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Diagonally implicit two derivative runge Kutta methods for solving first order initial value problems

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Abstract: Three Diagonally Implicit Two Derivative Runge-Kutta (DITDRK) methods for the numerical solution of first order Initial Value Problems (IVPs) are derived. We present fourth, fifth and sixth-order Diagonally Implicit Two Derivative Runge-Kutta methods designed with minimum number of function evaluations. The stability of the method derived are analyzed. The numerical experiments are carried out to show the efficiency of the derived methods in comparison with other existing Runge-Kutta (RK) methods of the same order and properties.

Keywords: Diagonally implicit methods, IVPs, ODEs, TDRK methods.

1 Introduction

Consider the numerical solution of the Initial Value Problems (IVPs) for first order Ordinary Differential Equations (ODEs) in the form of

$$y' = f(x,y), \qquad\qquad y(x_0) = y_0. \tag{1}$$

A numerous number of researchers have proposed several efficient Diagonally Implicit Runge-Kutta (DIRK) methods and Two Derivative Runge-Kutta (TDRK) methods with constant step-size in the derivation of their methods. In the evolution of TDRK methods, Chan and Tsai[1] introduced special explicit TDRK methods by including the second derivative which involves one evaluation of f and a few evaluations of g per step with stages up to five and of order up to seven as well as some embedded pairs. Chan et al.[2] then presented their study related to stiff ODEs problems on explicit and implicit TDRK methods and extend the applications of the TDRK methods to various Partial Differential Equations (PDEs). Zhang et al.[3] developed a new Trigonometrically Fitted TDRK method of algebraic order five, analyze the linear stability and phase properties of the new method. Chen et al.[4] constructed three practical exponentially fitted TDRK (EFTDRK) methods where the numerical experiments show the efficiency and accuracy of the developed methods compared to their prototype TDRK methods or RK methods of the same order and the traditional exponentially fitted RK method in the literature. In the previous year, Yakubu and Kwami[5] introduced a new class of implicit TDRK collocation methods especially for the numerical solution of systems of equations and their implementation in an efficient parallel computing environment. Meanwhile, Houwen and Sommeijer [6] derived homogeneous dispersion relations for the special class of Diagonally Implicit Runge-Kutta (DIRK) methods and a few high-order dispersive DIRK methods. Franco and Gómez [7] then developed fourth-order symmetric DIRK methods with four and five stages which have high order of dispersion

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(up to order six). In 2009, Ababneh et al. [8] introduced a new fifth-order DIRK integration formula for stiff initial value problems, designed to be L-stable method. A few years later, Jawias et al. [9] developed fourth-order fourth stage DIRK methods for linear ordinary differential equations and the stability aspect of the method is investigated. Yazdi and Mongeau[10] introduced a fourth-order implicit RK scheme with low-dispersion and low-dissipation property. Hence, in this paper, three DITDRK methods of fourth, fifth and sixth-order are constructed. In Section 2, an overview of DITDRK method is given. The three DITDRK methods are derived as welll as the stability of the methods derived are analyzed in Section 3. The numerical results, discussion and conclusion are dealt in Section 4, Section 5 and Section 6 respectively.

2 Diagonally implicit two derivative Runge-Kutta method

A TDRK method for the numerical integration of IVPs (1) is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + h^2 \sum_{j=1}^{s} \hat{a}_{ij} g(Y_j), \tag{2}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i) + h^2 \sum_{i=1}^{s} \hat{b}_i g(Y_i), \tag{3}$$

where $i = 1,\ldots,s$.
The TDRK parameters $a_{ij}, \hat{a}_{ij}, b_i, \hat{b}_i$ and c_i are assumed to be real and s is the number of stages of the method. The s-dimensional vectors $b, \hat{b}, c$ and $s\times s$ matrix, A and $\hat{A}$ are introduced where $b = [b_1, b_2, \ldots, b_s]^T, \hat{b} = \left[\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_s\right]^T, c = [c_1, c_2, \ldots, c_s]^T, A = [a_{ij}]$ and $\hat{A} = [\hat{a}_{ij}]$ respectively.

The TDRK method with the coefficients in 2 and 3 are presented using the Butcher table as follows,

$$\begin{array}{c|c||c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array}$$

Diagonally implicit methods with a minimal number of function evaluations can be developed by considering the methods in the form

$$Y_i = y_n + hc_i f(x_n, y_n) + h^2 \sum_{j=1}^{i} \hat{a}_{ij} g(x_n + hc_j, Y_j), \tag{4}$$

$$y_{n+1} = y_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} \hat{b}_i g(x_n + hc_i, Y_i), \tag{5}$$

where $i = 1,\ldots,s$.
The above method is denoted as special DITDRK method. The unique part of this method is that it involves only one evaluation of f and many evaluation of g per step compared to many evaluation of f per step in traditional RK methods. Its Butcher tableau is given as follows,

$$\begin{array}{c||c} c & \hat{A} \\ \hline & \hat{b}^T \end{array}$$

The order conditions for special DITDRK methods are given in Table 1.

Order	Conditions				
1	$b^T e = 1$				
2	$\hat{b}^T e = \frac{1}{2}$				
3	$\hat{b}^T c = \frac{1}{6}$				
4	$\hat{b}^T c^2 = \frac{1}{12}$				
5	$\hat{b}^T c^3 = \frac{1}{20}$	$\hat{b}^T \hat{A}c = \frac{1}{120}$			
6	$\hat{b}^T c^4 = \frac{1}{30}$	$\hat{b}^T c\hat{A}c = \frac{1}{180}$	$\hat{b}^T \hat{A}c^2 = \frac{1}{360}$		
7	$\hat{b}^T c^5 = \frac{1}{42}$	$\hat{b}^T c^2\hat{A}c = \frac{1}{252}$	$\hat{b}^T c\hat{A}c^2 = \frac{1}{504}$	$\hat{b}^T \hat{A}c^3 = \frac{1}{840}$	$\hat{b}^T \hat{A}^2 c = \frac{1}{5040}$

Table 1: Order conditions for special DITDRK methods.

Meanwhile, the comparison of total number of order conditions between DIRK and DITDRK methods are given in Table 2.

Order	DIRK	DITDRK
1	1	-
2	2	1
3	4	2
4	8	3
5	17	5
6	37	8

Table 2: Comparison of Total Number of Order Conditions between DIRK and DITDRK methods

2.1 Stability Analysis of DITDRK Method

The stability function of TDRK method is given as follows,

$$R(z) = 1 + zb^T(I - zA - z^2\hat{A})^{-1}e + z^2\hat{b}^T(I - zA - z^2\hat{A})^{-1}e. \tag{6}$$

Meanwhile for special implicit TDRK method, we consider the following test equation

$$y' = \lambda y, \qquad y'' = \lambda^2 y, \qquad \lambda > 0. \tag{7}$$

Applying equation 7 to equation 4 and 5 produces the difference equation

$$y_{n+1} = H(z)y_n, \qquad z = iv, \qquad v = \lambda h, \tag{8}$$

where

$$H(z) = (1 + z^2\hat{b}(I - z^2\hat{A})^{-1}e) + (z + z^3\hat{b}(I - z^2\hat{A})^{-1}c) \tag{9}$$

is the stability polynomial of the DITDRK method.

