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New Trends in Mathematical Sciences

Approximation of fixed points for nonexpansive operators by means of the fastest Krasnoselskij iteration in Hilbert spaces

Toseef Ahmed Malik[1] *and Masood Ahmed Choudhary*[2]

[1]Department of Mathematics, University of Jammu, Jammu, India
[2]Department of Mathematics, Govt. Hr. Sec. School Mohore, Reasi, J and K, India

Abstract: The interest of this paper is to survey some old and new results on the approximation of fixed points for nonexpansive and pseudocontractive type operators by means of the fastest Krasnoselskij iteration. We consider here generalized pseudocontractions which are also Lipschitzian, a class for which we can use the Krasnoselskij iteration in order to approximate their fixed points.

1 Introduction

In the last three decades many papers have been published on the iterative approximation of fixed point for certain classes of operators, using the Mann and Ishikawa iteration methods, see for example Berinde [4], for recent survey. These papers were motivated by the fact that, under weaker contractive type condition, the Picard iteration (or the method of successive approximation), $x_{n+1} = Tx_n$, $n \geq 0$ need not converge to the fixed point of the operators in question.

In order to approximation fixed points of certain classes of operators which satisfy weak contractive type conditions that do not guarantee the convergence of Picard iterative process (or the method of successive approximation), certain mean value fixed point iterations, namely Krasnoselskij, Mann and Ishikawa iteration methods are useful to approximate fixed points. Though these iterative procedures have been introduced mainly in order to approximate fixed points of those operators for which the Picard iteration does not converge, even so there are results on important class of contractive mappings, that is, the class of quasi-contraction, for which all Picard, Krasnoselskij, Mann and Ishikawa iterations converge. One of the most studied class of quasi-contractive type operators is that of Zamfirescu operators, for which all important fixed point iteration procedures, i.e, the Picard, Mann and Ishikawa iterations, are known to converge to the unique fixed point of the operator T. Zamfirescu showed in [10] that an operator which satisfies the contractive conditions has a unique fixed point that can be approximated using the Picard iteration. Later, Rhoades [8], proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operators.

Verma [9] approximated fixed points of strictly pseudo-contractive operators in Hilbert spaces by both Krasnoselskij and Mann type iterative methods. When for a certain class of mappings, two or more fixed points iteration procedures can be used to approximate their fixed points, it is of theoretical and practical importance to compare the rate of convergence of these methods and to find out, if possible which of them converges faster. Finally, we shall obtain a result on the fastest

* Corresponding author e-mail: Tsfmlk5@gmail.com; msdchdhr706@gmail.com

iteration in the family of the krasnoselskij iterative scheme. As far as we known, there are only a few papers devoted to this very important numerical problem. It is also well known that if T is assumed to be only a nonexpasive map, then the Picard iteration $\{T^n x_0\}_{n \geq 0}$ need no longer converge. The key idea in introducing Krasnoselskij iteration is that fact that, if T_λ is the averaged mapping associated to T, then if T is nonexpasive, so is T_λ, and both have the same fixed point set. Moreover, T_λ has much more asymptotic behavior than the original mapping T.

2 Preliminaries

Definition 1. *Let H be a real vector space. An inner product is a functional $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ (the set of real numbers) satisfying:*

(p1) $\langle x, x \rangle \geq 0$, *for all $x, y \in H$;*
(p2) $\langle x, y \rangle = \langle y, x \rangle$, *for all $x, y \in H$;*
(p3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, *for each $x, y, z \in H$ and all $\alpha, \beta \in \mathbb{R}$.*

If $\langle \cdot, \cdot \rangle$ is an inner product on H, then the function $x \to \langle x, x \rangle^{1/2}$ defines a norm on H, called the norm induced by the inner product. The pair $(H, \langle \cdot, \cdot \rangle)$ is called a prehilbertian space.

A sequence $\{x_n\}$ in H is called Cauchy sequence iff $\langle x_n - x_m , x_n - x_m \rangle^{1/2} = \|x_n - x_m\| \to 0$ as $n, m \to 0$. A prehilbertian space H is said to be complete if every Cauchy sequence in H converges to a point of H. A prehilbertian space that is complete (with respect to the metric corresponding to the norm induced by the scalar product) is called Hilbert space.

Definition 2. *Let H be a real Hilbert space with $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$. An operator $T : H \to H$ is said to be generalized pseudo-contraction if, for all x, y in H, there exists a constant $r > 0$ such that*

$$\|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2 \tag{1}$$

which is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq r \|x - y\|^2, for \ all \ x, y \in H \tag{2}$$

or to $\langle (I - T)x - (I - T)y, x - y \rangle \geq (1 - r) \|x - y\|^2,$ \tag{3}

where I is the identity map. Clearly, if T is generalized pseudo-contraction with $r < 1$, then $I - T$ is strongly monotone. For $r = 1$ in (2.1), T is called pseudo-contraction, a concept introduced and studied by Browder and Petryshyn [6] and thereafter by many authors, in connection with the problem of approximating fixed points, see Berinde [4].

