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Equations of geodesics in two dimensional Finsler space with special (α,β)- metric

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Abstract: The equation of geodesic in a two-dimensional Finsler space is given by Matsumoto and Park for Finsler space with a (α,β)- metric in the year 1997 and 1998. Further Park and Lee studied the above case for generalized Kropina metric in the year 2000. Recently Chaubey and his co-authors studied the same for some special (α,β)- metric in 2013 and 2014. In continuation of this the purpose of present paper is to express the differential equations of geodesics in a two-dimensional Finsler space with some special Finsler (α,β)- metric.

Keywords: Finsler space, geodesic equations,(α,β) - metric, two dimensional Finsler space.

1 Introduction

In 1994,M. Matsumoto[6] studied the equation of geodesic in two dimensional Finsler spacesin detail. After that 1997, Matsumoto and Park[1] obtained the equation of geodesics in two dimensional Finsler spaces with the Randers metric $(L=\alpha+\beta)$ and the Kropina metric $L=(\frac{\alpha^2}{\beta})$, and in 1998, they have [2] obtained the equation of geodesic in two-dimensional Finsler space with the slope metrics, i.e. Matsumoto metric given by $L=\frac{\alpha^2}{(\alpha-\beta)}$, by considering β as an infinitesimal of degree one and neglecting infinitesimal of degree more than two they obtained the equations of geodesic of two-dimensional Finsler space in the form $y''=f(x,y,y')$, where (x,y) are the co-ordinate of two-dimensional Finsler space. Further Park and Lee [3] studied the above case for generalized Kropina metric in the year 2000. In continuation of this Chaubey and his co-authors [7,8] are studied the same case for the different special (α,β) - metric and illustrated their main results in the different figures. In the present paper we have shown that under the same conditions, the geodesic of the two-dimensional space with following metrics: $L=\alpha+\beta+\frac{\beta^2}{\alpha-\beta}$, $L=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$, and $L=\alpha+\beta+\frac{\beta^{n+1}}{\alpha^n}$.

2 Preliminaries

Let $F^2=(M^2,L)$ be a two dimensional Finsler space with a Finslermetric function$L(x^1,x^2;y^1,y^2)$. We denote $\frac{\partial f}{\partial x^i}=f_i,\frac{\partial f}{\partial y^i}=f_{(x)}(i=1,2)$ for any Finsler function $f(x^1,x^2;y^1,y^2)$. Hereafter, the suffices i,j run over 1, 2.

Since $L(x^1,x^2;y^1,y^2)$ is (1) p-homogeneous in (y^1,y^2) we have $L_{(j)(i)}y^i=0$ which imply the existence of a function, so called the Weierstrass invariant $W(x^1,x^2;y^1,y^2)$[1,2,8] given by

$$\frac{L_{(1)(1)}}{(y^2)^2}=-\frac{L_{(1)(2)}}{y^1y^2}=-\frac{L_{(2)(2)}}{(y^1)^2}=W(x^1,x^2;y^1,y^2). \tag{1}$$

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In a two-dimensional associated Riemannian space $R^2 = (M^2, \alpha)$ with respect to $L = \alpha$ and ?$\alpha^2 = a_{ij}(x^1, x^2)y^i y^j$, the Weierstrass invariant W_r of R^2 is written as

$$W_r = \frac{1}{\alpha^3} a_{11}a_{22} - (a_{12})^2.$$

Further L_j are still (l) p-homogeneous in (y^1, y^2), so that we get

$$L_{j(i)}y^i = L_j. \tag{2}$$

The geodesic equations in F^2 along curve $C : x^i = x^i(t)$ are given by [1].

$$L_i - \frac{dL_i}{dt} = 0. \tag{3}$$

Substituting (2) in (3), we get

$$L_{1(2)} - L_{2(1)} + (y^1\dot{y}^2 - y^2\dot{y}^1)W = 0, \tag{4}$$

which is called the Weierstrass form of geodesic equation in F^2,[1,2]where $\dot{y}^i = dy^i/dt$ .For the metric function $L(x, y; \dot{x}, \dot{y})$ and (4) becomes to

$$\frac{\partial^2 L}{\partial \dot{y}\partial x} - \frac{\partial^2 L}{\partial \dot{x}\partial y} + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\frac{\partial^2 L}{(\partial \dot{y})^2} = 0. \tag{5}$$