3 Derivation of DITDRK methods

In this section, we will derive the DITDRK methods of order four, five and six. For DITDRK methods, the following simplifying assumption is imposed:

$$\sum_{i=1}^{s} \hat{a}_{ij} = \frac{1}{2}{c_i}^2, \text{ for } i = 1, ..., s. \tag{10}$$

The order conditions given in Table 1. as well as the simplifying assumption 10 need to be satisfied in order for a method to be a DITDRK method. In this paper, two stages fourth-order, three stages fifth-order and four stages sixth-order DITDRK methods are considered.

3.1 Two stages fourth-order DITDRK method

We consider a two stage DITDRK method given by the following Butcher table,

$$\begin{array}{c||cc} c_1 & \hat{a}_{11} & \\ c_2 & \hat{a}_{21} & \hat{a}_{22} \\ \hline & \hat{b}_1 & \hat{b}_2 \end{array}$$

Table 3: Butcher Table for Two Stages DITDRK Method

According to the order conditions inTable 1, we have

$$\hat{b}_2 + \hat{b}_3 - \frac{1}{2} = 0, \tag{11}$$

$$\hat{b}_2 c_2 + \hat{b}_3 c_3 - \frac{1}{6} = 0, \tag{12}$$

$$\hat{b}_2 {c_2}^2 + \hat{b}_3 {c_3}^2 - \frac{1}{12} = 0. \tag{13}$$

Solving equation 11, 13, we obtain $\hat{b}_1, \hat{b}_2$ and c_2 in term of c_1

$$\hat{b}_1 = \frac{1}{(36{c_1}^2 - 24c_1 + 6)}, \tag{14}$$

$$\hat{b}_2 = \frac{1}{3}\left(\frac{9{c_1}^2 - 6c_1 + 1}{6{c_1}^2 - 4c_1 + 1}\right), \tag{15}$$

$$c_2 = \frac{1}{2}\left(\frac{2c_1 - 1}{3c_1 - 1}\right). \tag{16}$$

Our aim is to choose c_1 such that the principal local truncation error coefficient, $\left\|\tau^{(5)}\right\|_2$ have a very small value. Wrong choices of c_1 may cause a huge global error difference. By plotting the graph of $\left\|\tau^{(5)}\right\|_2$ against c_1, a small value of c_1 is chosen in the range of $[0.0, 1.0]$ and hence, the value of c_1 lies between $[0.1, 0.3]$. We choose $c_1 = \frac{1}{5}$ for an optimized pair. All the coefficients are showed in the following Butcher tableau and it is denoted as DITDRK(2,4).

$$
\begin{array}{c||cc}
\frac{1}{5} & \frac{1}{50} & \\
\frac{3}{4} & \frac{209}{800} & \frac{1}{50} \\
\hline
 & \frac{25}{66} & \frac{4}{33}
\end{array}
$$

Table 4: Butcher table for DITDRK(2,4) Method

With the norms of the principal local truncation error of

$$\left\| \tau^{(5)} \right\|_2 = 4.374801584 \times 10^{-3} \tag{17}$$

where the stability polynomial is

$$H(v) = \frac{1}{(v^2-50)^2}\left(\frac{17}{2}v^5 + \frac{331}{6}v^4 + \frac{950}{3}v^3 + 1150\,v^2 + 2500\,v + 2500\right). \tag{18}$$

The stability region of the DITDRK(2,4) method is plotted in Figure 1 with the stability interval of the method derived is $v \in (-3.347, 0.000)$.

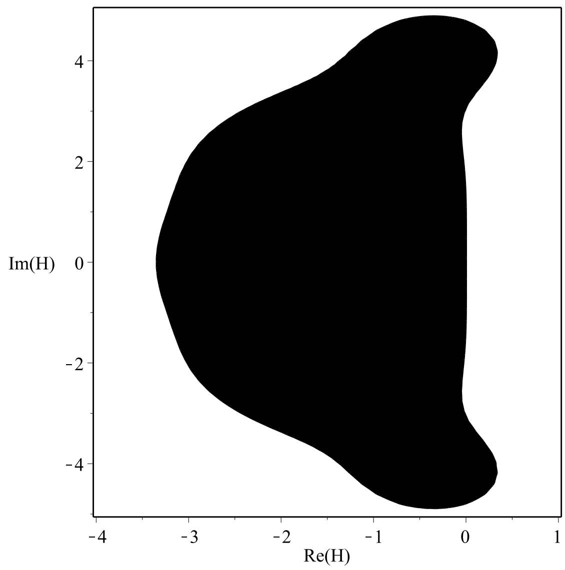

Fig. 1: Stability region of DITDRK(2,4) method

3.2 Three stages fifth-order DITDRK method

We consider a three stages DITDRK method given by the following Butcher table,

$$
\begin{array}{c||ccc}
c_1 & \hat{a}_{11} & & \\
c_2 & \hat{a}_{21} & \hat{a}_{22} & \\
c_3 & \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \\
\hline
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3
\end{array}
$$

Table 5: Butcher Table for Two Stages DITDRK Method

For simplicity, we let $\hat{b}_1 = 0$. According to the order conditions in Table 1, we have

$$\hat{b}_2 + \hat{b}_3 - \frac{1}{2} = 0, \tag{19}$$

$$\hat{b}_2 c_2 + \hat{b}_3 c_3 - \frac{1}{6} = 0, \tag{20}$$

$$\hat{b}_2 {c_2}^2 + \hat{b}_3 {c_3}^2 - \frac{1}{12} = 0, \tag{21}$$

$$\hat{b}_2 {c_2}^3 + \hat{b}_3 {c_3}^3 - \frac{1}{20} = 0. \tag{22}$$

Solving equation 19, 22, we obtain $\hat{a}_{32}, \hat{b}_2, \hat{b}_3, c_2$ and c_3 in term of c_1

$$\begin{aligned}
\hat{a}_{32} = &\left(\frac{-5.4957550765359254872}{120\,c_1 - 18.606123086601862822}\right)(-5.0000000000000000001\,c_1 \\
&-9.9999999999999999999\,{c_1}^2 + 30\,{c_1}^3 + 1), \qquad (23)\\
\hat{b}_2 =&0.31804138174397716939, \qquad (24)\\
\hat{b}_3 =&0.18195861825602283060, \qquad (25)\\
c_2 =&0.15505102572168219018, \qquad (26)\\
c_3 =&0.64494897427831780983. \qquad (27)
\end{aligned}$$

Our aim is to choose c_1 such that the principal local truncation error coefficient, $\left\|\tau^{(6)}\right\|_2$ have a very small value. Wrong choices of c_1 may cause a huge global error difference. By plotting the graph of $\left\|\tau^{(6)}\right\|_2$ against c_1, a small value of c_1 is chosen in the range of $[0.0, 1.0]$ and hence, the value of c_1 lies between $[0.2, 0.4]$. We choose $c_1 = \frac{1}{3}$ for an optimized pair. All the coefficients are listed below and it is denoted as DITDRK(3,5).

