New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2019.361

Uniqueness of meromorphic functions whose differential polynomials share a nonconstant polynomial

Dilip Chandra Pramanik and Jayanta Roy

Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India

Received: 24 August 2018, Accepted: 4 January 2019
Published online: 21 June 2019.

Abstract: In this paper, we study the uniqueness of meromorphic functions whose differential polynomials share a nonconstant polynomial generalizing some earlier results.

Keywords: Meromorphic function, small function, uniqueness, sharing value, differential polynomial.

1 Introduction

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna Theory of meromorphic functions as explained in [4,9,18]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any nonconstant meromorphic function $f(z)$, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = \circ(T(r,f))$ as $r \to \infty$, $r \notin E$. A meromorphic function $a(z)$ is said to be small with respect to $f(z)$ if $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex numbers.

For any two nonconstant meromorphic functions f and g, and $a \in S(f) \cap S(g)$, we say that f and g share "a" IM(CM) provided that $f - a$ and $g - a$ have the same zeros ignoring(counting) multiplicities.

The following theorem in the value distribution theory is well known [3,15].

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [12], Yang and Hua [20] obtained a unicity theorem respectively to the above theorem.

Theorem 2. *Let $f(z)$ and $g(z)$ be two nonconstant entire (meromorphic) functions, $n \geq 6$ $(n \geq 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM then either $f(z) = c_1 \exp(cz)$, $g(z) = c_2 \exp(-cz)$, where c_1, c_2 and c are three constant satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Theorem 3. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let n, k be two positive integers with $n > 2k+4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM then either $f(z) = c_1 \exp(cz)$, $g(z) = c_2 \exp(-cz)$, where c_1, c_2 and c are three constant satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

* Corresponding author e-mail: dcpramanik.nbu2012@gmail.com

Theorem 4. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > 2k+8$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 CM then $f(z) \equiv g(z)$.*

Then Fang and Qiu [10] considered the fixed point sharing uniqueness problem and obtained the following theorem.

Theorem 5. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and $n \geq 6$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constant satisfying $4(c_1c_2)^{n+1}c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Later in 2004, Lin and Yi [16] proved the following theorem:

Theorem 6. *Let f and g be two nonconstant entire functions and $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.*

Zhang [6] extended the above two theorems and got the following results:

Theorem 7. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*

(1) *$k = 1$, $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constant satisfying $4(c_1c_2)^n(nc)^2 = -1$, or*
(2) *$f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem 8. *Let f and g be two nonconstant entire functions and let n, k be two positive integers with $n > 2k+6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, then $f \equiv g$.*

Regarding Theorems $1.7-1.8$, Xu et al [7] considered the case of meromorphic functions. They proved.

Theorem 9. *Let f and g be two nonconstant meromorphic functions and let n, k be two positive integers with $n > 3k+10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constant satisfying $4(c_1c_2)^n(nc)^2 = -1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem 10. *Let n, k be two positive integers with $n > 3k+12$, and f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

In view of above theorems, Sahoo [13] obtained the following result in 2010 for some more general nonlinear differential polynomial.

Theorem 11. *Let f and g be two nonconstant meromorphic functions and let n, k and m be three positive integers with $n > 9k+4m+13$. Let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ where $a_0 \neq 0$, $a_1, \cdots, a_m \neq 0$ are complex constants. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share z IM, f and g share ∞ IM. Then $f = tg$ for a constant t such that $t^d = 1$, where $d = gcd(n+m, n+m-1, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \ldots, m\}$; or f and g satisfy the algebraic equation $R(f,g) = 0$ where $R(f,g)$ is given by*

$$R(f,g) = f^n(a_mf^m + \ldots + a_1f + a_0) - g^n(a_mg^m + \ldots + a_1g + a_0). \tag{1}$$

It is natural question to ask what happen if sharing fixed point in above theorem is replaced by sharing a nonconstant polynomial.

Keeping in mind the above question X. B. Zhang and J. F. Xu [19] obtained the following result:

Theorem 12. *Let f and g be two nonconstant meromorphic functions, $p(z)$ be a nonconstant polynomial of degree $deg(p) = l \leq 5$, n, k and m be three positive integers with $n > 3k+m+7$. Let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ be a nonzero polynomial. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share p CM, f and g share ∞ IM, then one of the following two cases holds:*

(1) $f = tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, ..., m\}$;
(2) f *and* g *satisfy the algebraic equation* $R(f,g) = 0$ *where* $R(f,g)$ *is given by (1).*
(3) $P(z)$ *is reduced to a nonzero monomial, namely* $P(z) = a_i z^i \not\equiv 0$ *for some* $i \in \{0, 1, ..., m\}$; *if* $p(z)$ *is not a constant, then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are three constants satisfying* $a_i^2 (c_1 c_2)^{n+i} ((n+i)c)^2 = -1$, *if* $p(z)$ *is a constant* b, *then* $f(z) = c_3 e^{cz}$, $g(z) = c_4 e^{-cz}$, c_3, c_4 *and* c *are three constants satisfying* $(-1)^k a_i^2 (c_3 c_4)^{n+i} ((n+i)c)^{2k} = b^2$.

In 2016, Sahoo et al [14] removed the restriction on the degree of the polynomial $p(z)$ and proved the following theorems:

Theorem 13. *Let* f *and* g *be two transcendental meromorphic functions,* $p(z)$ *be a nonconstant polynomial of degree* l, *and let* $n(\geq 1)$, $k(\geq 1)$ *and* $m(\geq 0)$ *be three integers with* $n > max\{3k+m+6, k+2l\}$. *In addition we suppose that either* k, l *are co-prime or* $k > l$ *when* $l \geq 2$. *Let* $P(w)$ *be defined as in the above theorem. If* $(f^n P(f))^{(k)}$ *and* $(g^n P(g))^{(k)}$ *share* $p(z)$ *CM;* f *and* g *share* ∞ *IM, then the following conclusions hold:*

