 and Corollary 8.

Definition 10. *A space $(X;\tau)$ is said to be P_p-space (resp., locally P_p-indiscrete) if every P_p-open set is open (resp., closed) in X.*

Theorem 8. *If a function $f : X \to Y$ is contra P_p-continuous and X is P_p-space, then f is contra-continuous.*

Proof. Let F be a closed set in Y. Since f is contra P_p-continuous, $f^{-1}(F)$ is P_p-open in X. Since X is P_p-space, $f^{-1}(F)$ is open in X. Hence f is contra-continuous.

Theorem 9. *Let X be locally P_p-indiscrete. If a function $f : X \to Y$ is contra P_p-continuous, then f is continuous.*

Proof. Let F be a closed set in Y. Since f is contra P_p-continuous, $f^{-1}(F)$ is P_p-open in X. Since X is locally P_p-indiscrete, $f^{-1}(F)$ is closed in X. Hence f is continuous.

Theorem 10. *Let $f : X \to Y$ be a contra P_p-continuous function. If A is α-open and preclosed subset of X, then $f \mid A : A \to Y$ is contra P_p-continuous in the subspace A.*

Proof. Let F be any closed set of Y. Since f is contra P_p-continuous, then by Theorem 4, $f^{-1}(F)$ is P_p-open in X. Since A is α-open and preclosed subset of X, then by Corollary 1, $(f \mid A)^{-1} = f^{-1}(F) \cap A$ is a P_p-open subspace of A. Therefore, by Theorem 4, $f \mid A : A \to Y$ is contra P_p-continuous.

Theorem 11. *A function $f : X \to Y$ is contra P_p-continuous, if for each $x \in X$, there exists a preclopen A of X containing x such that $f \mid A : A \to Y$ is contra P_p-continuous in the subspace A.*

Proof. Let $x \in X$, then by hypothesis, there exists a preclopen A containing x such that $f \mid A : A \to Y$ is contra P_p-continuous. Let F be any closed subset of Y containing $f(x)$. By Theorem 4, there exists a P_p-open U in A containing x such that $(f \mid A)(U) \subset F$. Since A is preclopen, then by Proposition 3, U is P_p-open in X and hence $f(U) \subset V$. Therefore, by Theorem 4, f is contra P_p-continuous.

Theorem 12. *If $X = R \cup S$, where R and S are preclopen sets, and $f : X \to Y$ is a function such that both $f \mid R$ and $f \mid S$ are contra P_p-continuous, then f is contra P_p-continuous.*

Proof. Let F be any closed subset of Y. Then $f^{-1}(F) = (f \mid R)^{-1}(F) \cup (f \mid S)^{-1}(F)$. Since $f \mid R$ and $f \mid S$ are contra P_p-continuous, then by Theorem 4 $(f \mid R)^{-1}$ and $(f \mid S)^{-1}$ are P_p-open sets in R and S, respectively. Since R and S are precolpen sets in X, then by Proposition 3 $(f \mid R)^{-1}$ and $(f \mid S)^{-1}$ are P_p-open sets in X. Since the union of two P_p-open sets is P_p-open, hence $f^{-1}(F)$ is P_p-open sets in X. Therefore, by Theorem 4, f is contra P_p-continuous.

Theorem 13. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X, there exists a function f of X into Uryshon topological space Y such that $f(x_1) \neq f(x_2)$ and f is contra P_p-continuous at x_1 and x_2, then X is a P_p-T_2 space.*

Proof. Let x_1 and x_2 be any distinct points in X. By hypothesis, there is a Uryshon space Y and a function $f : X \to Y$ such that $f(x_1) \neq f(x_2)$ and f is contra P_p-continuous at x_1 and x_2, Let $y_i = f(x_i)$ for $i = 1,2$. Then $y_1 \neq y_2$. Since Y is Uryshon, there exist open sets U_{y_1} and U_{y_2} containing y_1 and y_2 respectively in Y such that $Cl(U_{y_1}) \cap Cl(U_{y_2}) = \phi$. Since f is contra P_p-continuous at x_1 and x_2, there exist P_p-open sets V_{x_1} and V_{x_2} containing x_1 and x_2 respectively in X such that $f(V_{x_i}) \subset Cl(U_{y_i})$ for $i = 1,2$. Hence, we have $V_{x_1} \cap V_{x_2} = \phi$. Therefore, X is a P_p-T_2 space.