Definition 3. *The operator $T : H \to H$ is called Lipschitzian (or Lipschitz continuous) if there exist a constant $s > 0$ such that*

$$\|Tx - Ty\| \leq s \cdot \|x - y\|, for \ all \ x, y \in H. \tag{4}$$

By the Cauchy Schwarz inequality

$$|\langle Tx - Ty, x - y \rangle| \leq \|Tx - Ty\| \, \|x - y\|.$$

It is clear that any Lipschitzian operator T, that is, any operator for which there exists $s > 0$ such that $\|Tx - Ty\| \leq s\|x - y\|$, $x, y \in H$ is also a generalized pseudo-contractive operator, with $r = s$ and T is non-expansive if T is $1 - Lipschitzian.

Example 1. [4] Let H be the real line $\mathbb{R}$ endowed with the Euclidean inner product and norm, $C = [\frac{1}{2},\ 2]$ and $T : C \to C$ a self map defined by $Tx = \frac{1}{x}$, for all $x \in C$. Then T is Lipschitzian with constant $s = 4$ and also generalized pseudo-contractive with constant $r = 4$. Moreover, T is also generalized pseudo-contractive with any constant $r > 0$ arbitrary. It is easy to see that T has a unique fixed point, $F_T = \{1\}$, and that, for any initial choice $x_0 = a \neq 1$, the Picard iteration yields the oscillatory sequence $a,\ 1/a,\ a,\ 1/a,\ldots$.

Definition 4. *A subset C of a real vector space H is called convex if for any pairs of points $x,\ y \in C$, the closed segment with the extremities $x,\ y$, that is, the set $\{\lambda x + (1 - \lambda)y : \lambda \in [0,1]\}$ is contained in C. A subset C of H is called bounded if there exists $M > 0$ such that $\|x\| \leq M$, for all $x \in C$.*

Definition 5. *Let H^* be the dual space of a real Hilbert space. The multivalued mapping $J : H \to P(H^*)$ defined by $Jx = \{f \in H^* : \langle f, x \rangle = \|x\| \cdot \|f\|,\ \|x\| = \|f\|\}$ is called the normalized duality mapping of H.*

In Hilbert space H, a sequence $\{x_n\}_{n=0}^{\infty} \subset H$ converges strongly to x if and only if $\|x_n - x\| \to 0$, as $n \to \infty$ which is denoted by $x_n \to x$, we shall often consider the weak convergence, corresponding to the weak topology in H. We say that $\{x_n\}_{n=0}^{\infty} \subset H$ converges weakly to x if for any $f \in H^*$, $\langle f, x_n \rangle \to \langle f, x \rangle$, as $n \to \infty$ and we denote this by $x_n \rightharpoonup x\ (n \to \infty)$.

Since conditions of pseudo-contractive type are very useful additional assumptions in approximating fixed points of Lipschitzian mappings, we summarize in the sequel the most important concepts of this kind.

Definition 6. *Let H be an arbitrary Hilbert space. A mapping T with domain $D(T)$ and range $R(T)$ in H is called:*

(a) *strongly pseudocontraction if there exists $k > 0$ such that for all $x, y \in D(T)$ there exists $j(x,y) \in J(x-y)$ such that*

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq k \cdot \|x-y\|^2;$$

(b) *pseudocontractive if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that*

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq 0,$$

where J is the normalized duality mapping.

Pseudo-contractive mappings are firmly connected with another important class of operators, that is, the class of accretive operators.

Definition 7. *A mapping T with domain and range in H is called:*

(a) *strongly accretive if there exists a positive number k such that for each $x, y \in D(T)$ there is a $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y) \rangle \geq k \cdot \|x-y\|^2;$$

(b) *accretive if for each $x, y \in D(T)$ we have*

$$\langle Tx - Ty, j(x-y) \rangle \geq 0.$$

Remark.

(i) By comparing Definitions (2.7) and (2.8), we remark that an operator T is (strongly) pseudo-contractive if and only if $(I - T)$ is (strongly) accretive.

(ii) As a consequence of a result of Kato [1], the concept of pseudo-contractive and accretive operators can be equivalently defined as follows:

(a) T is strongly pseudo-contractive if there exists $t > 1$ such that, for all $x, y \in D(T)$ and $r > 0$, the following inequality holds

$$\|x - y\| \geq \|(1+r)(x+y) - rt(Tx - Ty)\|$$

(b) T Is pseudo-contractive if $t = 1$ in the previous inequality.

(c) T is strongly accretive if there exists $k > 0$ such that the inequality

$$\|x - y\| \leq \|x - y + r[(T - kI)x - (T - kI)y]\|$$

holds for all $x, y \in D(T)$ and $r > 0$.

(d) T is accretive if $k = 0$ in in the previous inequality.