(i) *If* $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ *is not a monomial, then either* $f = tg$ *for a constant* t *that satisfies* $t^d = 1$, *where* $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, ..., m\}$; *or* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$ *where* $R(f,g)$ *is given by (1). In particular* $m = 1$ *and* $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$, *then* $f = g$.
(ii) *When* $P(w) = a_m w^m$, *or* $P(w) = c_0$, *then either* $f = tg$ *for some* t *such that* $t^{n+m^*} = 1$, *or* $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, *where* $Q(z)$ *is a polynomial without constant such that* $Q'(z) = p(z)$, b_1, b_2 *and* b *are three constants satisfying* $a_m^2 (b_1 b_2)^{n+m} ((n+m)b)^2 = -1$ *or* $c_0^2 (b_1 b_2)^n (nb)^2 = -1$, *where* m^* *is same as in Lemma 13.*

Theorem 14. *Let* f *and* g *be two transcendental meromorphic functions,* $p(z)$ *be a nonconstant polynomial of degree* l, *and let* $n(\geq 1)$, $k(\geq 1)$ *and* $m(\geq 0)$ *be three integers with* $n > max\{9k+4m+11, k+2l\}$. *In addition we suppose that either* k, l *are co-prime or* $k > l$ *when* $l \geq 2$. *Let* $P(w)$ *be defined as in the above theorem. If* $(f^n P(f))^{(k)}$ *and* $(g^n P(g))^{(k)}$ *share* $p(z)$ *IM;* f *and* g *share* ∞ *IM, then the conclusion of Theorem 13 hold.*

In this paper we investigate on the above theorem to remove sharing ∞ IM and obtain the following results:

Theorem 15. *Let* f *and* g *be two transcendental meromorphic functions with* $\sigma(f) < \infty$, *whose zeros and poles are of multiplicity at least* s, *where* s *is positive integer. Let* $p(z)$ *is a polynomial of degree* l *and*

$$P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0,$$

where $a_0, a_1, \cdots, a_m$ *are constants, where* $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ *be three integers and* $k > l$ *when* $l \geq 2$ *satisfying*

$$n > max\left\{2k+2l, \frac{3k+8}{s} + m, 2k(\sigma(f)-1) - (m+2l)\right\}.$$

If $(f^n P(f))^{(k)}$ *and* $(g^n P(g))^{(k)}$ *share* $p(z)$ *CM, then one of the following holds:*

(i) *If* $P(w)$ *is not a monomial and (3) hold.Then either* $f = tg$ *for some* t *such that* $t^d = 1$, *where* $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, ..., m\}$; *or* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$ *where* $R(f,g)$ *is given by (1).*
(ii) $P(w) = a_m w^m$, *or* $P(w) = c_0$, *then either* $f = tg$ *for a constant* t *such that* $t^{n+m^*} = 1$, *or* $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, *where* $Q(z)$ *is a polynomial without constant such that* $Q'(z) = p(z)$, b_1, b_2 *and* b *are three constants satisfying* $a_m^2 (b_1 b_2)^{n+m} ((n+m)b)^2 = -1$ *or* $c_0^2 (b_1 b_2)^n (nb)^2 = -1$, *where* m^* *is same as in Lemma 13.*

Theorem 16. *Let f and g be two transcendental meromorphic functions with $\sigma(f) < \infty$, whose zeros and poles are of multiplicity at least s, where s is positive integer. Let $p(z)$ is a polynomial of degree l and $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ be a nonzero polynomial and $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ be three integers and $k > l$ when $l \geq 2$ satisfying*

$$n > max\left\{2k+2l, \frac{9k+14}{s} + 4m, 2k(\sigma(f)-1) - (m+2l)\right\}.$$

If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $p(z)$ IM, then the conclusions of Theorem 15 hold.

2 Lemmas

Let F_1 and G_1 be nonconstant meromorphic functions defined in a complex plane $\mathbb{C}$. We denote by H the following function:

$$H = \left(\frac{F_1^{(2)}}{F_1^{(1)}} - 2\frac{F_1^{(1)}}{F_1 - 1}\right) - \left(\frac{G_1^{(2)}}{G_1^{(1)}} - 2\frac{G_1^{(1)}}{G_1 - 1}\right). \tag{2}$$

Lemma 1. [2] *Let $f(z)$ be a nonconstant meromorphic function and let $a_0(z), a_1(z), ...,$ $a_n(z)(\not\equiv 0)$ be a small function with respect to f. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. [4] *Let $f(z)$ be a nonconstant meromorphic function in the complex plane. If the order of f is finite, then*

$$m(r, \frac{f'}{f}) = O(\log r),\ r \to \infty.$$

Lemma 3. [4] *Let $h(z)$ be a nonconstant entire function and let $f(z) = e^{h(z)}$. Let λ and μ be the order and lower order of $h(z)$, respectively . We have*
(i) If $\mu < \infty$, then μ is a positive integer, $h(z)$ is a polynomial of degree μ, and $\lambda = \mu$.
(ii) If $\mu < \infty$, then $h(z)$ is transcendental and $\lambda = \mu$.

Lemma 4. [17] *Let $f(z)$ be a nonconstant meromorphic function and s, k be two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 5. [2] *Let $f(z)$ be a nonconstant meromorphic function and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

Lemma 6. [5] *If $N(r, 0; f^{(k)}/f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ that are not the zeros of f, where a is zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, \frac{1}{f^{(k)}}/f \neq 0) \leq k\overline{N}(r, f) + N(r, \frac{1}{f}| < k) + k\overline{N}(r, \frac{1}{f}| \geq k) + S(r, f).$$

Lemma 7. *Let f and g be two nonconstant transcendental meromorphic functions whose zeros and poles are of multiplicity at least s, where s is positive integers. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, where $a_0(\neq 0), a_1, \cdots, a_m(\neq 0)$ are constants, and let $n(\geq 1), k(\geq 1), m(\geq 0)$ are integers and $p(z)$ is a polynomial of degree l. If*

$$\lambda > \frac{1}{s} + \frac{m+n}{(m+n)s+2k} \tag{3}$$

where λ is the number of distinct roots of $P(w) = 0$ then

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} \neq p^2(z)$$

Proof.

