New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2019.352

Double Laplace iterative method for solving nonlinear partial differential equations

Ranjit R. Dhunde[1] *and Govind L. Waghmare*[2]

[1]Department of Mathematics, Datta Meghe Institute of Engineering, Technology and Research, Wardha (M.S. - India)
[2]Department of Mathematics, Government Institute of Science, Nagpur (M.S. - India)

Received: 26 May 2018, Accepted: 25 November 2018
Published online: 30 April 2019.

Abstract: In this article, the method based on double Laplace transform in combination with new iterative method is used to solve general nonlinear partial differential equation subject to the initial and boundary conditions. The effectiveness of the method is illustrated with examples of nonlinear dissipative wave equation, KdV equations, nonlinear heat equation and Gas-Dynamic equation.

Keywords: Double Laplace transform, inverse double Laplace transform, new iterative method, nonlinear partial differential equation.

1 Introduction

In physical sciences, we come across linear and nonlinear partial differential equations. Linear partial differential equations can be solved using single and double Laplace transform. Adomian decomposition method [1,2,3,4,5,6,7], variational iteration method [8,9], homotopy perturbation method [10] and reduced differential transform method [11, 12] are used to solve nonlinear partial differential equations. One cannot solve nonlinear partial differential equations using Laplace transform. So in [13,14] Laplace transform is combined with homotopy perturbation method and in [15] with variational iteration method to solve nonlinear partial differential equations.

Eltayeb and Kilicmann [16]; Debnath [17] applied double Laplace transform for solving some linear partial differential equations. In [18,19], nonlinear telegraph and Klein-Gordon equations are solved using double Laplace transform coupled with new iterative method [20]. Recently in 2017, Eltayeb [21] combined double Laplace transform with Adomian decomposition method to solve nonlinear partial differential equations.

We consider a general nonlinear partial differential equation which covers almost all the nonlinear partial differential equations solved in [13,14,15,18,19,21], of the form :

$$\sum_{n=0}^{N} a_n \frac{\partial^n u(x,t)}{\partial t^n} + \sum_{m=1}^{M} b_m \frac{\partial^m u(x,t)}{\partial x^m} + \mathbf{N}u(x,t) = h(x,t), (x,t) \in {\mathbb{R}_+}^2. \tag{1}$$

where $a_n, \quad 0 \le n \le N; b_m, \quad 1 \le m \le M$ are given constant coefficients and N, M are positive integers, $\mathbf{N}u(x,t)$ is nonlinear term and $h(x,t)$ is the source function in the form $h(x,t) = h_1(x,t) + h_2(x,t)$.

* Corresponding author e-mail: ranjitdhunde@rediffmail.com

Associated with (1), we consider the initial conditions

$$\frac{\partial^n u(x,0)}{\partial t^n} = f_n(x), n = 0,1,2,...,N-1, x \in \mathbb{R}_+, \tag{2}$$

and the boundary conditions

$$\frac{\partial^m u(0,t)}{\partial x^m} = g_m(t), m = 0,1,2,...,M-1, t \in \mathbb{R}_+. \tag{3}$$

Further, we assume that the functions $h, f_n, n = 0,1,2,...,N-1$ and $g_m, m = 0,1,2,...,M-1$ are such that problem (1) with initial conditions (2) and boundary conditions (3) having a solution.

In this article, we solve general nonlinear PDE (1) subject to the initial conditions (2) and the boundary conditions (3) using double Laplace iterative method. Nonlinear telegraph and Klein-Gordon equations considered in [18,19] are the particular cases of PDE (1).

2 A brief introduction of double Laplace transforms

Let $f(x,t)$ be a function of two variables x and t defined in the positive quadrant of the xt-plane. The double Laplace transform of the function $f(x,t)$ as given by Ian N. Sneddon [22] is defined by

$$L_x L_t \{f(x,t)\} = \overline{f}(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t) dt dx, \tag{4}$$

whenever that integral exist. Here p and s are complex numbers.