3 Some fixed point iteration procedures:

Let $T : H \to H$ be a self map. For any $x_0 \in H$, the sequence $\{x_n\}_{n=0}^{\infty} \subset H$ given by

$$x_n = Tx_{n+1} = T^n(x_0), \quad n = 1, 2.........$$ (5)

is called the *Picard iteration or the sequence of successive approximations* with the initial value at x_0.

Let H be a Hilbert space and $T : H \to H$ a self map. For any $x_0 \in H$ and a constant $\lambda \in [0,1]$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2.........$$ (6)

is called *Krasnoselskij iteration or Krasnoselskij iterative method* and is denoted by $K_n(x_0, \lambda, T)$.

Let H be a Hilbert space and $T : H \to H$ a self map. Let $x_0 \in H$ be arbitrary and $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ a sequence of real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset H$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2.......$$ (7)

is called the *Mann iteration or Mann iterative procedure* and will be denoted by $M_n(x_0, \alpha_n, T)$. The sequence $\{x_n\}_{n=0}^{\infty} \subset H$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n = 0, 1, 2.......$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n = 0, 1, 2............$$ (8)

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0,1]$, and $x_0 \in H$ is arbitrary, is called the *Ishikawa iteration or Ishikawa iterative procedure* and is denoted by $I(x_0, \alpha_n, \beta_n, T)$.

Remark. For $\alpha_n = \lambda$ (constant), the iteration (3.3) reduces to the so called *Krasnoselskij iteration* while for $\alpha_n = 1$ we obtain the *Picard iteration or the sequence of successive approximation* (3.1), as it is commonly known obviously, for $\beta_n = 0$, the *Ishikawa iteration* (3.4) reduces to *Mann iteration* (3.3).

Example 2. Let $C = [\frac{1}{2}, 2] \subseteq \mathbb{R}$ be endowed with the usual norm and $T : C \to C$, defined by $Tx = \frac{1}{x}$, $x \in C$. Then

(i) T has a unique fixed point, that is $F_T = \{1\}$;

(ii) The Picard iteration does not converge 1, for any $x_0 \neq 1$ in $[\frac{1}{2}, 2]$;

(iii) The Krasnoselskij iteration converges to the fixed point of T, for λ satisfying $0 < \lambda < 2(1-r)/(17-2r)$, where $0 < r < 1$.

The convergence theorems below, stated for the Krasnoselskij and Mann iterative procedures respectively, by Verma [9] will be used in the proof of the main results.

Theorem 1. [2] *Let C be a non-empty closed convex subset of a real Hilbert space H and $T : C \to C$ a generalized pseudo-contractive and Lipschitzian operator with the corresponding constants $s > 0$ and $r > 0$ satisfying*

$$0 < r < 1, \ r \leq s. \tag{9}$$

Then

(i) *T has a unique fixed point p;*

(ii) *For each x_0 in C, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n = 0.1, 2.....$ converges strongly to p, for all $\lambda \in (0,a) \cap (0,1)$,*

$$\text{where} \quad a = 2(1-r)/(1-2r+s^2) \tag{10}$$

Theorem 2. [9] *Let H be a real Hilbert space and C be a non-empty closed convex subset of H. Let $T : C \to C$ be Lipschitzian and generalized pseudo-contractive operator with the corresponding constants $s > 1$ and $r > 0$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be an increasing sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \tag{11}$$

Then

(i) *T has a unique fixed point p in C;*

(ii) *The Mann iteration $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, 2...$ converges strongly to p for any $x_0 \in C$ and all t in $(0,a)$ that satisfy $0 \leq (1-t)^2 - 2t(1-t)r + t^2 s^2 < 1$, where a is given by (3.6).*

Example 3. Let $C = [\frac{1}{2}, 2]$ and $T : C \to C$ a function given by $Tx = \frac{1}{x}$, $x \in C$. Here $s = 4$ and $r > 0$ arbitrary. Taking, for example $r = 0.5$, we get $2(1-r)/(1-2r+s^2) = \frac{1}{16}$, and so by Theorem (3.3), the sequence $\{x_n\}$ given by $x_{n+1} = (1-\lambda)x_n + \lambda \cdot \frac{1}{x_n}, \quad n = 0, 1, 2.......$ converges strongly to the fixed point $p = 1$ of T for all values of λ in the interval $(0, \frac{1}{16})$.

In order to compare two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ that converges to a certain fixed point p of a given operator T, Rhoades [8] considered that $\{u_n\}$ is better than $\{v_n\}$ if

$$\|u_n - p\| \leq \|v_n - p\|, \ \forall n \tag{12}$$

Definition 8. [5] *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converges to a and b, respectively, and assume that there exists*

$$l = \lim_{n \to \infty} \left| \frac{a_n - a}{b_n - b} \right|. \tag{13}$$

(a) *If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b;*

(b) *If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.*

Remark.

(i) In the case (a) we use the notation $a_n - a = o(b_n - b)$;

(ii) if $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$, that is, $b_n - n = o(a_n - a)$.