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} = p^2(z) \tag{4}$$

Let $P(z) = a_m(z-d_1)^{l_1}.(z-d_2)^{l_2} \cdots (z-d_\lambda)^{l_\lambda}$, where $\sum_{j=1}^{\lambda} l_j = m$, $1 \leq \lambda \leq m$, $d_i \neq d_j$, $i \neq j$ and $1 \leq i, j \leq \lambda$, d_j‘s are nonzero constants and l_j‘s are positive integers, $j = 1,2,..,\lambda$. Let $z_0 \notin \{z : p(z) = 0\}$ be a zero of f with multiplicity $p_0(\geq s)$. Then z_0 is a pole of g with multiplicity $q_0(\geq s)$ say. From (4) we get, $np_0 - k = (m+n)q_0 + k$ so, $np_0 - k \geq (m+n)s + k$ and $p_0 \geq \frac{(m+n)s+2k}{n}$. Let $z_1 \notin \{z : p(z) = 0\}$ be a zero of $P(f)$ of order p_1 and a zero of $f - d_i$ of order q_i for some $i = 1,2,...\lambda$. Then $p_1 = l_i q_i$ for some $i = 1,2,...\lambda$, and z_1 is a pole of g with multiplicity $\overline{q}(\geq s)$ say, so from (4) we get

$$\begin{aligned} q_i l_i - k &= (n+m)\overline{q} + k \\ &\geq (n+m)s + k \\ i.e., q_i &\geq \frac{(n+m)s+2k}{l_i}, \end{aligned}$$

for $i = 1,2,..,\lambda$. Let $z_2 \notin \{z : p(z) = 0\}$ be a zero of $(f^n P(f))^{(k)}$ of order p_2 but not a zero of $f^n P(f)$. Then from (4) z_2 is a pole of g of order $\xi(\geq s)$. Then

$$p_2 = (n+m)\xi + k \geq (n+m)s + k.$$

Suppose that $z_3 \notin \{z : p(z) = 0\}$ be a pole of f then from (4) z_3 is a zero of $(g^n P(g))$ or a zero of $(g^n P(g))^{(k)}$. Therefore

$$\overline{N}(r,f) \leq \overline{N}(r,\frac{1}{g}) + \sum_{j=1}^{\lambda} \overline{N}(r,\frac{1}{g-d_i}) + \overline{N}(r,0;B^k/B \neq 0) + S(r,g),$$

where $\overline{N}(r,0;B^{(k)}/B \neq 0)$ denotes the reduced counting function of those zeros of $B^{(k)}$ that are not the zeros of B and $B = g^n P(g)$. Now by Lemma 6 we have

$$\begin{aligned} \overline{N}(r,0;B^{(k)}/B \neq 0) &\leq \frac{1}{(n+m)s+k} N(r,\frac{1}{B^{(k)}}/B \neq 0) \\ &\leq \frac{1}{(n+m)s+k} \left\{ k\overline{N}(r,B) + N(r,\frac{1}{B}| < k) + k\overline{N}(r,\frac{1}{B}| \geq k) \right\} \\ &\leq \frac{k}{(n+m)s+k} \left\{ \overline{N}(r,B) + N_k(r,\frac{1}{B}) \right\} \\ &\leq \frac{k}{(n+m)s+k} \left\{ \overline{N}(r,\frac{1}{g}) + \sum_{j=1}^{\lambda} \overline{N}(r,\frac{1}{g-d_i}) + \overline{N}(r,g) \right\} \end{aligned}$$

So,

$$\begin{aligned}\overline{N}(r,f) &\le \left(1+\frac{k}{(n+m)s+k}\right)\left(\frac{n}{(m+n)s+2k}+\frac{m}{(m+n)s+2k}\right)T(r,g)\\ &+\frac{k}{((n+m)s+k)s}T(r,g)+S(r,f)+S(r,g)\\ &\le \frac{1}{s}T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

By second fundamental theorem

$$\begin{aligned}\lambda T(r,f) &\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\sum_{j=1}^{\lambda}\overline{N}(r,\frac{1}{f-d_i})+S(r,f)\\ &\le \left(\frac{n}{(m+n)s+2k}+\frac{m}{(m+n)s+2k}\right)T(r,f)+\frac{1}{s}T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly,

$$\lambda T(r,g) \le \left(\frac{n}{(m+n)s+2k}+\frac{m}{(m+n)s+2k}\right)T(r,g)+\frac{1}{s}T(r,f)+S(r,f)+S(r,g).$$

Adding above two inequality we get

$$\left\{\lambda-\frac{1}{s}-\frac{m+n}{(m+n)s+2k}\right\}(T(r,f)+T(r,g)) \le S(r,f)+S(r,g)$$

which contradict given assumption. This completes the proof of the Lemma.

Lemma 8. *Let f and g be two nonconstant meromorphic functions with $\sigma(f)<\infty$, $p(z)$ be nonconstant polynomial of degree l and n, k, m be three positive integer with $n > max\{2k+2l, 2k(\sigma(f)-1)-(m+2l)\}$. In addition we assume $k > l$ when $l \ge 2$, and if*

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = p^2(z) \tag{5}$$

where $P(w)=a_mw^m$ or $P(w)=c_0$ then $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and c are three constants satisfying ${a_m}^2(b_1b_2)^{n+m}((n+m)b)^2=-1$ or ${c_0}^2(b_1b_2)^n(nb)^2=-1$ and $Q(z)$ is a polynomial without constant such that $Q'(z)=p(z)$.

Proof. We first prove that

$$f \ne 0, g \ne 0. \tag{6}$$

Let $P(w)=a_mw^m$. Then from (5) we get

$$\left(f^{n+m}\right)^{(k)}\left(g^{n+m}\right)^{(k)} = p^2(z) \tag{7}$$

Suppose that z_0 is a zero of f say multiplicity r but $p(z_0)\ne 0$ then z_0 is a pole of g say multiplicity s_0. Then we have from (7), $(n+m)r-k=(n+m)s_0+k \Rightarrow (n+m)(r-s_0)=2k$, which is contradiction for $n>2k+2l$. Now suppose that z_0 is a zero of f say multiplicity r_1 if z_0 is not a pole of g then z_0 must be zero of $p(z)$ of multiplicity l_0, say, then we have from (7) $(n+m)r_1-k>2l_0$, which is again contradiction. If z_0 is a pole of g say multiplicity s_1 then we have $(n+m)r_1-k=(n+m)s_1+k+2l_0$, $(n+m)(r_1-s_1)=2k+2l_0$ which is impossible. So, f has no zeros. Similarly it can be shown that g also has no zeros. Thus (6) is proved. Next we prove that