$\frac{1}{3}$	$\frac{1}{18}$		
0.15505102572168219018	$-0.04353514526688267948$4	$\frac{1}{18}$	
0.64494897427831780983	$-0.01883228990936789538$6	0.17125632406513946377	$\frac{1}{18}$
	0	0.31804138174397716939	0.18195861825602283060

Table 6: Butcher table for two stages DITDRK method

$$
\begin{aligned}
\hat{b}_1 &= 0, & c_3 &= 0.64494897427831780983,\\
\hat{b}_2 &= 0.31804138174397716939, & \hat{a}_{11} &= \hat{a}_{22} = \hat{a}_{33} = \tfrac{1}{18},\\
\hat{b}_3 &= 0.18195861825602283060, & \hat{a}_{21} &= -0.043535145266882679484,\\
c_1 &= \tfrac{1}{3}, & \hat{a}_{31} &= -0.018832289909367895386,\\
c_2 &= 0.15505102572168219018, & \hat{a}_{32} &= 0.17125632406513946377.
\end{aligned}
$$

With the norms of the principal local truncation error of

$$\left\|\tau^{(6)}\right\|_2 = 1.9460734978232808834 \times 10^{-3} \tag{28}$$

where the stability polynomial is

$$
\begin{aligned}
H(v) = &\frac{1}{(v^2-18)^3}\Big(0.3372755410\,v^7 + 4.411826624\,v^6 + 59.40000000\,v^5\\
&+188.9999999\,v^4 - 1944.0\,v^2 - 5832\,v - 5832\Big).
\end{aligned} \tag{29}
$$

The stability region of the DITDRK(3,5) method is plotted in Figure 2 with the stability interval of the method derived is $v \in (-2.666, 0.000)$.

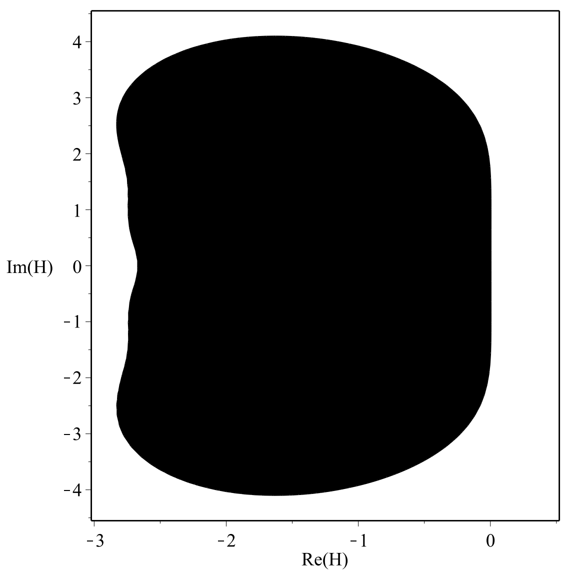

Fig. 2: Stability region of DITDRK(3,5) method

3.3 Four stages sixth-order DITDRK method

We consider a four stages DITDRK method given by the following Butcher table, For simplicity, we let $\hat{a}_{32} = \hat{a}_{42} = 0$.

$$
\begin{array}{c||cccc}
c_1 & \hat{a}_{11} & & & \\
c_2 & \hat{a}_{21} & \hat{a}_{22} & & \\
c_3 & \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & \\
c_4 & \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & \hat{a}_{44} \\
\hline
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4
\end{array}
$$

Table 7: Butcher table for two stages DITDRK method

According to the order conditions in Table 1, we have

$$\hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 - \frac{1}{2} = 0, \tag{30}$$

$$\hat{b}_1 c_1 + \hat{b}_2 c_2 + \hat{b}_3 c_3 + \hat{b}_4 c_4 - \frac{1}{6} = 0, \tag{31}$$

$$\hat{b}_1 {c_1}^2 + \hat{b}_2 {c_2}^2 + \hat{b}_3 {c_3}^2 + \hat{b}_4 {c_4}^2 - \frac{1}{12} = 0, \tag{32}$$

$$\hat{b}_1 {c_1}^3 + \hat{b}_2 {c_2}^3 + \hat{b}_3 {c_3}^3 + \hat{b}_4 {c_4}^3 - \frac{1}{20} = 0, \tag{33}$$

$$\frac{1}{2}\hat{b}_1 {c_1}^3 + \frac{1}{2}\hat{b}_2 c_1 {c_2}^2 - \frac{1}{2}\hat{b}_2 {c_1}^3 + \frac{1}{2}\hat{b}_2 {c_1}^2 c_2 + \frac{1}{2}\hat{b}_3 c_1 {c_3}^2 - \frac{1}{2}\hat{b}_3 {c_1}^3 + \frac{1}{2}\hat{b}_3 {c_1}^2 c_3 + \frac{1}{2}\hat{b}_4 c_1 {c_4}^2 - \hat{b}_4 c_1 a_{4,3} - \frac{1}{2}\hat{b}_4 {c_1}^3 + \hat{b}_4 a_{4,3} c_3 \frac{1}{2}\hat{b}_4 {c_1}^2 c_4 \frac{1}{120} = 0. \tag{34}$$

Solving equation 30, 34, we obtain $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4, \hat{a}_{43}, c_2, c_3$ and c_4 in term of c_1.
Our aim is to choose c_1 such that the principal local truncation error coefficient, $\left\|\tau^{(7)}\right\|_2$ have a very small value. Wrong choices of c_1 may cause a huge global error difference. By plotting the graph of $\left\|\tau^{(7)}\right\|_2$ against c_1, a small value of c_1 is chosen in the range of $[0.0, 1.0]$ and hence, the value of c_1 lies between $[0.3, 0.5]$. We choose $c_1 = 0.04$ for an optimized pair. All the coefficients are listed below and it is denoted as DITDRK(4,6).