Corollary 10. *If f is contra P_p-continuous injection of a topological space X into a Uryshon space Y, then X is a P_p-T_2 space.*

*Proof.*Let x_1 and x_2 be any distinct points in X. Then by hypothesis, f is contra P_p-continuous of X into a Uryshon space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence, by Theorem 13, X is a P_p-T_2 space.

Proposition 5. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two contra P_p-continuous functions. If Y is Uryshon, then the set $E = \{(x_1,x_2) \in X_1 \times X_2 : f(x_1) = g(x_2)\}$ is P_p-closed in the product space $X_1 \times X_2$.*

Proof. In order to show that E is P_p-closed, we show that $(X_1 \times X_2) \setminus E$ is P_p-open. Let $(x_1,x_2) \notin E$. Then $f(x_1) \neq g(x_2)$. Since Y is Uryshon, there exist open sets U_1 and U_2 of Y containing $f(x_1)$ and $g(x_2)$ respectively, such that $Cl(U_1) \cap Cl(U_2) = \phi$. Since f and g are contra P_p-continuous, $f^{-1}(Cl(U_1))$ and $g^{-1}(Cl(U_2))$ are P_p-open sets containing x_1 and x_2 in $X_i(i = 1,2)$. Hence, by Proposition 4, $f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2))$ is P_p-open. Further $(x_1,x_2) \in f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2)) \subset (X_1 \times X_2) \setminus E$. It follows that $(X_1 \times X_2) \setminus E$ is P_p-open. Thus, E is P_p-closed in the product space $X_1 \times X_2$.

Corollary 11. *If $f : X \to Y$ is a contra P_p-continuous function and Y is a Urysohn space, then $E = \{(x_1,x_2) \mid f(x_1) = f(x_2)\}$ is P_p-closed in the product space $X \times X$.*

Corollary 12. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two contra P_p-continuous functions. If Y is ultra Hausdorff, then the set $E = \{(x_1,x_2) \in X_1 \times X_2 : f(x_1) = g(x_2)\}$ is P_p-closed in the product space $X_1 \times X_2$.*

Corollary 13. *If $f : X \to Y$ is a contra P_p-continuous function and Y is ultra Hausdorff, then $E = \{(x_1,x_2) \mid f(x_1) = f(x_2)\}$ is P_p-closed in X.*

Theorem 14. *Let $f : X \to Y$ be a contra P_p-continuous injection function. If Y is an ultra Hausdorff space, then X is a P_p-T_2 space.*

Proof. Let x_1 and x_2 be any distinct points in X, then $f(x_1) \neq f(x_2)$ and there exist clopen sets U and V containing $f(x_1)$ and $f(x_2)$ respectively, such that $U \cap V = \phi$. Since f is contra P_p-continuous, then $f^{-1}(U) \in P_pO(X)$ and $f^{-1}(V) \in P_pO(X)$ such that $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence, X is a P_p-T_2 space.

Proposition 6. *If $f_i : X_i \to Y_i$ is a contra P_p-continuous function for each $i = 1,2$. Let $f : X_1 \times Y_2 \to X_1 \times Y_2$ be a function defined as follows: $f(X_1,X_2) = (f_1(x_1), f_2(x_2))$. Then f is contra P_p-continuous.*

Proof. Let $R_1 \times R_2 \subset Y_1 \times Y_2$, where R_i is open set in Y_i for each $i = 1,2$. Then $f^{-1}(R_1 \times R_2) = f_1^{-1}(Y_1) \times f_2^{-1}(R_2)$. Since f_i is contra P_p-continuous for $f_i = 1,2$, then by Proposition 4, $f^{-1}(R_1 \times R_2)$ is P_p-closed in $X_1 \times X_2$.