$$N(r,f)=O(\log r); \quad (8)$$
$$N(r,g)=O(\log r);$$

$$(f^nP(f))^{(k)}=\frac{p(z)^2}{(g^nP(g))^{(k)}} \quad (9)$$

Since $N(r,(f^nP(f))(k))=N(r,f^nP(f))+k\overline{N}(r,f^nP(f)=(n+m)N(r,f)+k\overline{N}(r,f)+S(r,f)$
By Lemma 5

$$N(r,\frac{1}{(g^{n+m})^{(k)}})\leq N(r,\frac{1}{g^{n+m}})+k\overline{N}(r,g^{n+m})+O(\log r)=k\overline{N}(r,g)+O(\log r)$$

using above inequality and (9) we get $(n+m)N(r,f)+k\overline{N}(r,f)\leq k\overline{N}(r,g)+O(\log r)$. Similarly we get $(n+m)N(r,g)+k\overline{N}(r,g)\leq k\overline{N}(r,f)+O(\log r)$. Combining we get $N(r,f)+N(r,f)=O(\log r)$. Thus we obtain (8) which mean that f and g has at most finitely many poles. Now we prove that $\sigma(f)=\sigma(g)$. By K.Yamanoi [8] result of second fundamental theorem with taking $F=f^nP(f)$, $G=g^nP(g)$ we get

$$\begin{aligned}T(r,F^{(k)})&\leq\overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+(\varepsilon+O(1))T(r,F)\\&\leq\overline{N}(r,f)+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+(\varepsilon+O(1))T(r,F)\end{aligned}$$

Therefore $T(r,F^{(k)})-\overline{N}(r,\frac{1}{F^{(k)}})\leq\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+(\varepsilon+O(1))T(r,F)$ Using Lemma 4 we get, $T(r,F)\leq\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-p(z)}\right)+N_{k+1}(r,\frac{1}{F})+(\varepsilon+O(1))T(r,F)$

$$\begin{aligned}(n+m)T(r,f)&\leq\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{G^{(k)}-p(z)}\right)+(\varepsilon+O(1))T(r,F)\\&\leq\overline{N}(r,f)+(k+1)(n+m)T(r,g)+l\log r+(\varepsilon+O(1))T(r,f)\end{aligned}$$

$\Rightarrow(n+m-l-1)T(r,f)\leq(k+1)(n+m)T(r,g)+(\varepsilon+O(1))T(r,f)$
Since $\varepsilon<1$ then $T(r,f)=O(T(r,g))$,
Similarly, $T(r,g)=O(T(r,f))$ and hence

$$\sigma(f)=\sigma(g) \quad (10)$$

Then $f=\frac{e^{h(z)}}{r(z)}$, $g=\frac{e^{h_1(z)}}{q(z)}$ where $r(z)$ and $q(z)$ are polynomial with degree $deg(r(z))=r$, $deg(q(z))=q$, while $h(z)$ and $h_1(z)$ are nonconstant entire functions.

By Lemma 3, $h(z)$ and $h_1(z)$ are polynomial with $deg(h(z))=deg(h_1(z))=h=\sigma(f)$, Then we have

$$\left(f^{n+m}\right)^{(k)}=\frac{(m+n)e^{(n+m)(h(z))}}{r^{n+m+k}(z)}R_k(z)$$

and

$$\left(g^{n+m}\right)^{(k)}=\frac{(m+n)e^{(n+m)(h_1(z))}}{q^{n+m+k}(z)}Q_k(z),$$

where $R_k(z)$ and $Q_k(z)$ are two polynomials. From, (5) we get $h(z)+h_1(z)=C$, where C is a constant. Furthermore we have $deg(R_k(z))+deg(Q_k(z))=deg(r^{n+m+k}(z))+deg(q^{n+m+k}(z))+2l$, $\Rightarrow k(r+h-1)+k(q+h-1)=q(n+m+k)+r(n+m+k)+2l$.

$$\Rightarrow 2k(h-1)=(n+m)(q+r)+2l \tag{11}$$

If $N(r,f)+N(r,g)\neq 0$ then $(q+r)\geq 1$. From (11) we obtain $2k(h-1)\geq (n+m)+2l \Rightarrow n\leq 2k(h-1)-(m+2l)$ which contradict the given conditions. Therefore $N(r,f)+N(r,g)=0$, showing that both f and g are entire function and so $r=q=0$. From (11) we get $h=l+1$, $k=1$ or $h=2$, $k=l$

Case 1: For $k=1$, $h=l+1$. We get $h'(z)=bp(z)$, $h_1'(z)=-bp(z)$ where $b\neq 0$ is a constant. $\Rightarrow h(z)=bQ(z)+d_1$ and $h_1(z)=-bQ(z)+d_2$, where $Q(z)$ is a polynomial without constant term such that $Q'(z)=p(z)$ and d_1, d_2 are constants. Therefore $f=b_1e^{bQ(z)}$, $g=b_2e^{-bQ(z)}$, where b_1, b_2 are constant satisfying the condition ${a_m}^2(b_1b_2)^{n+m}((n+m)b)^2=-1$.

Case 2: $h=2$, $k=l$.
For this case we getting a contradiction by our assumption. The case $P(w)=c_0$ can be proved similarly. This completes the proof of the Lemma.