The inverse double Laplace transform $L_x{}^{-1} L_t{}^{-1}\left[\overline{f}(p,s)\right] = f(x,t)$ is defined as in [17] by the complex double integral formula

$$L_x^{-1} L_t^{-1}\left[\overline{f}(p,s)\right] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\overline{f}(p,s) ds, \tag{5}$$

where $\overline{f}(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $Re\ p \geq c$ and $Re\ s \geq d$, where c and d are real constants to be chosen suitably.
The double Laplace transform for the partial derivatives of an arbitrary integer order as in [23] are

$$L_x L_t\left[\frac{\partial^m f(x,t)}{\partial x^m}\right] = p^m \overline{f}(p,s) - \sum_{j=0}^{m-1} p^{m-1-j} L_t\left[\frac{\partial^j f(0,t)}{\partial x^j}\right], \tag{6}$$

$$L_x L_t\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = s^n \overline{f}(p,s) - \sum_{k=0}^{n-1} s^{n-1-k} L_x\left[\frac{\partial^k f(x,0)}{\partial t^k}\right]. \tag{7}$$

3 Double Laplace transform combined with iterative method

Applying the double Laplace transform on both sides of (1), we get

$$\begin{aligned}\sum_{n=0}^{N} a_n[s^n \overline{u}(p,s) - \sum_{k=0}^{n-1} s^{n-k-1} L_x\left[\frac{\partial^k u(x,0)}{\partial t^k}\right]] + \sum_{m=1}^{M} b_m[p^m \overline{u}(p,s) - \sum_{j=0}^{m-1} p^{m-j-1} L_t\left[\frac{\partial^j u(0,t)}{\partial x^j}\right]] + L_x L_t\left[\mathbf{N}u(x,t)\right] \\ = \overline{h_1}(p,s) + L_x L_t\left[h_2 u(x,t)\right].\end{aligned} \tag{8}$$

Further, applying single Laplace transform to the initial conditions (2) and the boundary conditions (3), we get

$$L_x\left[\frac{\partial^n u(x,0)}{\partial t^n}\right]=\overline{f_n}(p), L_t\left[\frac{\partial^m u(0,t)}{\partial x^m}\right]=\overline{g_m}(s),\\ n=0,1,2,...,N-1, and\ m=0,1,2,...,M-1. \tag{9}$$

By substituting (9) in (8), we get

$$\sum_{n=0}^{N}a_n[s^n\overline{u}(p,s)-\sum_{k=0}^{n-1}s^{n-k-1}\overline{f_k}(p)]+\sum_{m=1}^{M}b_m[p^m\overline{u}(p,s)-\sum_{j=0}^{m-1}p^{m-j-1}\overline{g_j}(s)]=\overline{h_1}(p,s)+L_xL_t\left[h_2(x,t)-\mathbf{N}u(x,t)\right]. \tag{10}$$

Simplifying, we obtain

$$\overline{u}(p,s)=[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}[\sum_{n=0}^{N}a_n(\sum_{k=0}^{n-1}s^{n-k-1}\overline{f_k}(p))+\sum_{m=1}^{M}b_m(\sum_{j=0}^{m-1}p^{m-j-1}\overline{g_j}(s))+\overline{h_1}(p,s)]\\ +[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}L_xL_t\left[h_2(x,t)-\mathbf{N}u(x,t)\right]. \tag{11}$$

Applying inverse double Laplace transform to (11), we obtain

$$u(x,t)=L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}[\sum_{n=0}^{N}a_n(\sum_{k=0}^{n-1}s^{n-k-1}\overline{f_k}(p))+\sum_{m=1}^{M}b_m(\sum_{j=0}^{m-1}p^{m-j-1}\overline{g_j}(s))+\overline{h_1}(p,s)]]\\ +L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}L_xL_t\left[h_2(x,t)-\mathbf{N}u(x,t)\right]]. \tag{12}$$

Now we apply the new iterative method

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \tag{13}$$

Substituting (13) in (12), we get

$$\sum_{i=0}^{\infty}u_i(x,t)=L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}[\sum_{n=0}^{N}a_n(\sum_{k=0}^{n-1}s^{n-k-1}\overline{f_k}(p))+\sum_{m=1}^{M}b_m(\sum_{j=0}^{m-1}p^{m-j-1}\overline{g_j}(s))+\overline{h_1}(p,s)]]\\ +L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}L_xL_t\left[h_2(x,t)-\mathbf{N}(\sum_{i=0}^{\infty}u_i(x,t))\right]]. \tag{14}$$