Let $\Gamma = \{\gamma^i_{jk}(x^1, x^2)\}$ be the Levi-Civita connection of the associated Riemannian space R^2. We introduce the linear Finsler connection $\Gamma = (\gamma^i_j k, \gamma^i_0 j, 0)$ and the h- and c-covariant differentiation in Γ^* are denoted by $(; i, (i))$ respectively, where the index (0) means the contraction with y^i. Then we have $y^i_{;j} = 0, \alpha_{;i} = 0$ and $\alpha_{(i);j} = 0$.

3 Equation of Geodesics in a two dimensional Finsler with (α, β) - metric space

In [2,4,5] a two dimensional Finsler space $F^2 = M^2, L(\alpha, \beta)$ with an (α, β)- metric, here $\beta = b_i(x^1, x^2)y^i$. For metric function $L(\alpha, \beta)$, we have

$$L_{(;i)} = L_\beta \beta_;, L_{(i)} = L_\alpha \alpha_{(i)} + L_\beta b_{(i)}, \tag{6}$$

where $\alpha_{(i)} = \frac{a_{ir}y^r}{\alpha}$ and the subscriptions α, β of L are the partial derivatives of L with respect to α, β respectively. Then we have in Γ^* .

$$L_{(j);i} = L_{(j)i} - L_{(j)(i)}\gamma^r_{0i} - L_r \gamma^r_{ji},$$

from which

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + L_{(2)(r)}\gamma^r_{(01)} - L_{(1)(r)}\gamma^r_{(02)}. \tag{7}$$

From (1) and (7) we have

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + W(y^1\gamma^2_{00} - y^2\gamma^1_{00}). \tag{8}$$

On other hand, from (6) we have

$$L_{(j);i} = L_{\alpha\beta}\beta_{(;i)}(;i)\beta_{(j)} + L_{\beta\beta}\beta_{(;i)}b_j j + L_\beta \beta_{(;i)}b_{(j;i)}. \tag{9}$$

Similarly to the case of $L(x^1x^2; y^1y^2)$ and $\alpha(x^1, x^2)$, we get the Weierstrass invariant $w(\alpha, \beta)$ as follows:

$$w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}. \tag{10}$$

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Substituting (10) in (9), we have

$$L_{(j);i} = \alpha w \beta_{;i}(\alpha b_j - \beta\alpha_j) + L_\beta b_{j;i}. \tag{11}$$

From (8) and (11) we have

$$L_{1(2)} - L_{2(1)} = \alpha w\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta(L_{(1;2)} - L_{(2;1)}) + W(y^1\gamma^2_{00} - y^2\gamma^1_{00}). \tag{12}$$

If we put $y^i_{;0} = \dot{y}^i + \gamma^i_{00}$, we get

$$y^1\dot{y}^2 - y^2\dot{y}^1 = y^1y^2_{;0} - y^2y^1_{;0} - (y^1\gamma^2_{00} - y^2\gamma^1_{00}). \tag{13}$$

Substituting (12) and (13) in (4), we have

$$\alpha w\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}) + W(y^1y^2_{;0} - y^2y^1_{;0}) = 0 \tag{14}$$

where $\beta_{;i} = b_{r;i}y^r$. The relation of W,W_r and w is written as follows:

$$W = (L_\alpha + \alpha w\gamma^2)W_r \tag{15}$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $b^2 = a^{ij}b_ib_j$. Therefore (14) is expressed as follows:

$$(L_\alpha + \alpha w\gamma^2)(y^1y^2_{;0} - y^2y^1_{;0})W_r - L_\beta(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}) + \alpha w\{b_{0;1}(\alpha b_2 - \beta\alpha_{(2)}) - b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})\} = 0. \tag{16}$$

Thus we have the following.