$\hat{b}_1 = 0.13130544171070143149,$ $c_3 = 0.36387079261672095548,$

$\hat{b}_2 = -0.21901457455909206227,$ $c_4 = 0.68621064060803474484,$

$\hat{b}_3 = 0.39071842080786578693,$ $\hat{a}_{11} = \hat{a}_{22} = \hat{a}_{33} = \hat{a}_{44} = 0.0008,$

$\hat{b}_4 = 0.19699071204052484377,$ $\hat{a}_{21} = 0.13930167526933376369,$

$c_1 = 0.04,$ $\hat{a}_{31} = 0.065400976859760374705,$

$c_2 = 0.52934237553654017600,$ $\hat{a}_{32} = \hat{a}_{42} = 0,$

$\hat{a}_{41} = 0.13198765087240971901,$ $\hat{a}_{43} = 0.10265487076943499257.$

With the norms of the principal local truncation error of

$$\left\|\tau^{(7)}\right\|_2 = 9.1170660663350809855 \times 10^{-4} \tag{35}$$

where the stability polynomial is

$$\begin{aligned} H(v) = & \frac{1}{(v^2-1250)^4}\left(-90718.44443\, v^9 - 2519235.112\, v^8 + 97899513.92\, v^7\right. \\ & +3070003682.0\, v^6 + 19052343750.0\, v^5 + 97828385430.0\, v^4 + \\ & 399088541400.0\, v^3 + 1212890625000.0\, v^2 + 2441406250000.0\, v + \\ & \left. 2441406250000.0\right). \end{aligned} \tag{36}$$

The stability region of the DITDRK(4,6) method is plotted in Figure 3 with the stability interval of the method derived is $v \in (-3.860, 0.000)$.

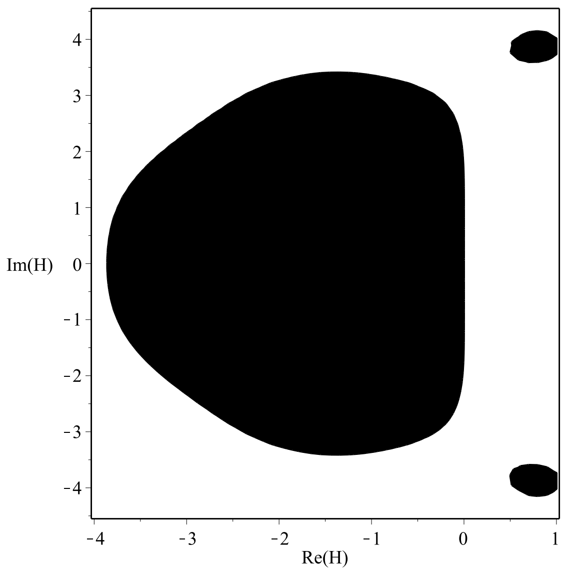

Fig. 3: Stability region of DITDRK(4,6) method

4 Problems tested and numerical results

In this section, the performance of the proposed methods are compared with existing RK methods by considering the following problems. All problems below are tested using C code for solving first order ODEs.

,

Problem 1.(Inhomogeneous problem, Vyver[11]) Vyver [11] problem 2

$$\begin{aligned} y_1{}' &= y_2, & y_1(0) &= 1, & x &\in [0,10], \\ y_2{}' &= -100y_1 + 99\sin(x), & y_2(0) &= 11. \end{aligned}$$

Exact solution is

$$y_1(x) = \cos(10x) + \sin(10x) + \sin(x), \qquad y_2(x) = -10\sin(x) + 10\cos(10x) + \cos(x).$$

Problem 2. (Jawias et al.[9]) problem 20

$$y' = y - x^2 + 1, \qquad y(0) = 0.5, \qquad x \in [0, 10].$$

Exact solution is $y(x) = (x+1)^2 - 0.5e^x$.

$$y(x) = (x+1)^2 - 0.5e^x.$$

Problem 3. (An "almost" Periodic Orbit problem, Stiefel and Bettis[13]) problem 3

$$\begin{aligned}
y_1' &= y_2, & y_1(0) &= 1, & x &\in [0, 10],\\
y_2' &= -y_1 + 0.001\cos(x), & y_2(0) &= 1,\\
y_3' &= y_4, & y_3(0) &= 0,\\
y_4' &= -y_3 + 0.001\sin(x), & y_4(0) &= 0.995.
\end{aligned}$$

Exact solution is

$$\begin{aligned}
y_1(t) &= \cos(x) + 0.0005x\sin(x), & y_2(x) &= -\sin(x) + 0.0005x\cos(x) + 0.0005x\sin(x),\\
y_3(t) &= \sin(x) - 0.0005x\cos(x), & y_4(x) &= \cos(x) + 0.0005x\sin(x) - 0.0005\cos(x).
\end{aligned}$$

Problem 4. (Prothero-Robinson problem, Chan and Tsai[1])

$$y' = \lambda(y - \varphi) + \varphi', \qquad y(0) = \varphi(0), \qquad Re(\lambda) < 0, \qquad x \in [0, 10],$$

where $\varphi(x)$ is a smooth function. We take $\lambda = -1$ and $\varphi(x) = \sin(x)$.
Exact solution is $y(x) = \varphi(x)$.

Problem 5. (Senu [14]) problem 6

$$\begin{aligned}
y_1' &= y_2, & y_1(0) &= 1.1, & x &\in [0, 10],\\
y_2' &= -16y_1 + 116e^{-10x}, & y_2(0) &= -10,\\
y_3' &= y_4, & y_3(0) &= 1,\\
y_4' &= -16y_3 + 116e^{-10x}, & y_4(0) &= -9.6.
\end{aligned}$$

Exact solution is

$$y_1(x) = 0.1\cos(4x) + e^{-10x}, \qquad y_2(x) = -0.4\sin(4x) - 10e^{-10x},$$
$$y_3(x) = 0.1\sin(4x) + e^{-10x}, \qquad y_4(x) = 0.4\cos(4x) - 10e^{-10x}.$$

Problem 6. (Ismail and Salih[15]) problem 41

$$y' = 15 - 3y, \qquad y(0) = 0, \qquad x \in [0, 10].$$

Exact solution is $y(x) = 5(1 - e^{-3x})$.

$$y(x) = 5(1 - e^{-3x}).$$

The following notations are used in Figures 4,21,

- **–DITDRK(2,4):** New DITDRK method of fourth-order two stages derived in this paper
- **–DIRKL(3,4):** Existing fourth-order three stages DIRK method. (Lambert[16])
- **–DIRKJ(4,4)**: Existing fourth-order four stages DIRK method. (Jawias et al. [9])
- **–DIRKF(4,4)**: Existing fourth-order four stages DIRK method. (Franco and Gomez [7])
- **–DIRKS(3,4)**: Existing fourth-order three stages DIRK method. (Sanz-Serna and Abia [17])
- **–DIRKK(5,4)**: Existing fourth-order five stages DIRK method. (Kalogiratou and Monavasilis [18])
- **–DITDRK(3,5):** New DITDRK method of fifth-order three stages derived in this paper
- **–DIRKK(6,5):** Existing fifth-order six stages DIRK method. (Kalogiratou and Monavasilis [18])
- **–DIRKK(7,5)**: Existing fifth-order seven stages DIRK method. (Kalogiratou and Monavasilis [18])
- **–DIRKKD(5,5)**: Existing fifth-order five stages DIRK method. (Kennedy and Carpenter [19])
- **–DIRKA(5,5)**: Existing fifth-order five stages DIRK method. (Ababneh et al.[8])
- **–DITDRK(4,6):** New DITDRK method of sixth-order four stages derived in this paper
- **–DIRKN(6,6)**: Existing sixth-order six stages DIRK method. (Cooper and Sayfy[20])