Suppose that for two fixed iterations procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \leq a_n, \ \ n = 0, 1, 2... \tag{14}$$

$$\|v_n - p\| \leq b_n, \ \ n = 0, 1, 2... \tag{15}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). Then, in view of Definition (3.6), we will adopt the following concepts.

Definition 9. [5] *Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (3.10) and (3.11), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ then it can be said that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.*

Example 4. [3] If we take $p = 0$, $u_n = \frac{1}{(n+1)}$, $v_n = \frac{1}{n}$, $n \geq 1$, then $\{u_n\}$ is better than $\{v_n\}$, but $\{u_n\}$ does not converge faster than $\{v_n\}$. Indeed we have

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1 \tag{16}$$

and hence $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence.

The example (3.9) shows that Definition (3.8) introduces a sharper concept of rate of convergence than the one considered by Rhoades [8].

In the next section we begin by proving the Browder-Gohde-Kirk fixed point theorem, which is a basic fixed point existence result for non-expansive operators. The proof will be given in a Hilbert space setting, suitable to many convergence theorems for the Krasnoselskij iterations.

4 The main result

Now, we are in position to state the main convergence theorems for Krasnoselskij iteration.

Theorem 3. *Let C be a closed bounded convex subset of a Hilbert space H and $T: C \to C$ be a non-expansive operator. Then T has at least one fixed point.*

Proof. For fixed element y_0 in C and a number s with $0 < s < 1$, we denote

$$T_s(x) = (1 - s)y_0 + sTx, \ \ x \in C.$$

Since C is convex and closed, we deduce that $T_s: C \to C$ is a s-contraction and it has a unique fixed point, say u_s (from the contraction mapping principle). On the other hand, since C is closed, convex and bounded in a Hilbert space H, it is weakly compact. Hence we find a sequence $\{s_j\}$ in $(0, 1)$ such that $s_j \to 1$ as $j \to \infty$ and $u_j = u_{s_j}$, converges weakly to

an element p of H. Since C is weakly closed, p lies in C. We shall prove that p is a fixed point of T. If u is any arbitrary point in H, we have $\|u_j - u\|^2 = \|(u_j - p) + (p - u)\|^2 = \|u_j - p\|^2 + \|p - u\|^2 + 2\langle u_j - p, \, p - u \rangle,$

$$where \quad 2\langle u_j - p, \, p - u \rangle \to 0 \quad as \, j \to \infty,$$

since $u_j - p$ converges weakly to zero in H. Letting $u = Tp$ above, we get

$$lim_{j\to\infty} \left(\|u_j - Tp\|^2 - \|u_j - p\|^2 \right) = \|p - Tp\|^2.$$

Moreover, since $s_j \to 1$ and $T_{s_j} u_j = u_j$, we have

$$\begin{aligned} Tu_j - u_j &= [s_j Tu_j + (1 - s_j)y_0] - u_j + (1 - s_j)[Tu_j - y_0] \\ &= (T_{s_j} u_j - u_j) + (1 - s_j)(Tu_j - y_0) \\ &= 0 + (1 - s_j)(Tu_j - y_0) \to 0 \quad as \, j \to \infty, \end{aligned}$$

and therefore $lim_{j\to\infty}\|Tu_j - u_j\| = 0$.

On the other hand, since T is non-expansive, we have

$$\|Tu_j - Tp\| \le \|u_j - p\|$$

and hence

$$\|u_j - Tp\| \le \|u_j - Tu_j\| + \|Tu_j - Tp\| \le \|u_j - Tu_j\| + \|u_j - p\|.$$

Thus $\quad lim\sup\left(\|u_j - Tp\| - \|u_j - p\|\right) \le lim_{j\to\infty}\|u_j - Tu_j\| = 0$, and due to the boundedness of C, we have also

$$lim\sup\left(\|u_j - Tp\|^2 - \|u_j - p\|^2\right) = lim\sup\left(\|u_j - Tp\| - \|u_j - p\|\right)\left(\|u_j - Tp\| + \|u_j - p\|\right) \le 0,$$

which yields

$$lim_{j\to\infty}\left(\|u_j - Tp\|^2 - \|u_j - p\|^2\right) = 0$$

and hence

$$\|p - Tp\|^2 = 0,$$

that is, p is a fixed point of T.

We can now consider a result on approximating fixed points of non-expansive mappings by means of the Krasnoselskij iteration. To this end, we start with the following concept:

Definition 10. *Let H be a Hilbert space and C a subset of H. A mapping $T: C \to H$ is called demicompact if it has the property that whenever $\{x_n\}$ is a bounded sequence in H and $\{Tx_n - x_n\}$ is strongly convergent, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is strongly convergent.*

Theorem 4. *Let C be a bounded closed convex subset of a Hilbert space H and $T: C \to C$ be a non-expansive and demicompact operator.*

(i) *Then the set F_T of fixed points of T is a nonempty convex set.*

(ii) *For any given x_0 in C and for any fixed number λ with $0 < \lambda < 1$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2... \qquad (17)$$

converges strongly to a fixed point of T.