Theorem 1. *In a two-dimensional Finsler space F^2 with an $(\alpha,\beta)-$ metric, the differential equation of a geodesic is given by (16).*

Suppose that α be positive$-$definite. Then we may refer to an isothermal coordinate system $(x^i) = (x,y)$ ([1,2]) such that

$$\alpha = aE, a = a(x,y) > 0, E = \sqrt{\dot{x}^2 + \dot{y}^2},$$

that is $a_{11} = a_{22} = a^2, a_{12} = 0$ and $(y^1,y^2) = (\dot{x},\dot{y})$. From $\alpha^2 = a_{ij}(x)y^iy^j$ we get $\alpha\alpha_{(i)(j)} = a_{ij} - a_{ir}a_{js}\frac{y^ry^s}{\alpha^2}$. Therefore we have $\alpha\alpha_{(1)(1)} = (\frac{a\dot{y}}{E})^2$ and $W_r = \frac{a}{E^3}$. Furthermore the Christoffel symbols are given by

$$\gamma^1_{11} = -\gamma^1_{22} = \gamma^2_{12} = \frac{a_x}{a}, \gamma^1_{12} = -\gamma^2_{11} = \gamma^2_{22} = \frac{a_y}{a},$$

where $a_x = \frac{\partial a}{\partial x}, a_y = \frac{\partial a}{\partial y}$ .Therefore we have

$$(y^1y^2_{;0} - y^2y^1_{;0})W_r = \frac{a}{E^3}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{1}{E}(a_x\dot{y} - a_y\dot{x}). \tag{17}$$

Next calculating $\gamma^2 = b^2\alpha^2 - \beta^2, b_{0;1}(\alpha b_2 - \beta\alpha_{(2)})$ and $b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})$ we have

$$\gamma^2 = (b_1)^2 + (b_2)^2(\dot{x}^2 + \dot{y}^2) - (b_1\dot{x} + b_2\dot{y})^2 = (b_1\dot{y} - b_2\dot{x})^2, \tag{18}$$

$$b_{r;1}(\alpha b_2 - \beta\alpha_{(2)})y^r = \frac{a}{E}b_{0;1}(b_2\dot{y} - b_1\dot{x})\dot{x} \tag{19}$$

$$b_{r;2}(\alpha b_1 - \beta\alpha_{(1)})y^r = \frac{a}{E}b_{0;2}(b_1\dot{y} - b_2\dot{x})\dot{y}. \tag{20}$$

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Substituting (17), (18), (19) and (20) in (16), we have

$$\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x})+E^2(a_x\dot{y}-a_y\dot{x})\}\{L\alpha+aEw(b_1\dot{y}-b_2\dot{x})^2\}-E^3L_\beta(b_{1y}-b_{2x})-E^3a^2w(b_{1\dot{y}}-b_{2\dot{x}})b_{0;0}=0, \tag{21}$$

where

$$b_{0;0}=b_r s y^r y^s=(b_{1x}\dot{x}+b_{1y}\dot{y})\dot{x}+(b_{2x}\dot{x}+b_{2y}\dot{y})+\frac{1}{a}\{(\dot{x}^2+\dot{y}^2)(a_xb_1+a_yb_2)-2(b_1\dot{x}+b_2\dot{y})(a_x\dot{x}+a_y\dot{y})\} \tag{22}$$

where $b_{ix}=\frac{\partial b_i}{\partial x}$ and $b_{iy}=\frac{\partial b_i}{\partial y}$. Thus we have the following.