The nonlinear term **N** is decomposed as

$$\mathbf{N}\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=\mathbf{N}(u_0(x,t))+\sum_{i=1}^{\infty}\left[\mathbf{N}\left(\sum_{l=0}^{i}u_k(x,t)\right)-\mathbf{N}\left(\sum_{l=0}^{i-1}u_k(x,t)\right)\right] \tag{15}$$

Substituting (15) in (14), we get

$$\sum_{i=0}^{\infty}u_i(x,t)=L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}[\sum_{n=0}^{N}a_n(\sum_{k=0}^{n-1}s^{n-k-1}\overline{f_k}(p))+\sum_{m=1}^{M}b_m(\sum_{j=0}^{m-1}p^{m-j-1}\overline{g_j}(s))+\overline{h_1}(p,s)]]\\ +L_x{}^{-1}L_t{}^{-1}[[\sum_{n=0}^{N}a_ns^n+\sum_{m=1}^{M}b_mp^m]^{-1}L_xL_t[h_2(x,t)-\mathbf{N}(u_0(x,t))-\sum_{i=1}^{\infty}[\mathbf{N}(\sum_{l=0}^{i}u_k(x,t))-\mathbf{N}(\sum_{l=0}^{i-1}u_k(x,t))]]]. \tag{16}$$

Then we define the recurrence relations as

$$u_0(x,t) = {L_x}^{-1}{L_t}^{-1}[[\sum_{n=0}^{N} a_n s^n + \sum_{m=1}^{M} b_m p^m]^{-1}[\sum_{n=0}^{N} a_n(\sum_{k=0}^{n-1} s^{n-k-1}\overline{f_k}(p)) + \sum_{m=1}^{M} b_m(\sum_{j=0}^{m-1} p^{m-j-1}\overline{g_j}(s)) + \overline{h_1}(p,s)]], \tag{17}$$

$$u_1(x,t) = {L_x}^{-1}{L_t}^{-1}[[\sum_{n=0}^{N} a_n s^n + \sum_{m=1}^{M} b_m p^m]^{-1} L_x L_t[h_2(x,t) - \mathbf{N}(u_0(x,t))]], \tag{18}$$

$$u_{q+1}(x,t) = -{L_x}^{-1}{L_t}^{-1}[[\sum_{n=0}^{N} a_n s^n + \sum_{m=1}^{M} b_m p^m]^{-1} L_x L_t[\mathbf{N}\left(\sum_{l=0}^{q} u_l(x,t)\right) - \mathbf{N}\left(\sum_{l=0}^{q-1} u_l(x,t)\right)]], \quad q \geq 1. \tag{19}$$

Therefore, the solution of (1) in series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots + u_q(x,t) + \ldots \tag{20}$$

4 Applications: nonlinear partial differential equations

In this section, in every example we consider particular cases of PDE 1.

Example 1. Consider the nonlinear Dissipative wave equation similar to [1]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial t}(uu_x) = 2e^{-t}\sin x - 2e^{-2t}\sin x\cos x, \tag{21}$$

with initial conditions

$$u(x,0) = \sin x, u_t(x,0) = -\sin x, \tag{22}$$

and boundary conditions

$$u(0,t) = 0, u_x(0,t) = e^{-t}. \tag{23}$$

Applying the double Laplace transform to (21) with the conditions (22) and (23), we obtain

$$\overline{u}(p,s) = \frac{1}{(s+1)(p^2+1)} - \frac{1}{(s^2-p^2)} L_x L_t[2e^{-2t}\sin x\cos x + \frac{\partial}{\partial t}(uu_x)]. \tag{24}$$

Applying inverse double Laplace transform to (24), we get

$$u(x,t) = e^{-t}\sin x - {L_x}^{-1}{L_t}^{-1}[\frac{1}{(s^2-p^2)} L_x L_t[2e^{-2t}\sin x\cos x + \frac{\partial}{\partial t}(uu_x)]]. \tag{25}$$

Now we apply the new iterative method.

Substituting (13) into (25) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = e^{-t}\sin x, \tag{26}$$

$$u_1(x,t) = -{L_x}^{-1}{L_t}^{-1}[\frac{1}{(s^2-p^2)} L_x L_t[2e^{-2t}\sin x\cos x + \frac{\partial}{\partial t}[(u_0)[(u_0)_x]]] = 0. \tag{27}$$

$$u_2(x,t) = -{L_x}^{-1}{L_t}^{-1}[\frac{1}{(s^2-p^2)} L_x L_t[\frac{\partial}{\partial t}[(u_0+u_1)[(u_0+u_1)_x]] - \frac{\partial}{\partial t}[(u_0)[(u_0)_x]]] = 0, \tag{28}$$

and so on.