(iii) *If $x_0 \in H$ and $\{x_n\}_{n=0}^{\infty} = M_n(x_0, \alpha_n, T)$ is the Mann iteration such that the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is bounded away from 0 and 1, then each of the sequence $\{x_{n+1} - x_n\}$ and $\{T x_n - x_n\}$ converges strongly to $0 \in H$.*

Proof.

(i) Since T is nonexpansive , so by theorem (4.1), T has fixed points in C, that is, $F_T \neq \phi$. Further, F_T is convex. i.e., when $x, y \in F_T$ and $\lambda \in [0, 1]$ we have

$$x_\lambda = (1 - \lambda)x + \lambda y \in F_T$$

Indeed, $\|T x_\lambda - x\| = \|T x_\lambda - T x\| \leq \|x_\lambda - x\|$ and $\|T x_\lambda - y\| \leq \|x_\lambda - y\|$,

$$\begin{aligned} \Rightarrow \quad \|x - y\| &\leq \|x - T x_\lambda + T x_\lambda - y\| \\ &\leq \|x - T x_\lambda\| + \|T x_\lambda - y\| \\ &\leq \|x - y\|. \end{aligned}$$

This shows that for some a, b with $0 \leq a, b \leq 1$, we have $x - T x_\lambda = a(x - x_\lambda)$ and $y - T x_\lambda = b(y - x_\lambda)$ from which it follows that $T x_\lambda = x_\lambda \in F_T$.

(ii) For any x_0 in C, the sequence $\{x_n\}_{n=0}^{\infty}$ given by (4.1) lies in C and is bounded. Let p be a fixed point of T, and so of averaged map T_λ, given by

$$T_\lambda = (1 - \lambda)I + \lambda I, \quad (I = identity \ operator) \qquad (18)$$

We first prove that the sequence $\{x_n - T x_n\}_{n \in \mathbb{N}}$ converges strongly to zero. Indeed
$x_{n+1} - p = (1 - \lambda)x_n + \lambda T x_n - p = (1 - \lambda)(x_n - p) + \lambda(T x_n - p)$.
On the other hand, for any constant a,

$$a(x_n - T x_n) = a(x_n - p) - a(T x_n - p).$$

Then $\|x_{n+1} - p\|^2 = (1 - \lambda)^2 \|x_n - p\|^2 + \lambda^2 \|T x_n - p\|^2 + 2\lambda(1 - \lambda)\langle T x_n - p, x_n - p \rangle$

$$and \ \ a^2 \|x_n - T x_n\|^2 = a^2 \|x_n - p\|^2 + a^2 \|T x_n - p\|^2 - 2a^2 \langle T x_n - p, x_n - p \rangle.$$

Hence, adding the corresponding sides of the preceding two inequalities and using the fact that T is nonexpansive and $T_p = p$, we get

$$\|x_{n+1} - p\|^2 + a^2 \|x_n - T x_n\|^2 \leq \left[2a^2 + \lambda^2 + (1 - \lambda)^2\right] \cdot \|x_n - p\|^2 + 2\left[\lambda(1 - \lambda) - a^2\right] \cdot \langle T x_n - p, x_n - p \rangle.$$

If we choose now an a such that $a^2 \leq \lambda(1 - \lambda)$, then from the last inequality we have

$$\|x_{n+1} - p\|^2 + a^2 \|x_n - T x_n\|^2 \leq \left(2a^2 + \lambda^2 + (1 - \lambda)^2 - 2\lambda(1 - \lambda) - 2a^2\right) \|x_n - p\|^2 = \|x_n - p\|^2$$

$\left(\text{by using the Cauchy Schwarz inequality, } \langle Tx_n - p, \, x_n - p \rangle \le \|Tx_n - p\| \cdot \|x_n - p\| \le \|x_n - p\|^2\right).$

Let $a^2 = \lambda(1 - \lambda) > 0$ and summing up the obtained inequality

$$a^2\|x_n - Tx_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

for $n = 0$ to $n = N$ we obtain

$$\lambda(1-\lambda)\sum_{n=0}^{N}\|x_n - Tx_n\|^2 \le \sum_{n=0}^{N}\left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right]$$
$$= \|x_0 - p\|^2 - \|x_{N+1} - p\|^2$$
$$\le \|x_0 - p\|^2,$$

which shows that $\sum_{n=0}^{\infty}\|x_n - Tx_n\|^2 < \infty$ and hence $\|x_n - Tx_n\| \to 0$, *as* $n \to \infty$. As T is demicompact, it follows that there exists a strongly convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to p \in F_T$. Since T is nonexpansive, $Tx_{n_i} \to Tp$ and $Tp = p$. This shows that the convergence of the entire sequence $\{x_n\}_{n=0}^{\infty}$ to p and now follows from the inequality $\|x_{n+1} - p\| \le \|x_n - p\|$, which can be deduce from the nonexpansiveness of T and is valid for each n.