Theorem 2. *In a two dimensional Finsler spaceF^2 with an (α,β)- metric, if we refer to an isothermal coordinate system (x,y) such that $\alpha=aE$, then the differential equation of a geodesic is given by (21) and (22).*

4 Equation of Geodesics in a two dimensional Finsler with special (α,β)-metric $L=\alpha+\beta+\frac{\beta^2}{(\alpha-\beta)}$

The (α,β)-metric $L(\alpha,\beta)=\alpha+\beta+\frac{\beta^2}{(\alpha-\beta)}$ is called special (α,β) metric

$$\begin{gathered}L_\alpha=1-\frac{\beta^2}{(\alpha-\beta)^2},\quad L_{\alpha\alpha}=\frac{2\beta^2}{(\alpha-\beta)^3},\\ L_{\alpha\beta}=-\frac{\alpha\beta}{(\alpha-\beta)^3},\quad L_{\beta\beta}=\frac{2\alpha^2}{(\alpha-\beta)^3},\\ w=\frac{L_{\alpha\alpha}}{\beta^2}=-\frac{L_{\alpha\beta}}{\alpha\beta}=\frac{L_{\beta\beta}}{\alpha^2}=\frac{2}{(\alpha-\beta)^3}.\end{gathered} \tag{23}$$

Substituting (23) in (21), we obtain the differential equation of a geodesic in an isothermal coordinate system (x,y) with respect to α as follows:

$$\begin{aligned}&\{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1\dot{y}-b_2\dot{x})^2\}\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x})+E^2(a_x\dot{y}-a_y\dot{x})\}-E^3\alpha^2(\alpha-\beta)(b_{1y}-b_{2x})\\&\quad-2E^3a^2(b_1\dot{y}-b_2\dot{x})b_{0;0}=0.\end{aligned} \tag{24}$$

In the particular case for the t of curve C is chosen x of (x,y) , then $\dot{x}=1,\dot{y}=y',\ddot{x}=0,\ddot{y}=y'',E=\sqrt{1+(y')^2}$.

$$\begin{aligned}&\{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1y'-b_2)^2\}\{ay''+(1+(y')^2)(a_xy'-a_y)\}-a(1+(y')^2)\{(1+(y')^2)\\&\quad\alpha(\alpha-\beta)(b_{1y}-b_{2x})-2\alpha(b_1y'-b_2)b_{0;0}\}=0\end{aligned} \tag{25}$$

$$b^*_{0;0}=(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'+\frac{1}{a}\{(1+(y')^2)(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')\}. \tag{26}$$

It seems quite complicated from, but y'' is given as a fractional expression in y'. Thus we have the following

Theorem 3. *Let F^2 be two-dimensional space with special Finsler metric. If we refer to a local coordinate system (x,y) with respect to α, then the differential equation of a geodesic $y=y(x)$ of F^2 is of the form*

$$y''=\frac{g(x,y,y')}{f(x,y,y')},$$

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where $f(x,y,y')$ is a quadratic polynomial in y' and $g(x,y,y')$ is a polynomial in y' of degree at most five.

In order to find the concrete form, we treat the case of which the associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a=1$ and $a_x=a_y=0$. If we take scalar function b such that $b_1=b_x, b_2=b_y$ then $b_{1y}-b_{2x}=0$.therefore (25) is reduces to

$$y''=\frac{(-2\alpha\{1+(y')^2)\}(b_xy'-b_y)(b_{1x}+b_{2y}y')(b_{2x}+b_{2y}y')y'}{\{\alpha(\alpha-\beta)(\alpha-2\beta)+2\alpha(b_1y'-b_2)^2\}}. \tag{27}$$

Thus we have the following

Corollary 1. *Let F^2 be a two -dimensional Finsler space with a special metric. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_1=\frac{\partial b}{\partial x}, b_2=\frac{\partial b}{\partial y}$ for a scalar b, then the differential space of geodesic $y=y(x)$ of F^2 is given by (27).*

5 Equation of Geodesics in a two dimensional Finsler with special (α,β) -metric $L=\alpha+\beta+\frac{\alpha^2}{\beta}+\frac{\alpha^3}{\beta^2}$