The performance of these numerical results are represented graphically in the following Figures 4,21:

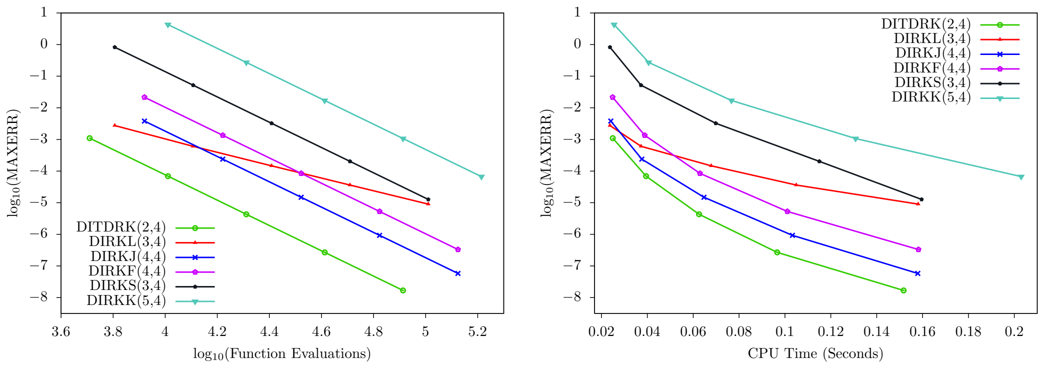

Fig. 4: The efficiency curve for Inhomogeneous problem(Problem 1) for DITDRK(2,4) method with $h = 1.0/2^i, i = 6, \ldots, 11$.

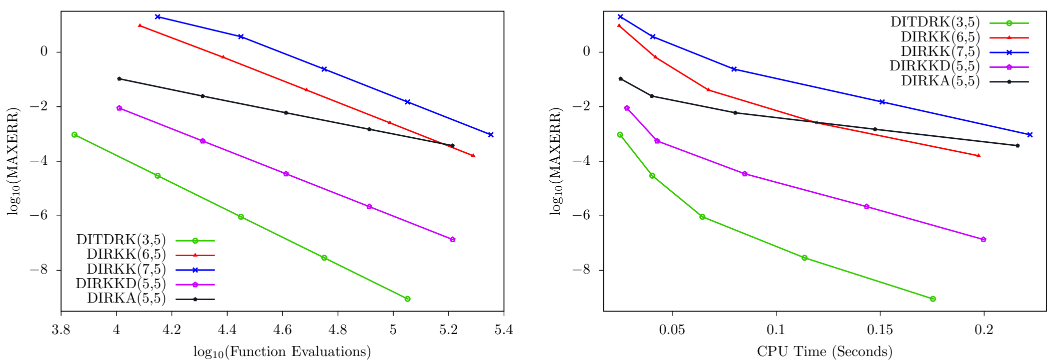

Fig. 5: The efficiency curve for Inhomogeneous problem (Problem 1) for DITDRK(3,5) method with $h = 1.0/2^i, i = 6, \ldots, 11$.

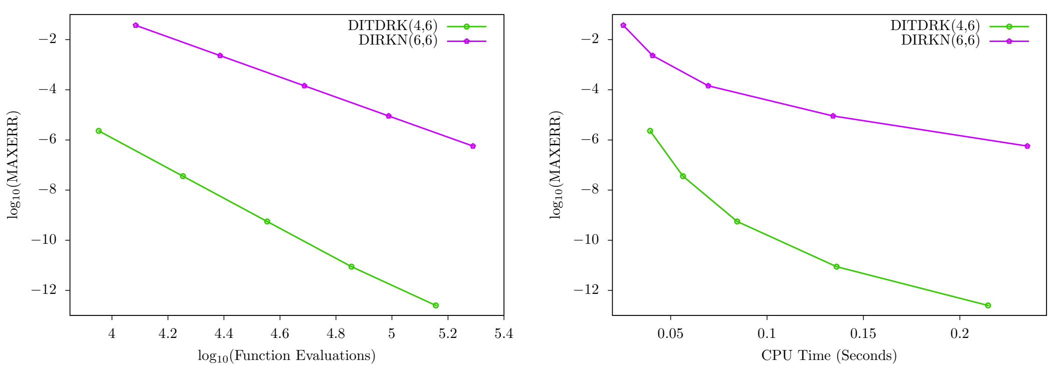

Fig. 6: The efficiency curve for Inhomogeneous problem (Problem 1) for DITDRK(4,6) method with $h = 1.0/2^i, i = 6, \ldots, 11$.

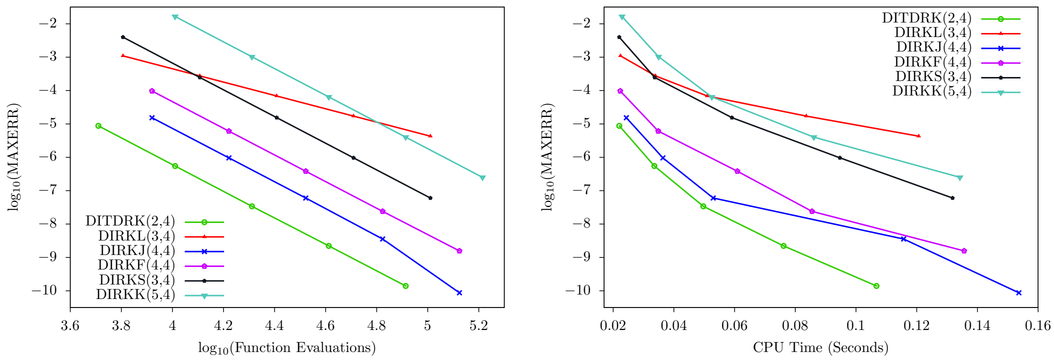

Fig. 7: The efficiency curve for Inhomogeneous problem Problem 2 for DITDRK(2,4) method with $h = 1.0/2^i, i = 6,\ldots,11$.