(iii) For $x_0 \in H$ and $M_n(x_0, \, \alpha_n, \, T)$ is a Mann iteration such that

$$\|x_{n+1} - x_n\| = \alpha_n\|Tx_n - x_n\|,$$

and hence, having in view that $0 < a \le \alpha_n \le b < 1$, if either one of the sequences $\{x_{n+1} - x_n\}$ or $\{Tx_n - x_n\}$ converges to 0 then the other does also. If $\lim \|x_n - p\| = 0$, then obviously $\lim \|x_{n+1} - x_n\| = 0$. Otherwise, since the sequence $(\|x_n - p\|)$ is non-increasing, we certainly have $\lim \|x_n - p\| = d > 0$. We define now the sequences $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$w_n = (x_n - p)/\|x_n - p\|, \ \ y_n = (Tx_n - p)/\|x_n - p\|, \ \ and, \ \ respectively \ \ z_n = (x_{n+1} - p)/\|x_n - p\|.$$

Since we have that

$$x_{n+1} - p = (1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p),$$

by dividing it by $\|x_n - p\|$ it results that

$$z_n = (1 - \alpha_n)w_n + \alpha_n y_n.$$

Since $\|w_n\| = 1$, $\|y_n\| \le 1$ for all n, and if $\lim \|z_n\| = 1$, then $\lim \|w_n - y_n\| = 0$, which implies that $\|z_n\| \to d/d = 1$, we have that $\lim \|w_n - y_n\| = 0$, which gives $\lim \|Tx_n - x_n\| = 0$, and this completes the proof.

Remark.

(i) From the proof of the theorem (4.3) it shows that T_λ given by (4.2) is asymptotically regular, i.e., $\|T_\lambda^n x - T_\lambda^{n+1} x\| \to 0$, *as* $n \to \infty$, for any $x \in C$, that is, $x_n - x_{n+1} \to 0$, *as* $n \to \infty$, for any $x_0 \in C$. The existence of the previous limit alone does not imply genearlly the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to fixed point of T as there is one more additional assumption of demicompactness of T in theorem (4.3). There are other possible additional assumptions to ensure the convergence of $\{x_n\}_{n=0}^{\infty}$ under the hypothesis of asymptotic regularity. For example, in the case of the real line, $C = [a, b]$ the closed and bounded interval and $T: C \to C$ a continuous function, Hilam [7] showed that the Picard iteration associated to T converges if and only if it is asymptotically regular;

(ii) The Krasnoselskij iteration is in fact the Picard iteration corresponding to the averaged operator T_λ associated to T and defined by (4.2).

If in theorem (4.3) we remove the assumption that T is demicompact, then the Krasnoselskij iteration does not longer converge strongly, in general, but it converges at least weakly to a fixed point, which is shown by the next theorem (4.5).

Theorem 5. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a non-expansive and operator. Then, for any x_0 in C, the Krasnoselskij iteration $\{x_n\}_{n=0}^\infty$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2...$$

converges weakly to a fixed point of T.

Proof. Let F_T be the set of all fixed points of T in C. Since T is nonexpansive, for each $p \in F_T$ and each n we have

$$\|x_{n+1} - p\| \le \|x_n - p\|,$$

which shows that the function $g(p) = \lim_{n \to \infty} \|x_n - p\|$ is well defined and is a lower semicontinuous convex function on F_T. Let

$$d_0 = \inf\{g(p) : p \in F_T\}.$$

For each $\varepsilon > 0$, the set

$$F_\varepsilon = \{y : g(y) \le d_0 + \varepsilon\}$$

is closed, convex, nonempty and bounded and hence, weakly compact. Therefore $\cap_{\varepsilon > 0} F_\varepsilon \ne 0$, and in fact

$$\cap_{\varepsilon > 0} F_\varepsilon = \{y : g(y) = d_0\} \equiv F_0.$$

Since F_0 contains exactly one point. Indeed, since F_0 is convex and closed, for $p_0, p_1 \in F_0$, and $p\lambda = (1 - \lambda)p_0 + \lambda p_1$,

$$\begin{aligned}
g^2(p\lambda) &= \lim_{n \to \infty} \|p\lambda - x_n\|^2 = \lim_{n \to \infty} \left(\|\lambda(p_1 - x_n) + (1 - \lambda)(p_0 - x_n)\|^2 \right) \\
&= \lim_{n \to \infty} \left(\lambda^2 \|p_1 - x_n\|^2 + (1 - \lambda)^2 \|p_0 - x_n\|^2 + 2\lambda(1 - \lambda)\langle p_1 - x_n, \ p_0 - x_n \rangle \right) \\
&= \lim_{n \to \infty} \left(\lambda^2 \|p_1 - x_n\|^2 + (1 - \lambda)^2 \|p_0 - x_n\|^2 + 2\lambda(1 - \lambda)\|p_1 - x_n\| \cdot \|p_0 - x_n\| \right) \\
&\quad + \lim_{n \to \infty} \{2\lambda(1 - \lambda)[\langle p_1 - x_n, \ p_0 - x_n \rangle - \|p_1 - x_n\| \cdot \|p_0 - x_n\|]\} \\
&= g^2(p) + \lim_{n \to \infty} \{2\lambda(1 - \lambda)[\langle p_1 - x_n, \ p_0 - x_n \rangle - \|p_1 - x_n\| \cdot \|p_0 - x_n\|]\}.
\end{aligned}$$