Definition 11. *The graph $G(f)$ of a function $f : X \to Y$ is contra P_p-closed in $X \times Y$ if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exists $U \in P_p(X,x)$ and a closed V in Y containing y such that $(U \times V) \cap G(f) = \phi$.*

Lemma 3. *The graph $G(f)$ of a function $f : X \to Y$ is contra P_p-closed in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in P_pO(X)$ containing x and $V \in C(Y)$ containing y such that $f(U) \cap V = \phi$.*

Theorem 15. *If $f : X \to Y$ is a contra P_p-continuous function and Y is Urysohn, then $G(f)$ is contra P_p-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \phi$. Since f is contra P_p-continuous, there exists a $U \in P_pO(X,x)$ such that $f(U) \subset Cl(V)$ and $f(U) \cap Cl(W) = \phi$. Hence, $G(f)$ is contra P_p-closed in $X \times Y$.

Theorem 16. *If $f : X \to Y$ is a P_p-continuous function and Y is T_1, then $G(f)$ is contra P_p-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and there exists an open set V of Y, such that $f(x) \in V$ and $y \notin V$. Since f is P_p-continuous, there exists $U \in P_pO(X,x)$ such that $f(U) \subset V$. Therefore, $f(U) \cap (Y \setminus V) = \phi$ and $Y \setminus V \in C(Y,y)$. This shows that $G(f)$ is contra P_p-closed in $X \times Y$.

Theorem 17. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and $g : X \to X \times Y$ be a graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra P_p-continuous, then f is contra P_p-continuous.*

Proof. Let V be an open set in Y. Then $X \times V$ is an open set in $X \times Y$. Since g is contra P_p-continuous, $g^{-1}(X \times V)$ is P_p-closed in X. Also $g^{-1}(X \times V) = f^{-1}(V)$ which is P_p-closed in X. Hence, f is contra P_p-continuous.

Theorem 18. *Let $f : X \to Y$ has a contra P_p-closed graph. If f is injective, then X is P_p-T_1.*

Proof. Let x_1 and x_2 be any two distinct points of X. Then, we have $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Then, there exist P_p-open U in X containing x_1 and $F \in C(Y, f(x_2))$ such that $f(U) \cap F = \phi$. Hence, $U \cap f^{-1}(F) = \phi$. Therefore, we have $x_2 \notin U$. This implies that X is P_p-T_1.

Definition 12. *A topological space X is said to be P_p-normal if each pair of disjoint closed sets can be separated by disjoint P_p-open sets.*

Theorem 19. *If a function $f : X \to Y$ is contra P_p-continuous, closed injection and Y is ultra normal, then X is P_p-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since f is closed injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Hence, $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in P_p(X)$ for $i = 1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is P_p-normal

Theorem 20. *If a function $f : X \to Y$ is contra P_p-continuous injection and Y is weakly Hausdorff, then X is P_p-T_1.*

Proof. Suppose that Y weakly Hausdorff. For any distinct points x_1 and x_2 in X, there exist regular closed sets U and V in Y such that $f(x_1) \in U$, $f(x_2) \notin U$, $f(x_1) \notin V$ and $f(x_2) \in V$. Since f is contra P_p-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are P_p-open subsets of X such that $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$. This shows that X is P_p-T_1.

Theorem 21. *Let $f : X \to Y$ be a contra P_p-continuous surjective function and A is α-open and preclosed subset of X. If f is a closed function, then the function $g : A \to f(A)$, which is defined by $g(x) = f(x)$ for each $x \in A$, is contra P_p-continuous.*

Proof. Putting $H = f(A)$. Let $x \in A$ and F be any closed set in H containing $g(x)$. Since H is closed in Y and F is closed in H, then F is closed in Y. Since f is contra P_p-continuous, then by Theorem 4, there exists a P_p-open U in X containing x such that $f(U) \subset F$. Taking $W = U \cap A$, since A is α-open and preclosed subset of X. Then by Corollary 1, W is P_p-open in A containing x and $g(W) \subset F_Y \cap H = F_H$. Then $g(W) \subset F_H$. Therefore, by Theorem 4, g is contra P_p-continuous.

We shall obtain some conditions for the composition of two functions to be contra P_p-continuous.