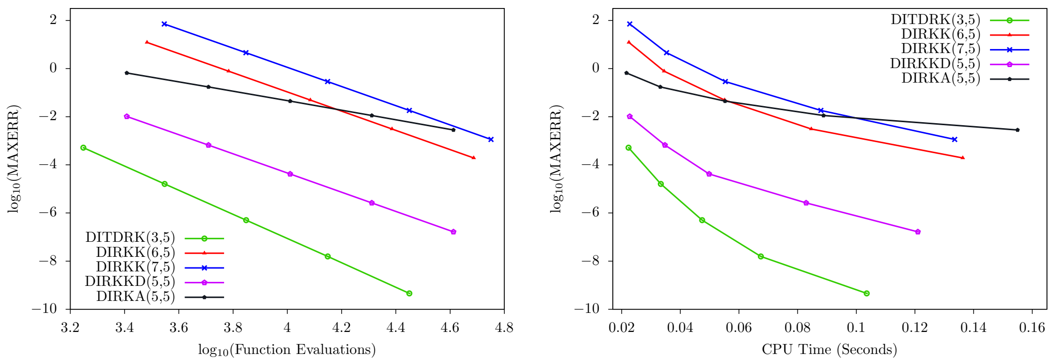

Fig. 8: The efficiency curve for Inhomogeneous problem (Problem 2) for DITDRK(3,5) method with $h = 1.0/2^i, i = 4,\ldots,8$.

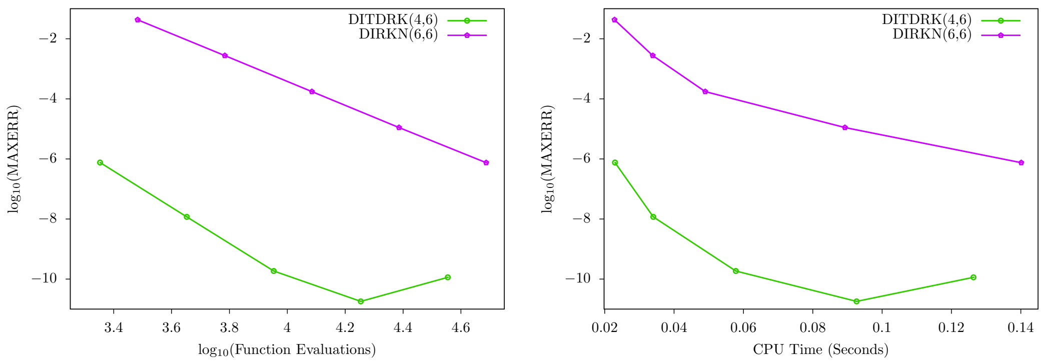

Fig. 9: The efficiency curve for Inhomogeneous problem (Problem 2) for DITDRK(4,6) method with $h = 1.0/2^i, i = 4,\ldots,8$.

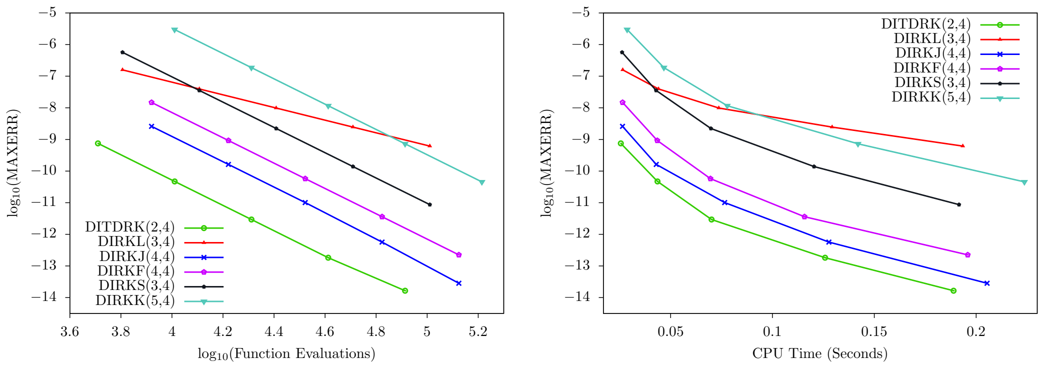

Fig. 10: The efficiency curve for Inhomogeneous problem (Problem 3) for DITDRK(2,4) method with $h = 1.0/2^i, i = 6,\ldots,11$.

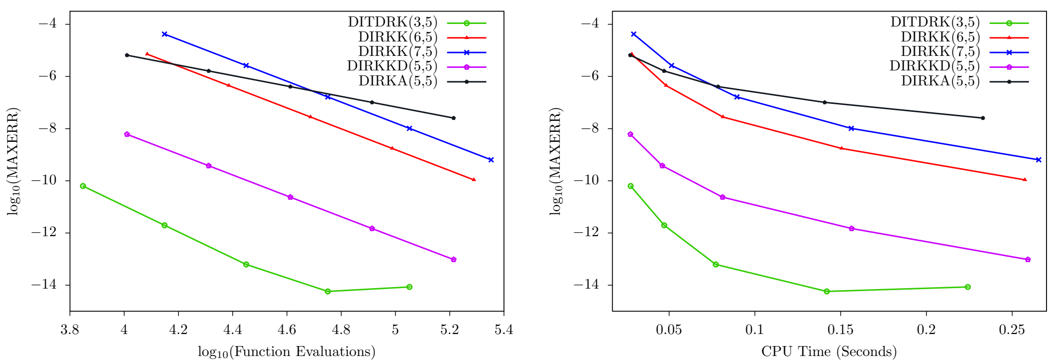

Fig. 11: The efficiency curve for Inhomogeneous problem (Problem 3) for DITDRK(3,5) method with $h = 1.0/2^i, i = 6,\ldots,11$.

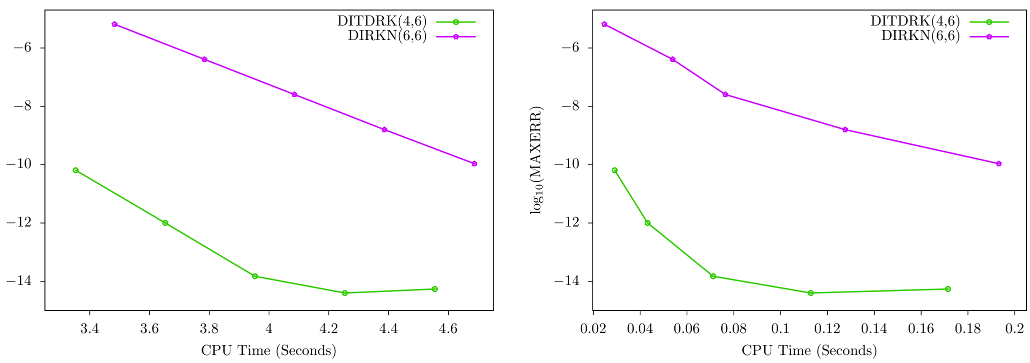

Fig. 12: The efficiency curve for Inhomogeneous problem (Problem 3) for DITDRK(4,6) method with $h = 1.0/2^i, i = 4,\ldots,8$.