Hence $\lim_{n \to \infty} \{2\lambda(1 - \lambda)[\langle p_1 - x_n, \ p_0 - x_n \rangle - \|p_1 - x_n\| \cdot \|p_0 - x_n\|]\} = 0$.
Since $\|p_1 - x_n\| \to d_0$ and $\|p_0 - x_n\| \to d_0$, the latter relation implies that

$$\begin{aligned}
\|p_1 - p_0\|^2 &= \|(p_1 - x_n) + (x_n - p_0)\|^2 \\
&= \|p_1 - x_n\|^2 + \|x_n - p_o\|^2 - 2\langle p_1 - x_n, \ p_0 - x_n \rangle \\
&\to d_0^2 - d_0^2 - 2d_0^2 = 0,
\end{aligned}$$

giving a contradiction.
Now in order to show that $x_n = U_\lambda^n x_0 \rightharpoonup p_0$, is suffices to assume that $x_{n_j} \rightharpoonup p$ for an infinite subsequence and then prove

that $p = p_0$. Since $p \in F_T$ and considering the definition of g and the fact that $x_{n_j} \to p$, we have

$$\begin{aligned}
\|x_n - p_0\|^2 &= \|x_{n_j} - p + p - p_0\|^2 \\
&= \|x_{n_j} - p\| + \|p - p_0\|^2 - 2\langle x_{n_j} - p, \ p - p_0 \rangle \\
&\to g^2(p) + \|p - p_0\|^2 = g^2(p_0) = d_0^2.
\end{aligned}$$

Since $g^2(p) \geq d_0^2$, the last inequality implies that

$$\|p - p_0\| \leq 0,$$

which implies that $p = p_0$.

We present here convergence theorem for the Krasnoselskij iteration scheme in the class of generalized pseudocontractive operators which are also Lipschitzian.

Theorem 6. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a Lipschitzian generalized pseudocontractive operator with corresponding constants $s \geq 1$ and $0 < r < 1$. Then*

   (i) *T has a unique fixed point p in C;*
  (ii) *For any $x_0 \in C$ and $\lambda \in (0,a)$ with 'a' given by (3.6), the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \lambda, T)$ converges strongly to p;*
 (iii) *For any $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying (3.7), the Mann iteration $\{x_n\}_{n=0}^{\infty} = M_n(x_0, \alpha_n, T)$ converges strongly to p;*
 (iv) *For any Mann iteration converging to p, with $0 \leq \alpha_n \leq b < 1$, there exists Krasnoselskij that converges faster to p.*

Proof. For all $\lambda \in [0,1]$, consider the operator T_λ on C given by

$$T_\lambda x = (1 - \lambda)x + \lambda Tx, \ \ x \in C \tag{19}$$

since C is convex, we have $T_\lambda(C) \subset C$, for all $\lambda \in [0,1]$. From the generalized pseudocontractive and Lipschitzian conditions on T and

$$\begin{aligned}
\|T_\lambda x - T_\lambda y\|^2 &= \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2 \\
&= (1-\lambda)^2 \|x-y\|^2 + 2\lambda(1-\lambda)\langle Tx - Ty, \ x - y \rangle + \lambda^2 \|Tx - Ty\|^2 \\
\|T_\lambda x - T_\lambda y\|^2 &\leq \left[(1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2 \right] \|x-y\|^2,
\end{aligned}$$

$$so \ \|T_\lambda x - T_\lambda y\| \leq \theta \|x - y\|, \ \ for \ all \ x, \ y \ in \ C, \tag{20}$$

where $0 < \theta = \left[(1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2 \right]^{1/2}$, *as $\lambda < a$.*

Since C is a closed subset of a Hilbert space, C is a complete metric space. Then by Banach contraction principle, T_λ has a unique fixed point say, q in C and the Picard iteration associated to T_λ,

$$x_{n+1} = T_\lambda x_n, \ \ n \geq 0 \tag{21}$$

converges strongly to q, for any $x_0 \in C$. By using theorem (4.3), $\{x_n\}_{n=0}^{\infty}$ is exactly the Krasnoselskij iteration $K_n(x_0, \lambda, T)$ associated to T and the fact that $F(T) = F(T_\lambda)$, for all $\lambda \in (0,1)$, that is, $p = q$ is the unique fixed point of

T, so we get (i) and (ii).