Lemma 9. *[4] Let f_1 and f_2 be two nonconstant meromorphic functions. If $c_1f_1+c_2f_2=c_3$, where c_1, c_2, c_3 are nonzero constants, then*

$$T(r,f_1)\leq \overline{N}(r,f_1)+\overline{N}(r,\frac{1}{f_1})+\overline{N}(r,\frac{1}{f_2})+S(r,f_1).$$

Lemma 10. *Let f and g be two nonconstant meromorphic functions having zeros and poles of order at least s. Let k, m, n, be three integers with $n>\frac{2k+1}{s}+m$ and let $P(w)=a_mw^m+a_{m-1}w^{m-1}+\cdots+a_1w+a_0$ or $P(w)\equiv c_0$, where $a_0\neq 0, a_1,\cdots,a_m\neq 0$ are complex constants. If $(f^nP(f))^{(k)}=(g^nP(g))^{(k)}$, then $f^nP(f)=g^nP(g)$.*

Proof. From the assumption, we get $f^nP(f)=g^nP(g)+r(z)$ where $r(z)$ is a polynomial of degree at most $k-1$. If $r(z)\not\equiv 0$ then by Lemma 9 we have $T(r,\frac{f^nP(f)}{r(z)})\leq \overline{N}(r,\frac{f^nP(f)}{r(z)})+\overline{N}(r,\frac{r(z)}{f^nP(f)})+\overline{N}(r,\frac{r(z)}{g^nP(g)})+S(r,f)+S(r,g)$. Therefore,

$$\begin{aligned}T(r,f^nP(f))&\leq T(r,\frac{f^nP(f)}{r(z)})+(k-1)\log r+O(1)\\&\leq \overline{N}(r,\frac{f^nP(f)}{r(z)})+\overline{N}(r,\frac{r(z)}{f^nP(f)})+\overline{N}(r,\frac{r(z)}{g^nP(g)})\\&+(k-1)\log r+S(r,f)+S(r,g)\\&\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{P(f)})\\&+\overline{N}(r,\frac{1}{P(g)})+2(k-1)\log r+S(r,f)+S(r,g),\end{aligned}$$

by Lemma 1 and $T(r,f)\geq s\log r$+ O(1), we have

$$(n+m)T(r,f)\leq \left(m+\frac{2}{s}+\frac{2(k-1)}{s}\right)T(r,f)+(m+\frac{1}{s})T(r,g)+S(r,f)+S(r,g)$$

Similarly, $(n+m)T(r,g)\leq \left(m+\frac{2}{s}+\frac{2(k-1)}{s}\right)T(r,g)+(m+\frac{1}{s})T(r,f)+S(r,f)+S(r,g)$. By combining above two we get, $\left\{(n+m)-(2m+\frac{2k+1}{s})\right\}(T(r,f)+T(r,g))\leq S(r,f)+S(r,g)$, which is a contradiction. Hence $r(z)\equiv 0$ and so, $f^nP(f)=g^nP(g)$.

Lemma 11. [20] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, a be a finite nonzero constant. If f and g share a CM, then one of the following cases holds:*

(i) $T(r,f) \le N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$; *same holds for* $T(r,g)$;
(ii) $fg = a^2$;
(iii) $f = g$.

Lemma 12. [1] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If f and g share 1 IM and $H \not\equiv 0$, then*

$$T(r,f) \le N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + 2\overline{N}(r,\frac{1}{f}) + 2\overline{N}(r,f) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + S(r,f) + S(r,g).$$

Lemma 13. [13] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $n(\ge 1)$, $k(\ge 1)$, $m(\ge 0)$ be three integers. Suppose that $F_1 = \frac{(f^nP(f))^{(k)}}{p(z)}$ and $G_1 = \frac{(g^nP(g))^{(k)}}{p(z)}$. If there exists two nonzero constants c_1 and c_2 such that $\overline{N}(r,\frac{1}{F-c_1}) = \overline{N}(r,\frac{1}{G_1})$ and $\overline{N}(r,\frac{1}{G_1-c_2}) = \overline{N}(r,\frac{1}{F_1})$, then $n \le 3k + m^* + 3$, where* $m^* = \begin{cases} m, & \text{if } P(f) \ne c_0; \\ 0, & \text{if } P(f) = c_0; \end{cases}$

3 Proof of the main Theorems

Proof of Theorem 15:

Proof. We discuss the following cases separately

Case (i): Let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0(\ne 0), a_1, \cdots, a_m(\ne 0)$ are constants, is not a monomials. Suppose that $F = (f^nP(f))^{(k)}$, $G = (g^nP(g))^{(k)}$ and $F^* = f^nP(f)$, $G^* = g^nP(g)$ and also $F_1 = \frac{F}{p(z)}$, $G_1 = \frac{G}{p(z)}$. Since F_1, G_1 share 1 CM by Lemma 11 one of the following subcases holds:

(a) $T(r,F_1) \le N_2(r,\frac{1}{F_1}) + N_2(r,\frac{1}{G_1}) + N_2(r,F_1) + N_2(r,G_1) + S(r,F_1) + S(r,G_1)$ same holds for $T(r,G_1)$;
(b) $F_1G_1 = p^2(z)$;
(c) $F_1 = G_1$.

Subcase (a): We have

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G). \tag{12}$$

By Lemma 4 with $s = 2$, we obtain

$$T(r,F^*) \le T(r,F) - N_2(r,\frac{1}{F}) + N_{k+2}(r,\frac{1}{F^*}) + S(r,F), \tag{13}$$

$$N_2(r,\frac{1}{G}) \le N_{k+2}(r,\frac{1}{G^*}) + k\overline{N}(r,G) + S(r,G). \tag{14}$$

Using (12), (14) in (13) we get,

$$\begin{aligned} T(r,F^*) &\le N_{k+2}(r,\frac{1}{F^*}) + N_{k+2}(r,\frac{1}{G^*}) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) + S(r,F) + S(r,G) \\ &\le (k+2)\overline{N}(r,\frac{1}{f}) + N\left(r,\frac{1}{P(f)}\right) + (k+2)\overline{N}(r,\frac{1}{g}) + N\left(r,\frac{1}{P(g)}\right) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) + S(r,f) + S(r,g) \\ &\le \frac{k+2}{s}N(r,\frac{1}{f}) + N\left(r,\frac{1}{P(f)}\right) + \frac{k+2}{s}N(r,\frac{1}{g}) + N\left(r,\frac{1}{P(g)}\right) + \frac{k+2}{s}N(r,g) + \frac{2}{s}N(r,f) + S(r,f) + S(r,g) \end{aligned}$$

Now by first fundamental theorem and Lemma 1 we get

$$(n+m)T(r,f) \leq \left(\frac{2k+4}{s}+m\right)T(r,g)+\left(\frac{k+4}{s}+m\right)T(r,f)+S(r,f)+S(r,g).$$

Similarly,

$$(n+m)T(r,g) \leq \left(\frac{2k+4}{s}+m\right)T(r,f)+\left(\frac{k+4}{s}+m\right)T(r,g)+S(r,f)+S(r,g).$$

Combining above two inequality, we get

$$(n+m)\{T(r,f)+T(r,g)\} \leq \left(\frac{3k+8}{s}+2m\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$$
$$\Rightarrow \left\{n-\left(\frac{3k+8}{s}+m\right)\right\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction for $n > \left(\frac{3k+8}{s}+m\right)$.