The (α,β)-metric $L(\alpha,\beta)=\alpha+\beta+\frac{\alpha^2}{\beta}+\frac{\alpha^3}{\beta^2}$ is called special (α,β) metric

$$\begin{aligned} L_\alpha &= 1+\frac{2\alpha}{\beta}+\frac{3\alpha^2}{\beta^2},\ L_{\alpha\alpha}=\frac{2}{\beta}+\frac{6\alpha}{\beta^2},\ L_\beta=1-\frac{\alpha^2}{\beta^2}-\frac{2\alpha^2}{\beta^3},\\ L_{\beta\beta} &= \frac{2\alpha^2}{\beta^3}+\frac{6\alpha^3}{\beta^4},\ L_{\alpha\beta}=-\frac{\alpha(2\beta+6\alpha)}{\beta^3},\\ w &= \frac{L_{\alpha\alpha}}{\beta^2}=-\frac{L_{\alpha\beta}}{\alpha\beta}=\frac{L_{\beta\beta}}{\alpha^2}=\frac{\beta+6\alpha}{\beta^4}. \end{aligned} \tag{28}$$

Substituting (28) in (21), we obtain the differential equation of a geodesic in an isothermal coordinate system (x,y) with respect to α as follows:

$$\begin{aligned} &\{\beta^2(\beta^2+2\alpha\beta)+\alpha(2\beta+6\alpha)(b_1\dot{y}-b_2\dot{x})^2\}\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x})+E^2(a_x\dot{y}-a_y\dot{x})\}-E^3\beta(\beta^3-3\alpha^2\beta-2\alpha^3)(b_{1y}-b_{2x})\\ &\qquad -E^3a^2(2\beta+6\alpha)(b_1\dot{y}-b_2\dot{x})b_{0;0}=0. \end{aligned} \tag{29}$$

In the particular case for the t of curve C is chosen x of (x,y) , then $\dot{x}=1,\dot{y}=y',\ddot{x}=0,\ddot{y}=y'',E=\sqrt{1+(y')^2}$.

$$\begin{aligned} &\{\beta^2(\beta^2+2\alpha\beta)+\alpha(2\beta+6\alpha)(b_1y'-b_2)^2\}\{ay''+(1+(y')^2)(a_xy'-a_y)\}-a(1+(y')^2)(\{(1+(y')^2)\beta\\ &\qquad (\beta^3-3\alpha^2\beta-2\alpha^3)(b_{1y}-b_{2x})+\alpha(2\beta+6\alpha)(b_1y'-b_2)b_{0;0}=0) \end{aligned} \tag{30}$$

$$b^*_{0;0}=(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'+\frac{1}{a}\{(1+(y')^2)(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')\}. \tag{31}$$

It seems quite complicated from, buty'' is given as a fractional expression in y'. Thus we have the following.

Theorem 4. *Let F^2 be two-dimensional space with special Finsler metric. If we refer to a local coordinate system (x,y) with respect to α, then the differential equation of a geodesic$y=y(x)$ of F^2 is of the form*

$$y''=\frac{g(x,y,y')}{f(x,y,y')},$$

where $f(x,y,y')$ is a quadratic polynomial in y' and $g(x,y,y')$ is a polynomial in y' of degree at most five.

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In order to find the concrete form, we treat the case of which the associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a=1$ and $a_x=a_y=0$. If we take scalar function b such that $b_1=b_x, b_2=b_y$ then $b_{1y}-b_{2x}=0$.Therefore (30) is reduces to

$$y''=\frac{(6\alpha+2\beta)\alpha(1+(y')^2)(b_xy'-b_y)(b_{1x}+b_{2y}y')(b_{2x}+b_{2y}y')y'}{\beta^2(\beta^2+2\alpha\beta+3\alpha^2)+\alpha(2\beta+6\alpha)(b_1y'-b_2)^2}. \tag{32}$$

Thus we have the following.