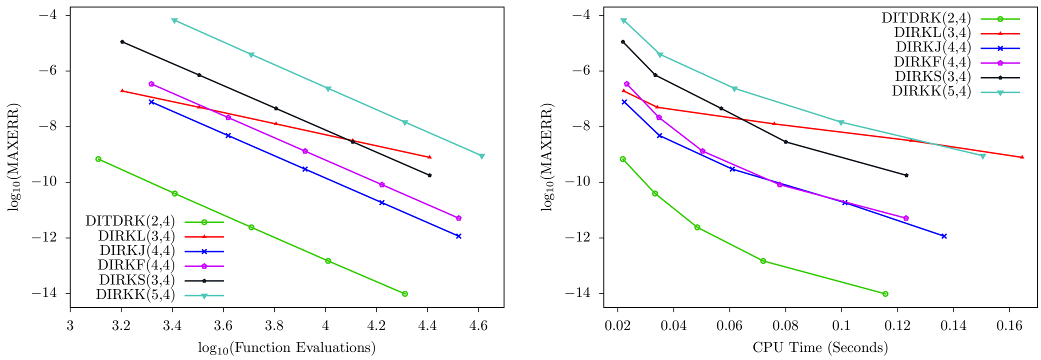

Fig. 13: The efficiency curve for Inhomogeneous problem (Problem 4) for DITDRK(2,4) method with $h = 1.0/2^i, i = 4,\ldots,8$.

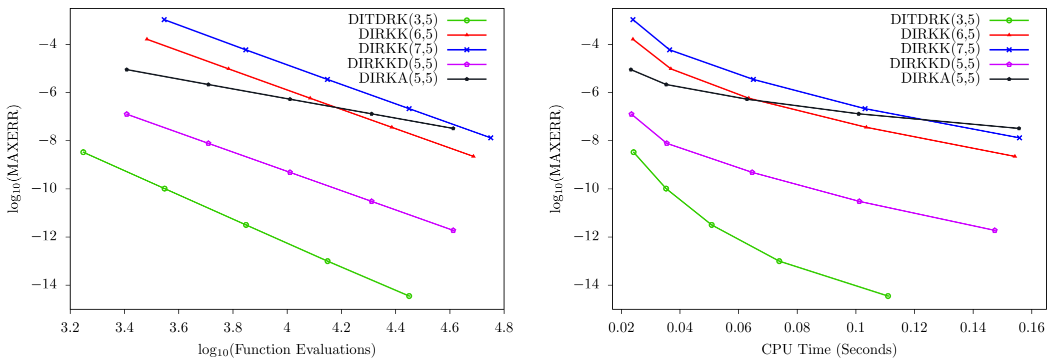

Fig. 14: The efficiency curve for Inhomogeneous problem (Problem 4) for DITDRK(3,5) method with $h = 1.0/2^i, i = 4,\ldots,8$.

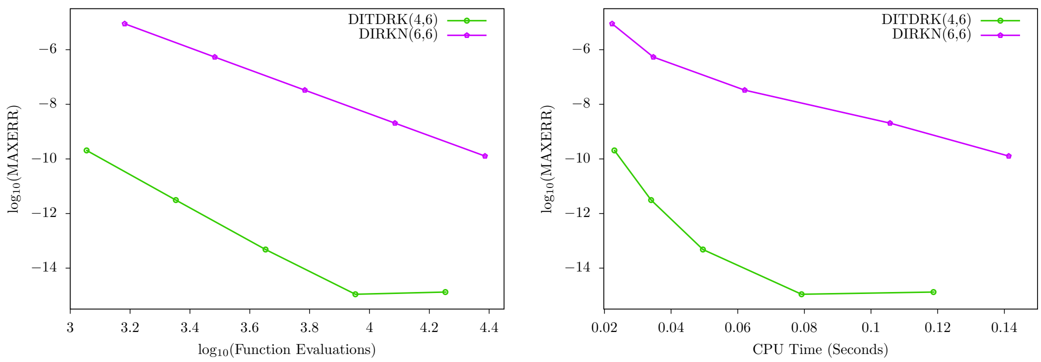

Fig. 15: The efficiency curve for Inhomogeneous problem (Problem 4) for DITDRK(4,6) method with $h = 1.0/2^i, i = 3,\ldots,7$.

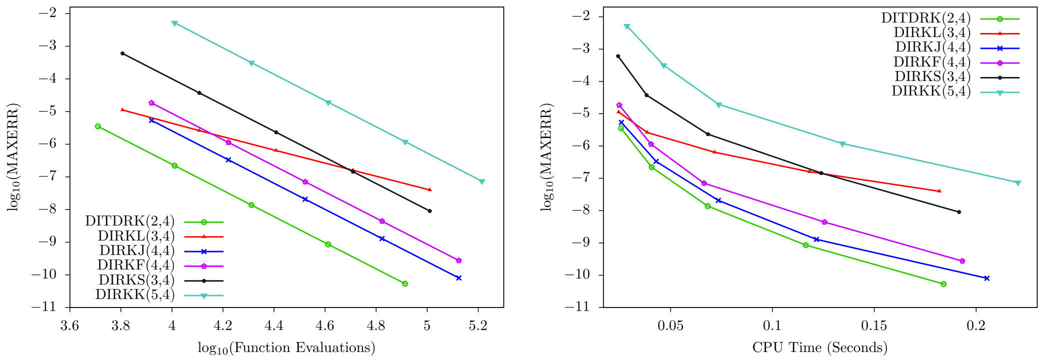

Fig. 16: The efficiency curve for Inhomogeneous problem (Problem 5) for DITDRK(2,4) method with $h = 1.0/2^i, i = 6, \ldots, 11$.

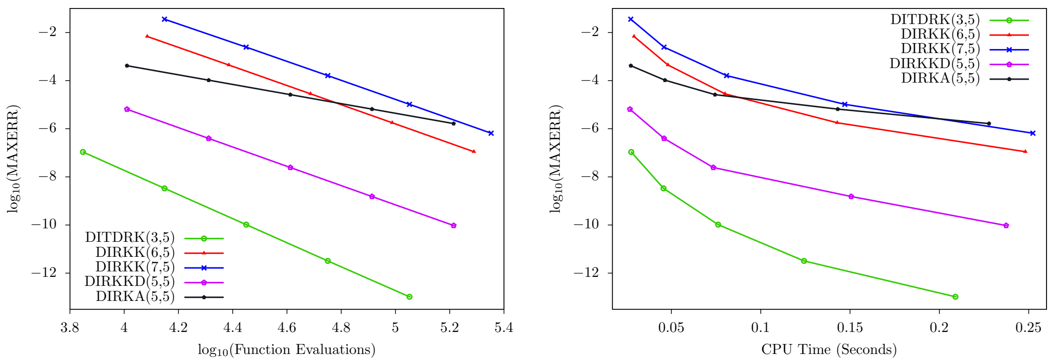

Fig. 17: The efficiency curve for Inhomogeneous problem (Problem 5) for DITDRK(3,5) method with $h = 1.0/2^i, i = 6, \ldots, 11$.