Subcase (b): Now by (b) we have

$$(f^nP(f))^{(k)}.(g^nP(g))^{(k)} = p^2(z),$$

which is a contradiction by Lemma 7.

Subcase (c): By (c) we get

$$(f^nP(f))^{(k)} = (g^nP(g))^{(k)}.$$

Here $n > \left(\frac{3k+8}{s}+m\right) > \left(\frac{2k+1}{s}+m\right)$. So by Lemma 10 we get

$$f^nP(f) = g^nP(g)$$

$$i.e., f^n(a_mf^m+...+a_1f+a_0) = g^n(a_mg^m+...+a_1g+a_0). \tag{15}$$

Let $h=\frac{f}{g}$. If h is a constant putting $f=gh$ in (15), we get

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+......+a_0g^n(h^n-1)=0$$

which implies $h^d=1$, where $d=gcd(n+m,n+m-1,....,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,...,m\}$. Thus $f=tg$ for some t such that $t^d=1$, where $d=gcd(n+m,n+m-1,....,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,...,m\}$. If h is not a constant then from (15) we see f and g satisfy the algebraic equation $R(f,g)=0$ where $R(f,g)$ is given by (1)

Case (ii): When $P(w)=a_mw^m$, or $P(w)=c_0$ $a_m\neq 0$, c_0 are complex constant. Proceeding as in case (i) above we obtain $F_1G_1=1$ or $F_1=G_1$.
If $F_1G_1=1$ then by Lemma 8 gives $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z)=p(z)$, b_1, b_2 and b are three constants satisfying ${a_m}^2(b_1b_2)^{n+m}((n+m)b)^2=-1$ or ${c_0}^2(b_1b_2)^n(nb)^2=-1$.
If $F_1=G_1$ we obtain $f=tg$ for a constant t such that $t^{n+m^*}=1$.

Proof of Theorem 16:

Proof. Case (i): Let $P(w)$ is not a monomial. Suppose that $F=(f^nP(f))^{(k)}$, $G=(g^nP(g))^{(k)}$ and $F^*=f^nP(f)$, $G^*=g^nP(g)$ and also $F_1=\frac{F}{p(z)}$, $G_1=\frac{G}{p(z)}$. Then F_1, G_1 share 1 IM. We assume that $H\not\equiv 0$ defined as in (2). So, from Lemma

12 we have

$$T(r,F_1) \le N_2(r,\frac{1}{F_1}) + N_2(r,\frac{1}{G_1}) + N_2(r,F_1) + N_2(r,G_1) + 2\overline{N}(r,\frac{1}{F_1})$$
$$+2\overline{N}(r,F_1) + \overline{N}(r,\frac{1}{G_1}) + \overline{N}(r,G_1) + S(r,f) + S(r,g).$$

i;e.,

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F})$$
$$+\overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,f) + S(r,g). \tag{16}$$

Now by Lemma 4 with $s = 2$, we get

$$T(r,F^*) \le T(r,F) - N_2(r,\frac{1}{F}) + N_{2+k}(r,\frac{1}{F^*}) + S(r,f). \tag{17}$$

and

$$N_2(r,\frac{1}{G}) \le k\overline{N}(r,G) + N_{2+k}(r,\frac{1}{G^*}) + S(r,g). \tag{18}$$

Using (16), (18) in (17) we have,

$$T(r,F^*) \le N_{2+k}(r,\frac{1}{F^*}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,f) + S(r,g)$$
$$\le N_{2+k}(r,\frac{1}{F^*}) + N_{2+k}(r,\frac{1}{G^*}) + k\overline{N}(r,G) + N_2(r,F) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F})$$
$$+\overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,f) + S(r,g)$$
$$\le (3k+4)\overline{N}(r,\frac{1}{f}) + 3N(r,\frac{1}{P(f)}) + (2k+3)\overline{N}(r,\frac{1}{g})$$
$$+2N(r,\frac{1}{P(g)}) + (2k+3)\overline{N}(r,g) + (2k+4)\overline{N}(r,f) + S(r,f) + S(r,g).$$

By using Lemma 1 and 1st fundamental theorem, we get

$$(n+m)T(r,f) \le \frac{3k+4}{s}T(r,f) + 3mT(r,f) + \frac{2k+3}{s}T(r,g) + 2mT(r,g) + \frac{2k+3}{s}T(r,g) + \frac{2k+4}{s}T(r,f) + S(r,f) + S(r,g)$$
$$\le \left(\frac{5k+8}{s} + 3m\right)T(r,f) + \left(\frac{4k+6}{s} + 2m\right)T(r,g) + S(r,f) + S(r,g). \tag{19}$$

Similarly,

$$(n+m)T(r,g) \le \left(\frac{5k+8}{s} + 3m\right)T(r,g) + \left(\frac{4k+6}{s} + 2m\right)T(r,f) + S(r,f) + S(r,g). \tag{20}$$

Combining (19) and (20) we get

$$\left\{n - (\frac{9k+14}{s} + 4m)\right\}(T(r,f) + T(r,g)) \le S(r,f) + S(r,g) \tag{21}$$

which is a contradiction as $n > max\left\{2k+2l, \frac{9k+14}{s}+4m, 2k(\sigma(f)-1)-(m+2l)\right\}$. Therefore $H \equiv 0$. This gives,

$$(\frac{F_1^{(2)}}{F_1^{(1)}} - 2\frac{F_1^{(1)}}{F_1-1}) = (\frac{G_1^{(2)}}{G_1^{(1)}} - 2\frac{G_1^{(1)}}{G_1-1}).$$

Integrating both sides of the above equality twice we get

$$\Rightarrow \frac{1}{F_1-1} = \frac{A}{G_1-1} + B, \tag{22}$$

where $A \neq 0$, B are constants. We now discuss the following three subcases:

Subcase (i): Let $B \neq 0$ and $A = B$. Then from (22) we get

$$\Rightarrow \frac{1}{F_1-1} = \frac{BG_1}{G_1-1}. \tag{23}$$

If $B = -1$ then from above equation we get

$$F_1G_1 = 1$$

i;e.,

$$(f^nP(f))^{(k)}.(g^nP(g))^{(k)} = p^2(z).$$

a contradiction by Lemma 7. If $B \neq -1$, from (23), we have $\frac{1}{F_1} = \frac{BG_1}{(1+B)G_1-1}$ and so $\overline{N}(r, \frac{1}{G_1-\frac{1}{1+B}}) = \overline{N}(r, \frac{1}{F_1})$. Now by Nevanlinna second fundamental theorem, we get

$$T(r,G_1) \leq \overline{N}(r, \frac{1}{G_1}) + \overline{N}(r, \frac{1}{G_1-\frac{1}{1+B}}) + \overline{N}(r,G_1) + S(r,G_1) \leq \overline{N}(r, \frac{1}{G_1}) + \overline{N}(r, \frac{1}{F_1}) + \overline{N}(r,G_1) + S(r,G_1).$$

Using Lemma 4

$$T(r,G) \leq N_{k+1}(r, \frac{1}{f^nP(f)}) + k\overline{N}(r,f) + T(r,G) + N_{k+1}(r, \frac{1}{g^nP(g)}) - (n+m)T(r,g) + \overline{N}(r,g) + S(r,g).$$

Therefore,

$$(n+m)T(r,g) \leq (k+1)\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{P(f)}) + k\overline{N}(r,f) + (k+1)\overline{N}(r, \frac{1}{g}) + N(r, \frac{1}{P(g)}) + \overline{N}(r,g) + S(r,f) + S(r,g),$$
$$(n+m)T(r,g) \leq \left(\frac{2k+1}{s}+m\right)T(r,f) + \left(\frac{k+2}{s}+m\right)T(r,g) + S(r,f) + S(r,g).$$

Similarly,

$$(n+m)T(r,f) \leq \left(\frac{2k+1}{s}+m\right)T(r,g) + \left(\frac{k+2}{s}+m\right)T(r,f) + S(r,f) + S(r,g).$$

Combining above two inequality, we get

$$\left(n-(\frac{3k+3}{s}+m)\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g).$$

Which contradict our assumption.

Subcase (ii):

Let $B \neq 0$ and $A \neq B$. Then from (22) we get $F_1 = \frac{(B+1)G_1-(B-A+1)}{BG_1+(A-B)}$ and so $\overline{N}(r, \frac{1}{G_1-\frac{B-A+1}{B+1}}) = \overline{N}(r, \frac{1}{F_1})$. Proceeding as in subcase (i) we get a contradiction.

Subcase (iii): Let $B = 0$ and $A \neq 0$. Then (22) gives $F_1 = \frac{G_1+A-1}{A}$ and $G_1 = AF_1 - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{1}{F_1-\frac{A-1}{A}}) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, \frac{1}{G_1-(1-A)}) = \overline{N}(r, \frac{1}{F_1})$. Using the Lemma 13 we have $n \leq 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F_1 = G_1$

$$(f^n P(f))^{(k)} = (g^n P(g))^{(k)}.$$

Here $n > \left(\frac{9k+14}{s} + 4m\right) > \left(\frac{2k+1}{s} + m\right)$. So by Lemma 10 we get

$$f^n P(f) = g^n P(g),$$

$$i.e., f^n(a_m f^m + ... + a_1 f + a_0) = g^n(a_m g^m + ... + a_1 g + a_0). \tag{24}$$

Let $h = \frac{f}{g}$.
If h is a constant putting $f = gh$ in (24), we get

$$a_m g^{n+m}(h^{n+m}-1) + a_{m-1} g^{n+m-1}(h^{n+m-1}-1) + ... + a_0 g^n(h^n - 1) = 0$$

which implies $h^d = 1$, where $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$. Thus $f = tg$ for some t such that $t^d = 1$, where $d = gcd(n+m, n+m-1, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$. If h is not a constant then from (24) we see f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by (1).

Case (ii): When $P(w) = a_m w^m$, or $P(w) = c_0$ $a_m \neq 0$, c_0 are complex constant. Proceeding as in case (i) above we obtain $F_1 G_1 = 1$ or $F_1 = G_1$. If $F_1 G_1 = 1$ then by Lemma 8 gives $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$, b_1, b_2 and b are three constants satisfying ${a_m}^2 (b_1 b_2)^{n+m} ((n+m)b)^2 = -1$ or ${c_0}^2 (b_1 b_2)^n (nb)^2 = -1$. If $F_1 = G_1$ we obtain $f = tg$ for a constant t such that $t^{n+m^*} = 1$

Following example is the supportive example of Theorem 15, when the polynomial $P(w)$ is not a monomial.

Example 1. Let $P(w) = a_4 w^4 + a_0$, where $a_0, a_4 \in \mathbb{C} \setminus \{0\}$ and $p(z) = z^3 - 3z^2 + z - 1$. Let

$$f(z) = \frac{(e^z - a)^2}{(e^z - b)^3} \text{ and } g(z) = -i\frac{(e^z - a)^2}{(e^z - b)^3},$$

where $a, b \in \mathbb{C} \setminus \{0\}$ with $a \neq b$. Let $k = 4, n = 16$. Clearly $s = 2$. Now

$$f^{16} P(f) = \frac{(e^z - a)^{32}}{(e^z - b)^{60}} \left\{ a_4 (e^z - a)^8 + a_0 (e^z - b)^{12} \right\}$$

and

$$g^{16} P(g) = \frac{(e^z - a)^{32}}{(e^z - b)^{60}} \left\{ a_4 (e^z - a)^8 + a_0 (e^z - b)^{12} \right\}.$$

Thus we see that f and g are two non-constant meromorphic functions having zeros and poles of multiplicity at least 2, and $[f^{16}P(f)]^{(4)}$ and $[g^{16}P(g)]^{(4)}$ share the polynomial $p(z)$ CM with

$$n > \max\left\{2k+2l,\ \frac{3k+8}{s}+m^*,\ 2k(\sigma(f)-1)-(m+2l)\right\}.$$

We thus see that one of the conclusion $f \equiv tg$ of Theorem 15 holds good where $t^d = (-i)^{gcd(20,16)} = (-i)^4 = 1$.