Therefore, we obtain the solution of (21) as follows:

$$u(x,t) = e^{-t}\sin x. \tag{29}$$

Example 2. Consider the nonlinear Dissipative wave equation similar to [1]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial t}(u^2) = -2\sin^2 x \sin t \cos t, \tag{30}$$

with initial conditions

$$u(x,0) = \sin x, u_t(x,0) = 0, \tag{31}$$

and boundary conditions

$$u(0,t) = 0, u_x(0,t) = \cos t. \tag{32}$$

Applying the double Laplace transform to (30) with the conditions (31) and (32), we obtain

$$\overline{u}(p,s) = \frac{s}{(s^2+1)(p^2+1)} - \frac{1}{(s^2-p^2)} L_x L_t [2\sin^2 x \sin t \cos t + \frac{\partial}{\partial t}(u^2)]. \tag{33}$$

Applying inverse double Laplace transform to (33), we get

$$u(x,t) = \cos t \sin x - L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2-p^2)} L_x L_t [2\sin^2 x \sin t \cos t + \frac{\partial}{\partial t}(u^2)]]. \tag{34}$$

Now we apply the new iterative method.

Substituting (13) into (34) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = \cos t \sin x, \tag{35}$$

$$u_1(x,t) = -L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2-p^2)} L_x L_t [2\sin^2 x \sin t \cos t + \frac{\partial}{\partial t}(u_0)^2)]] = 0, \tag{36}$$

$$u_2(x,t) = -L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2-p^2)} L_x L_t [\frac{\partial}{\partial t}(u_0+u_1)^2 - \frac{\partial}{\partial t}(u_0)^2]] = 0, \tag{37}$$

and so on.

Therefore, we obtain the solution of (30) as follows:

$$u(x,t) = \cos t \sin x. \tag{38}$$

Example 3. Consider the inhomogeneous KdV equation similar to [5]

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = -e^x(1+t) + te^x(1-te^x), \tag{39}$$

with initial condition

$$u(x,0) = 1, \tag{40}$$

and boundary conditions

$$u(0,t)=1-t,u_x(0,t)=u_{xx}(0,t)=-t. \tag{41}$$

Applying the double Laplace transform to (39) with the conditions (40) and (41), we obtain

$$\overline{u}(p,s)=\frac{1}{ps}-\frac{1}{s^2(p-1)}+\frac{1}{(s+p^3)}L_xL_t[te^x(1-te^x)+u\frac{\partial u}{\partial x}]. \tag{42}$$

Applying inverse double Laplace transform to (42), we get

$$u(x,t)=1-te^x+L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3)}L_xL_t[te^x(1-te^x)+u\frac{\partial u}{\partial x}]]. \tag{43}$$

Now we apply the new iterative method.

Substituting (13) into (43) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t)=1-te^x, \tag{44}$$

$$u_1(x,t)=L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3)}L_xL_t[te^x(1-te^x)+u_0\frac{\partial(u_0)}{\partial x}]]=0, \tag{45}$$

$$u_2(x,t)=L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3)}L_xL_t[(u_0+u_1)\frac{\partial}{\partial x}(u_0+u_1)-u_0\frac{\partial(u_0)}{\partial x}]]=0, \tag{46}$$

and so on.

Therefore, we obtain the solution of (39) as follows:

$$u(x,t)=1-te^x. \tag{47}$$

Example 4. Consider the inhomogeneous fifth order KdV equation similar to [6]

$$\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}=\cos x+2t\sin x+\frac{t^2}{2}\sin 2x, \tag{48}$$

with initial condition

$$u(x,0)=0, \tag{49}$$

and boundary conditions

$$u(0,t)=t,u_x(0,t)=0,u_{xx}(0,t)=-t,u_{xxx}(0,t)=0,u_{xxxx}(0,t)=t. \tag{50}$$

Applying the double Laplace transform to (48) with the conditions (49) and (50), we obtain

$$\overline{u}(p,s)=\frac{p}{(p^2+1)s^2}+\frac{1}{(s+p^3-p^5)}L_xL_t[\frac{t^2}{2}\sin 2x+u\frac{\partial u}{\partial x}]. \tag{51}$$

Applying inverse double Laplace transform to (51), we get

$$u(x,t)=t\cos x+L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3-p^5)}L_xL_t[\frac{t^2}{2}\sin 2x+u\frac{\partial u}{\partial x}]]. \tag{52}$$

Now we apply the new iterative method.