Corollary 2. *Let F^2 be a two -dimensional Finsler space with a special metric. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_1=\frac{\partial b}{\partial x}, b_2=\frac{\partial b}{\partial y}$ for a scalar b, then the differential space of geodesic $y=y(x)$ of F^2 is given by (32).*

6 Equation of Geodesics in a two dimensional Finsler with special (α,β) -metric $L=\alpha+\beta+\frac{\beta^{(n+1)}}{\alpha^n}$

The (α,β)-metric $L(\alpha,\beta)=\alpha+\beta+\frac{\beta^{(n+1)}}{\alpha^n}$is called special (α,β) metric.

$$\begin{aligned} L_\alpha &= 1-n\frac{\beta^{(n+1)}}{\alpha^(n+1)}, \quad L_{\alpha\alpha}=n(n+1)\frac{\beta^{(n+1)}}{\alpha^{(n+2)}}, \\ L_\beta &= 1+(n+1)\frac{\beta^n}{\alpha^n}, \quad L_{\beta\beta}=n(n+1)\frac{\beta^{(n-1)}}{\alpha^n}, \\ L_{\alpha\beta} &= -n(n+1)\frac{\beta^n}{\alpha^{(n+1)}}, \\ w &= \frac{L_{\alpha\alpha}}{\beta^2}=-\frac{L_{\alpha\beta}}{\alpha\beta}=\frac{L_{\beta\beta}}{\alpha^2}=n(n+1)\frac{\beta^{(n-1)}}{\alpha^{(n+2)}}. \end{aligned} \tag{33}$$

Substituting (33) in (21), we obtain the differential equation of a geodesic in an isothermal coordinate system (x,y) with respect to α as follows:

$$\begin{aligned} &\{(\alpha^{(n+1)}-n\beta^{(n+1)})+n(n+1)\beta^{(n-1)}(b_1\dot{y}-b_2\dot{x})^2\}\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x})+E^2(a_x\dot{y}-a_y\dot{x})\} \\ &-E^3\alpha(\alpha^n+(n+1)\beta^n)(b_{1y}-b_{2x})-E^3a^2n(n+1)\beta^{(n-1)}(b_1\dot{y}-b_2\dot{x})b_{0;0}=0. \end{aligned} \tag{34}$$

In the particular case for the t of curve C is chosen x of (x,y) , then $\dot{x}=1, \dot{y}=y', \ddot{x}=0, \ddot{y}=y'', E=\sqrt{1+(y')^2}$.

$$\begin{aligned} &\{(\alpha^(n+1)-n\beta^(n+1))+n(n+1)\beta^(n-1)(b_1y'-b_2)^2\}(ay''+(1+(y')^2)(a_xy'-a_y))-a(1+(y')^2)((1+(y')^2) \\ &(\alpha^n+(n+1)\beta^n))(b_{1y}-b_{2x})+n(n+1)\beta^(n-1)(b_1y'-b_2)b_{0;0}=0) \end{aligned} \tag{35}$$

$$b^*_{0;0}=(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'+\frac{1}{a}((1+(y')^2)(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')). \tag{36}$$

It seems quite complicated from, buty$''$ is given as a fractional expression in y'. Thus we have the following,

Theorem 5. *Let F^2 be two-dimensional space with special Finsler metric. If we refer to a local coordinate system (x,y) with respect to α, then the differential equation of a geodesicy $=y(x)$ of F^2 is of the form*

$$y''=\frac{g(x,y,y')}{f(x,y,y')},$$

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where $f(x,y,y')$ is a quadratic polynomial in y' and $g(x,y,y')$ is a polynomial in y' of degree at most five.

In order to find the concrete form, we treat the case of which the associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a=1$ and $a_x=a_y=0$. If we take scalar function b such that $b_1=b_x, b_2=b_y$ then $b_{1y}-b_{2x}=0$.Therefore (35) is reduces to

$$y''=\frac{n(n+1)\beta^(n-1)(1+(y')^2)(b_xy'-b_y)(b_{1x}+b_{2y}y')(b_{2x}+b_{2y}y')y'}{(\alpha^{(n+1)}-n\beta^{(n+1)})+n(n+1)\beta^{(n-1)}(b_1y'-b_2)^2} \tag{37}$$

Thus we have the following.

Corollary 3. *Let F^2 be a two -dimensional Finsler space with a special metric. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_1=\frac{\partial b}{\partial x}, b_2=\frac{\partial b}{\partial y}$ for a scalar b, then the differential space of geodesic $y=y(x)$ of F^2 is given by (37).*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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