The next example is the supportive example of Theorem 15 when the polynomial $P(w)$ is a monomial.

Example 2. Let $f = \tan z$, and $g(z) = -\tan z$, $p(z) = a_2 z^2 + a_0$, where $a_0, a_2 \in \mathbb{C} \setminus \{0\}$, $P(w) = w^2$. Let $n = 18$. Clearly $s = 1, l = 2, m^* = 2$ and

$$n > \max\left\{2k+2l,\ \frac{3k+8}{s}+m^*,\ 2k(\sigma(f)-1)-(m+2l)\right\}.$$

We also see that $[f^{18}P(f)]^{(2)} = [f^{20}]^{(2)} = [\tan^{20} z]^{(2)}$ and $[g^{18}P(g)]^{(2)} = [g^{20}]^{(2)} = [\tan^{20} z]^{(2)}$ share the polynomial $p(z)$ CM, and one of the conclusion $f \equiv tg$ of Theorem 15 holds good where $t^d = (-1)^{gcd(20,18)} = (-1)^2 = 1$.

The next example is the supportive example of Theorem 15 when the polynomial $P(w) = c_0$.

Example 3. Let

$$f(z) = \frac{2+3i}{1-5i}e^{3z^3+2z^2-z+6} \text{ and } g(z) = \frac{1-5i}{2+3i}e^{-(3z^3+2z^2-z+6)}.$$

Let $n = 13$, $P(w) = c_0 = \dfrac{i}{13}$ and $k = 1$. Here we see that $b_1 = \dfrac{2+3i}{1-5i}$ and $b_2 = \dfrac{1-5i}{2+3i}$ and $b = 1$. It is clear that

$$c_0^2 (b_1 b_2)^n (nb)^2 = -1.$$

Let $Q(z) = 3z^3 + 2z^2 - z + 6$ and $p(z) = 9z^2 + 4z - 1$. Clearly, $Q'(z) = p(z)$. Clearly

$$n > \max\left\{2k+2l,\ \frac{3k+8}{s}+m^*,\ 2k(\sigma(f)-1)-(m+2l)\right\}$$

is satisfied. We see that

$$\begin{aligned}
\left(f^{13}P(f)\right)' &= \frac{i}{13}\left(\frac{2+3i}{1-5i}\right)^{13}\left(e^{13(3z^3+2z^2-z+6)}\right)' \\
&= \frac{i}{13}\left(\frac{2+3i}{1-5i}\right)^{13} \cdot 13(9z^2+4z-1) \cdot e^{13(3z^3+2z^2-z+6)} \\
&= i\left(\frac{2+3i}{1-5i}\right)^{13} \cdot p(z)e^{13(3z^3+2z^2-z+6)}.
\end{aligned}$$

Similarly,

$$\left(g^{13}P(g)\right)' = -i\left(\frac{1-5i}{2+3i}\right)^{13} \cdot p(z)e^{-13(3z^3+2z^2-z+6)}.$$

Clearly $\left(f^{13}P(f)\right)'$ and $\left(g^{13}P(g)\right)'$ share the polynomial $p(z)$ CM. We see that f and g are of the forms $f(z) = b_1 e^{bQ(z)}$ and $g(z) = b_2 e^{-bQ(z)}$ with $c_0^2 (b_1 b_2)^n (nb)^2 = -1$.

Acknowledgement

The authors would like to thanks the reviewers for their valuable comments and suggestions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] A. Banerjee, Meromorphic functions sharing one value, Int. J. Math. Math. Sci. 22 (2005), 3587–3598.
[2] C. C. Yang, On deficiencies of differential polynomials II, Math. Z. 125 (1972), 107–112.
[3] H. H. Chen, M. L. Fang, On the value distrbution of $f^n f'$, Sci. China Ser. A 38 (1995), 789–798.
[4] H. X. Yi, C. C. Yang, Uniqueness theory of meromorphic functions(in Chinese), Science Press, Beijing, 1995.
[5] I. Lahiri, S. Dewan, Value distribution of the product of a meromorphic functions and its derivative, Kodai Math. J. 26 (2003), 95-100.
[6] J. L. Zhang, Uniqueness theorems for entire functions concerning fixed points, Comput. Math. Appl. 56 (2008), 3079–3087.
[7] J. F Xu, F. L*ü*, H. X. Yi, Fixed points and uniqueness of meromorphic functions, Comput. Math. Appl. 59 (2008), 9–17.
[8] K. Yamanoi, The second main theorem for small functions and related problems, Acta Math. 192 (2004), 225–294.
[9] L. Yang, Value distributions theory, Springer-Verlag, Berlin, 1993.
[10] M. L. Fang, H. L. Qiu, Meromorphic functions that share fixed points, J. Math. Anal. Appl. 268 (2002), 426–439.
[11] M. L. Fang, Uniqueness and value sharing of entire functions, Comput. Math. Appl. 44 (2002), 828–831.
[12] M. L. Fang, X. H. Hua, Entire functions that share one value, Nanjing Univ. J. Math. Biquarterly 13 (1996), 44–48.
[13] P. Sahoo, Meromorphic functions that share fixed points with finite weights, Bull. Math. Anal. Appl. 2 (2010), 106–118.
[14] P. Sahoo, Uniqueness and differential polynomials of meromorphic functions sharing a polynomial, Pales. J. Math. 5 (2016), 46–56.
[15] W. Bergweiler, A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoam. 11 (1995), 355–373.
[16] W. C. Lin, H. X. Yi, Uniqueness theorems for meromorphic functions concerning fixed points, Complex Var. Theory Appl. 49 (2004), 793–806.
[17] W. C. Lin, H. X. Yi, Uniqueness theorems for meromorphic functions, Indian J. Pure Appl. Math. 35 (2004), 121–132.
[18] W. K. Hayman, Meromorphic function, Clarendon Press, Oxford, 1964.
[19] X. B. Zhang, J. F. Xu, Uniqueness of meromorphic functions sharing a small function and its applications, compt. Math. Appl. 61 (2011), 722-730.
[20] X. H. Hua, C. C. Yang, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math. 22 (1997), 395–406.