(iii) Let $\{x_n\}_{n=0}^\infty$ be the Mann iteration with $\{\alpha_n\}_{n=0}^\infty \subset [0,1]$ satisfying (3.7), consider t, $0 < t < 1$, denote $a_n = \frac{1}{t}\alpha_n$, $n = 0, 1, 2, \dots$. Then the Mann iteration will be given by $x_{n+1} = (1 - ta_n)x_n + ta_n T x_n$, $n = 0, 1, 2, \dots$, we have

$$\|x_{n+1} - p\| = \|(1 - a_n)x_n + a_n[(1-t)x_n + tTx_n] - p\|$$

$$\|x_{n+1} - p\| \leq (1 - a_n)\|x_n - p\| + a_n\|(1-t)(x_n - p) + t(Tx_n - Tp)\|. \tag{22}$$

using the properties of T we find that

$$\|t(Tx_n - Tp) + (1-t)(x_n - p)\|^2 = (1-t)^2\|x_n - p\|^2 + 2t(1-t)\langle Tx_n - p, x_n - p\rangle + t^2\|Tx - p\|^2$$
$$\leq (1-t)^2\|x_n - p\|^2 + 2t(1-t)r\|x_n - p\|^2 + t^2 s^2\|x_n - p\|^2$$

$$= \left[(1-t)^2 + 2t(1-t)r + t^2 s^2\right]\|x_n - p\|^2 \tag{23}$$

By (4.6) and (4.7) we get

$$\|x_{n+1} - p\| \leq \left\{1 - a_n + a_n\left[(1-t)^2 + 2t(1-t)r + t^2 s^2\right]^{1/2}\right\}\|x_n - p\|$$
$$= (1 - (1-\theta)a_n)\|x_n - p\|$$

$$\leq \prod_{k=1}^n (1 - (1-\theta)a_k)\|x_1 - p\| \tag{24}$$

where $0 \leq \theta = \left[(1-t)^2 + 2t(1-t)r + t^2 s^2\right]^{1/2} < 1$, for all t such that $0 < t < 2(1-r)/(1-2r+s^2)$.

Since by (3.7), $\sum_{n=0}^\infty \alpha_n$ diverges, it follows that $\sum_{n=0}^\infty a_n$ diverges too, and in view of $\theta < 1$ we get

$$lim_{n\to\infty} \prod_{k=1}^n (1 - (1-\theta)a_k) = 0,$$

so by (4.8) shows that $\{x_n\}$ converges strongly to p.

(iv) Take $x = x_n$, $y = x_{n-1}$ in (4.4) we obtain $\|x_{n+1} - x_n\| \leq \theta\|x_n - x_{n-1}\|$. By induction, which yields

$$\|x_{n+1} - x_n\| \leq \theta^n\|x_1 - x_0\|$$

and hence by triangle rule we get

$$\|x_{n+p} - x_n\| \leq \theta^n(1 + \theta + \dots + \theta^{p-1})\|x_1 - x_0\| \tag{25}$$

valid for all n, $p \in \mathbb{N}$. Now letting $p \to \infty$ in (4.9) and by using part (ii), we have

$$\|x_n - x\| = \frac{\theta^n}{1-\theta}\|x_1 - x_0\|. \tag{26}$$

therefore, in view of (4.8) and (4.10), in order to compare Krasnoselskij and Mann iterations, we have to compare

$$\theta^n \ and \ \prod_{k=1}^{n}\left[1-(1-\theta)a_k\right].$$

Let $\{x_n\}_{n=0}^{\infty}$ be a certain Maan iteration converging to p, with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying $0 \le \alpha_n \le b < 1$. Then $a_k \le b/t$ (denote b/t by b) and for any m, $0 < m < 1$, we find $\theta \in (0,1)$ such that

$$b < \frac{1-(\theta/m)}{1-\theta},$$

clearly, which implies that

$$\theta < \frac{m(1-b)}{1-bm},$$

using the fact that $a_k \le b$, it gives

$$\frac{\theta}{1-(1-\theta)a_k} \le m < 1,$$

which shows that

$$\frac{\theta}{\prod_{k=1}^{n}\left[1-(1-\theta)a_k\right]} \le \lim_{n \to \infty} m^n = 0,$$

so the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty} = K_n(x_0, \ \theta, \ T)$ converges faster than the Mann iteration $\{x_n\}_{n=0}^{\infty} = M_n(x_0, \ \alpha_n, \ T)$.

In order to complete the proof, we still need to show that the intervals $(0,a)$ with a given by (3.6) and $\left(0, \frac{m(1-b)}{1-m}\right)$ have a nonempty intersection. But this is immediate, since $\frac{m(1-b)}{1-m} > 0$ and $0 < a = \frac{2(1-r)}{1-2r+s^2} \le 1$, under the hypothesis of the theorem.

Remark.(i) Part (iv) of theorem (4.6) shows that, in order to approximate the fixed points of a Lipschitzian and pseudocontractive operator T, it is always more convenient to use a certain Krasnoselskij iteration in the family (3.2) with $\lambda \in (0,a)$ and $'a'$ is given by (3.6).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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