Substituting (13) into (52) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = t\cos x, \tag{53}$$

$$u_1(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3-p^5)}L_xL_t[\frac{t^2}{2}\sin 2x + u_0\frac{\partial(u_0)}{\partial x}]], \tag{54}$$

$$u_2(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s+p^3-p^5))}L_xL_t[(u_0+u_1)\frac{\partial}{\partial x}(u_0+u_1) - u_0\frac{\partial(u_0)}{\partial x}]] = 0, \tag{55}$$

and so on.

Therefore, we obtain the solution of (48) as follows:

$$u(x,t) = t\cos x. \tag{56}$$

Example 5. Consider the nonlinear heat equation in [7]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}), \tag{57}$$

with initial condition

$$u(x,0) = x. \tag{58}$$

Applying the double Laplace transform to (57) with the condition (58), we obtain

$$\overline{u}(p,s) = \frac{1}{p^2 s} + \frac{1}{s}L_xL_t[\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})]. \tag{59}$$

Applying inverse double Laplace transform to (59), we get

$$u(x,t) = x + L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})]]. \tag{60}$$

Now we apply the new iterative method.

Substituting (13) into (60) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = x, \tag{61}$$

$$u_1(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[\frac{\partial}{\partial x}(u_0\frac{\partial u_0}{\partial x})]] = t, \tag{62}$$

$$u_2(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s)}L_xL_t[\frac{\partial}{\partial x}[(u_0+u_1)\frac{\partial(u_0+u_1)}{\partial x}] - \frac{\partial}{\partial x}(u_0\frac{\partial u_0}{\partial x})]] = 0, \tag{63}$$

and so on.

Therefore, we obtain the solution of (57) as follows:

$$u(x,t) = x + t. \tag{64}$$

Example 6. Consider the non-homogeneous advection problem in [13]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2, \tag{65}$$

with initial condition

$$u(x,0) = 0. \tag{66}$$

Applying the double Laplace transform to (65) with the condition (66), we obtain

$$\overline{u}(p,s) = \frac{2}{ps^3} + \frac{1}{p^2s^2} + \frac{1}{s}L_xL_t[t^3 + xt^2 - u\frac{\partial u}{\partial x}]. \tag{67}$$

Applying inverse double Laplace transform to (67), we get

$$u(x,t) = t^2 + xt + L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[t^3 + xt^2 - u\frac{\partial u}{\partial x}]]. \tag{68}$$

Now we apply the new iterative method.

Substituting (13) into (68) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = t^2 + xt, \tag{69}$$

$$u_1(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[t^3 + xt^2 - u_0\frac{\partial u_0}{\partial x}]] = 0, \tag{70}$$

$$u_2(x,t) = -L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[(u_0+u_1)\frac{\partial (u_0+u_1)}{\partial x} - u_0\frac{\partial u_0}{\partial x}]] = 0, \tag{71}$$

and so on.

Therefore, we obtain the solution of (65) as follows:

$$u(x,t) = t^2 + xt. \tag{72}$$

Example 7. Consider the nonlinear partial differential equation in [14]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + (\frac{\partial u}{\partial x})^2 = 2x + t^4, \tag{73}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = a, \tag{74}$$

and boundary conditions

$$u(0,t) = at, u_x(0,t) = t^2. \tag{75}$$

Applying the double Laplace transform to (73) with the conditions (74) and (75), we obtain

$$\overline{u}(p,s) = \frac{a}{ps^2} + \frac{2}{p^2s^3} + \frac{1}{(s^2+p^2)} L_x L_t [t^4 - (\frac{\partial u}{\partial x})^2]. \tag{76}$$

Applying inverse double Laplace transform to (76), we get

$$u(x,t) = at + xt^2 + L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2+p^2)} L_x L_t [t^4 - (\frac{\partial u}{\partial x})^2]]. \tag{77}$$

Now we apply the new iterative method.