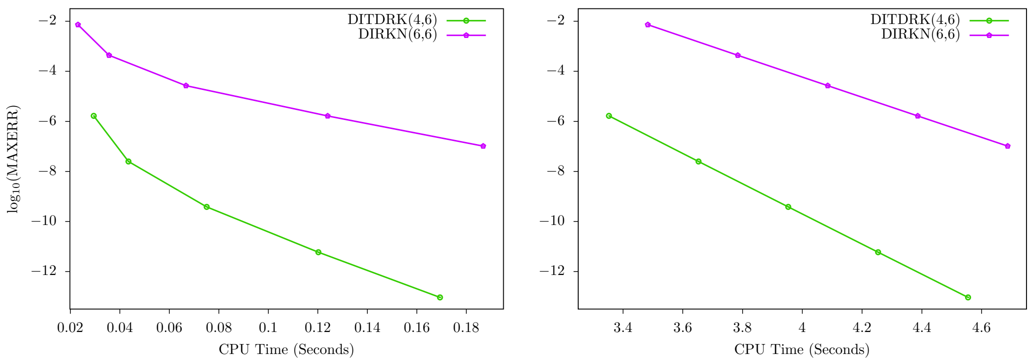

Fig. 18: The efficiency curve for Inhomogeneous problem (Problem 5) for DITDRK(4,6) method with $h = 1.0/2^i, i = 4, \ldots, 8$.

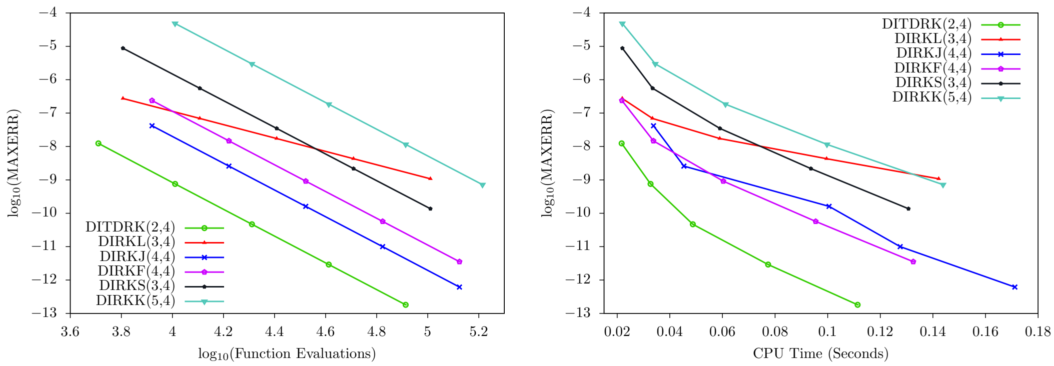

Fig. 19: The efficiency curve for Inhomogeneous problem (Problem 6) for DITDRK(2,4) method with $h = 1.0/2^i, i = 6,\dots,11$.

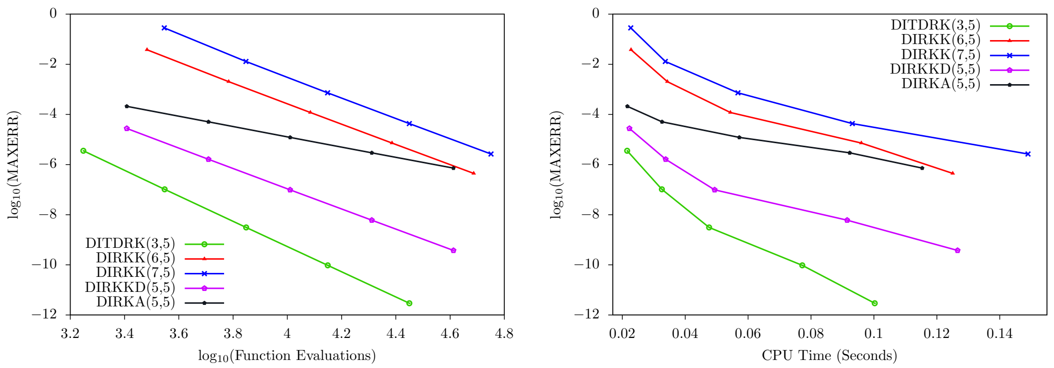

Fig. 20: The efficiency curve for Inhomogeneous problem (Problem 6) for DITDRK(3,5) method with $h = 1.0/2^i, i = 4,\dots,8$.

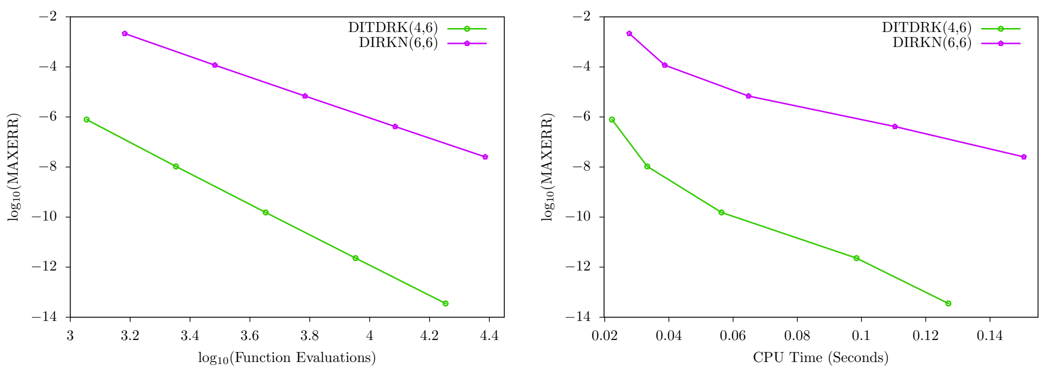

Fig. 21: The efficiency curve for Inhomogeneous problem (Problem 6) for DITDRK(4,6) method with $h = 1.0/2^i, i = 3,\dots,7$.

5 Discussion

The results show the typical properties of the DITDRK methods, DITDRK(2,4), DITDRK(3,5) and DITDRK(4,6) which have been derived earlier. The derived methods are compared with some well-known existing DIRK methods of the same order. The global error and the efficiency of the method over a long period of integration are plotted. Figures 4 and 21 represent the efficiency and accuracy of the method developed by plotting the graph of the logarithm of the maximum global error against the logarithm number of function evaluations for a longer periods of computations as well as the CPU times in seconds. From the plotted graphs, the derived methods has the smallest maximum global error and shorter CPU times compared to other existing DIRK methods of the same order.

6 Conclusion

In this research, fourth, fifth and sixth-order DITDRK methods have been developed. Based on the numerical results obtained, it can be concluded that the developed methods are more promising compared to other well-known existing DIRK methods in terms of accuracy, CPU times and the number of function evaluations per step.

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