Theorem 22. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition function $g \circ f : X \to Z$ is contra P_p-continuous if f and g satisfy one of the following conditions:*

(1) *f is contra P_p-continuous and g is continuous.*
(2) *f is P_p-continuous and g is contra-continuous.*
(3) *f is contra P_p-continuous and g is a strongly θ-continuous.*
(4) *f is contra P_p-continuous and g is a quasi θ-continuous.*

Proof.

(1) Let W be any open subset of Z. Since g is continuous $g^{-1}(W)$ is an open subset of Y. Since f is contra P_p-continuous, then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is a P_p-closed subset in X. Therefore, $g \circ f$ is contra P_p-continuous.
(2) Let W be any open subset of Z. Since g is contra-continuous, then $g^{-1}(W)$ is a closed subset of Y. Since f is P_p-continuous, then by Theorem 2, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is a P_p-closed subset in X. Therefore, $g \circ f$ is contra P_p-continuous.
(3) Let W be any open subset of Z. In view of strongly θ-continuity of g, $g^{-1}(W)$ is a θ-open subset of Y. Again, since f is contra P_p-continuous, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is a P_p-closed subset in X. Therefore, $g \circ f$ is contra P_p-continuous.
(4) Obvious

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] M. Caldas, S. Jafari and T. Noiri, Characterizations of Pre-R_0 and Pre-R_1 Topological Spaces, *Topology Proceedings*, Vol. 25, Summer (2000), 17-30.

[2] M. Caldas, S. Jafari and T. Noiri, Contra θ-continuity in topological spaces, *Questions, Answers in General Topology*, 33 (2015), 117-125.

[3] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *Internat. J. Math. and Math. Sci.*, 19 (2) (1996), 303-310.

[4] J. Dontchev, Survey on preopen sets, *The Proceedings of the Yatsushiro Topological Conference*, (1998), 1-18.

[5] S. Jafari and T. Noiri, Contra-super continuous functions, *Annales Univ Sci Budapest*, 42 (1999), 27-34.

[6] S. Jafari and T. Noiri, On contra-precontinuous functions, *Bull. Malaysian Math. Sc. Soc.*, 25 (2002), 115-128.

[7] A. Kar and P. Bhattacharyya, Some weak separation axioms, *Bull. Cal. Math. Soc.*, 82 (1990), 415-422.

[8] A. B. Khalaf and Sh. M. Mershkhan, P_p-open sets and P_p-continuous Functions, *Gen. Math. Notes*, 20 (2014), 34-51.

[9] P. E. Long and L. Herrington, Strongly θ-continuous functions, *J. Korean Math. Soc.*, 18 (1) (1981), 21-28.

[10] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and week precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, 53 (1982), 47-53.

[11] Sh. M. Mershkhan, Application of P_p-open set in Topological spaces, *M. Sc. Thesis, Zakho University* (2013).

[12] M. Mrsevic, On pariwise R_0 and pairwise R_1 bitopological spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, 30 (1986), 141-148.

[13] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, 15 (3)(1965), 961-970.

[14] T. Noiri, Supercontinuity and some strong forms of continuity, *Indian J. Pure Appl. Math.*, 15 (3) (1984), 241-250.

[15] T. Noiri and V. Popa, Weak forms of faint continuity, *Bull. Math. Soc. Sci. Math. Roumanie*, 34 (82) (1990), 263-270.

[16] M. Pal and P. Bhattacharyya, Feeble and strong forms of pre-irresolute function, *Bull. Malaysian Math. Soc.* (Second Series), 19 (1996), 63-75.

[17] T. Soundararajan, Weakly Hausdorff spaces and the cardinality of topological spaces, *In: General topology and its relation to modern analysis and algebra. III, Proc. Conf. Kanpur, (1968), Academia, Prague* (1971), p 3016.

[18] R. Staum, The algebra of bounded continuous functions into a non-archimedean field, *Pacific J. Math.* 50 (1974), 169-185.

[19] M. H. Stone, Applications of the theory of Boolean rings to topology, *Trans. Amer. Math. Soc.*, 41 (1937), 375-481.

[20] N. V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, 78 (2) (1968), 103-118.