Substituting (13) into (77) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = at + xt^2, \tag{78}$$

$$u_1(x,t) = L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2+p^2)} L_x L_t [t^4 - (\frac{\partial u_0}{\partial x})^2]] = 0, \tag{79}$$

$$u_2(x,t) = -L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2+p^2)} L_x L_t [(\frac{\partial (u_0+u_1)}{\partial x})^2 - (\frac{\partial u_0}{\partial x})^2]] = 0, \tag{80}$$

and so on.

Therefore, we obtain the solution of (73) as follows:

$$u(x,t) = at + xt^2. \tag{81}$$

Example 8. Consider the nonlinear partial differential equation in [15]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = x^2 t^2, \tag{82}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = x, \tag{83}$$

and boundary conditions

$$u(0,t) = 0, u_x(0,t) = t. \tag{84}$$

Applying the double Laplace transform to (82) with the conditions (83) and (84), we obtain

$$\overline{u}(p,s) = \frac{1}{p^2s^2} + \frac{1}{(s^2-p^2)} L_x L_t [x^2t^2 - u^2]. \tag{85}$$

Applying inverse double Laplace transform to (85), we get

$$u(x,t) = xt + L_x{}^{-1} L_t{}^{-1} [\frac{1}{(s^2-p^2)} L_x L_t [x^2t^2 - u^2]]. \tag{86}$$

Now we apply the new iterative method.

Substituting (13) into (86) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = xt, \tag{87}$$

$$u_1(x,t) = L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s^2-p^2)}L_xL_t[x^2t^2-(u_0)^2]] = 0, \tag{88}$$

$$u_2(x,t) = -L_x{}^{-1}L_t{}^{-1}[\frac{1}{(s^2-p^2)}L_xL_t[(u_0+u_1)^2-(u_0)^2]] = 0, \tag{89}$$

and so on.

Therefore, we obtain the solution of (82) as follows:

$$u(x,t) = xt. \tag{90}$$

Example 9. Consider the following non-homogeneous nonlinear Gas Dynamic equation in [24]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial(u^2)}{\partial x}-u(1-u) = -e^{t-x}, \tag{91}$$

with initial condition

$$u(x,0) = -e^{-x}. \tag{92}$$

Applying the double Laplace transform to (91) with the condition (92), we obtain

$$\overline{u}(p,s) = \frac{1}{ps}-\frac{1}{(s-1)(p+1)}-\frac{1}{s}L_xL_t[\frac{1}{2}\frac{\partial(u^2)}{\partial x}-u+u^2]. \tag{93}$$

Applying inverse double Laplace transform to (93), we get

$$u(x,t) = 1-e^{t-x}-L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[\frac{1}{2}\frac{\partial(u^2)}{\partial x}-u+u^2]]. \tag{94}$$

Now we apply the new iterative method.

Substituting (13) into (94) and applying (17), (18), (19), we obtain the components of the solution as follows:

$$u_0(x,t) = 1-e^{t-x}, \tag{95}$$

$$u_1(x,t) = -L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[\frac{1}{2}\frac{\partial(u_0)^2}{\partial x}-u_0+(u_0)^2]] = 0, \tag{96}$$

$$u_2(x,t) = -L_x{}^{-1}L_t{}^{-1}[\frac{1}{s}L_xL_t[\frac{1}{2}\frac{\partial(u_0+u_1)^2}{\partial x}-(u_0+u_1)+(u_0+u_1)^2-[\frac{1}{2}\frac{\partial(u_0)^2}{\partial x}-u_0+(u_0)^2]]] = 0, \tag{97}$$

and so on.

Therefore, we obtain the solution of (91) as follows:

$$u(x,t) = 1-e^{t-x}. \tag{98}$$

5 Conclusion

From the illustrative examples nonlinear dissipative wave equation, KdV equations, nonlinear heat equation and Gas-Dynamic equation, it is clear that double Laplace transform combined with new iterative method is one of the best method to solve wide range of nonlinear partial differential equations in Mathematical Physics.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Mathematical and Computer Modelling, vol. 13, no. 7, pp. 17-43, 1990.

[2] G. Adomian, *Nonlinear dissipative wave equations*, Applied Mathematics Letters, vol. 11, no. 3, pp. 125-126, 1998.

[3] D. Kaya, *A new approach to solve a nonlinear wave equation*, Bulletin of the Malaysian Mathematical Society, vol. 21, pp. 95-100, 1998.

[4] D. Kaya and M. Inc, *On the solution of the nonlinear wave equation by the decomposition method*, Bulletin of the Malaysian Mathematical Society, vol. 22, pp. 151-155, 1999.

[5] D. Kaya and M. Aassila, *An application for a generalized KdV equation by the decomposition method*, Physics Letters A, vol. 299, pp. 201-206, 2002.

[6] D. Kaya, *An explicit and numerical solutions of some fifth-order KdV equation by decomposition method*, Applied Mathematics and Computation, vol. 144, pp. 353-363, 2003.

[7] Serdal Pamuk, *Solution of the porous media equation by Adomian's decomposition method*, Physics Letters A, vol. 344, pp. 184-188, 2005.

[8] A. M. Wazwaz, *The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations*, Computer and Mathematics with Applications, vol. 54, pp. 926-932, 2007.

[9] A. M. Wazwaz, *The variational iteration method: A powerful scheme for handling linear and nonlinear diffusion equations*, Computer and Mathematics with Applications, vol. 54, pp. 933-939, 2007.

[10] M. Rafei, D. D. Ganji, *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method*, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 7, no. 3, pp. 321-329, 2006.

[11] Y. Keskin, G. Oturanc, *Reduced differential transform method for solving linear and nonlinear wave equations*, Iranian Journal of Science and Technology, Transaction A, vol. 34, no. A2, 2010.

[12] Y. Keskin, G. Oturanc, *Reduced differential transform method for generalized KdV equations*, Mathematical and Computational Applications, vol. 15, no. 3, pp. 382-393, 2010.

[13] Y. Khan, Q. Wu, *Homotopy perturbation transform method for nonlinear equations using He's polynomials*, Computers and Mathematics with Applications, vol. 61, pp. 1963-1967, 2011.

[14] H. K. Mishra,A. K. Nagar, *He-Laplace method for linear and nonlinear partial differential equations*, Journal of Applied Mathematics, vol. 2012, Article ID 180315, 16 pages, 2012.

[15] Eman M. A. Hilal,Tarig M. Elzaki, *Solution of nonlinear partial differential equations by new Laplace variational iteration method*, Journal of Function Spaces, vol. 2014, Article ID 790714, 5 pages, 2014.

[16] H. Eltayeb and A. Kilicman, *A note on double Laplace transform and telegraphic equations*, Abstract and Applied Analysis, volume 2013, Article ID 932578, 6 pages, 2013.

[17] L. Debnath, *The double Laplace transforms and their properties with applications to functional, integral and partial differential equations*, International Journal of Applied and Computational Mathematics, vol. 2, no. 2, pp. 223-241, 2016.

[18] Ranjit R. Dhunde, G. L. Waghmare, *Double Laplace transform combined with iterative method for solving non-linear telegraph equation*, Journal of the Indian Mathematical Society, volume 83, nos. 3-4, pp. 221-230, 2016.

[19] Ranjit R. Dhunde, G. L. Waghmare, *Analytical solution of the nonlinear Klein-Gordon equation using double Laplace transform and iterative method*, American Journal of Computational and Applied Mathematics 2016, 6(6); 195-201.

[20] Varsha Daftardar-Gejji, Hossein Jafari, *An iterative method for solving nonlinear functional equations*, Journal of Mathematical Analysis and Applications, vol. 316, pp. 753-763, 2006.

[21] Hassan Eltayeb, *A note on double Laplace decomposition method and nonlinear partial differential equations*, New Trends in Mathematical Sciences, vol. 5, no. 4, 156-164, 2017.

[22] Ian N. Sneddon, *The use of integral transforms*, Tata Mcgraw Hill Edition, 1974.

[23] A. Anwar, F. Jarad, D. Baleanu and F. Ayaz, *Fractional caputo heat equation within the double Laplace transform*, Romanian Journal Physics, volume 58, no. 1-2, pp.15-22, 2013.

[24] Y. Liu, F. Xu and X. Yin, *Variational approximate solutions of fractional nonlinear nonhomogeneous equations with Laplace transform*, Abstract and Applied Analysis, vol. 2013, Article ID 819268, 9 pages, 2013.