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Some results on two new subclasses of p-valent spirallike and convexlike functions

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Abstract: In the present paper, we introduce two new subclasses of p-valent spirallike and p-valent convexlike functions which are analytic in the open unit disk. We prove necessary and sufficient conditions for this newly defined classes and also point out some known consequences of our results.

Keywords: p-valent function, spirallike functions, convexlike functions.

1 Introduction

Let $A(p)$, $p = 1,2,3,...$ be the class of $p-$valent analytic functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1}$$

defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.We write $A(1) = A$. A function $f \in A(p)$ is said to be $p-$valently starlike of order α in U, if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \left(z \in U, 0 \leq \alpha < p, p \in \mathbb{N} = \{1,2,...\}\right).$$

The class of all $p-$valently starlike functions of order α is denoted by $S_p^*(\alpha)$. On the other hand, a function $f \in A(p)$ is said to be $p-$valently convex of order α in U, if it satisfies the inequality.

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \left(z \in U, 0 \leq \alpha < p, p \in \mathbb{N} = \{1,2,...\}\right).$$

The class of all $p-$ valently convex functions of order α is denoted by $C_p(\alpha)$. Let S denote the subclass of A consisting of analytic and univalent function $f(z)$ in the open unit disc U. A function $f(z)$ is said to be $\lambda-$spirallike of order α, if it is satisfies the inequality

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha \cos\lambda \left(0 \leq \alpha < 1, z \in U\right)$$

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for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$. The class of such functions is denoted by $S_p^{\alpha}(\lambda)$ [2].

In 2012, S.Owa, F.Sağsöz and M.Kamali [4]introduced and studied the class $S_\beta(\alpha)$,consists of functions in $f \in S$ for which

$$\left|\frac{1}{e^{i\beta}F(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}\left(\beta \in \mathrm{R}, 0 < \alpha < 1, z \in U\right),$$

where $F(z) = \frac{zf'(z)}{f(z)}$.They given the following Theorem 1.

Theorem 1. [4]. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{n + \left|n - 2\alpha e^{-i\beta}\right|\right\}|a_n| \le 1 - \left|1 - 2\alpha e^{-i\beta}\right|$$

for some $|\beta| < 1$ and $0 < \alpha < \cos\beta$, then $f(z) \in S_\beta(\alpha)$.

A function $f \in A(p)$ belongs to the class $S_\beta(\alpha, p)$ if it is satisfies the inequality

$$\left|\frac{f^{(p-1)}(z)}{e^{i\beta}zf^{(p)}(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, (z \in U)$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the pth derivative of $f(z)$. On the other hand , a function $f \in A(p)$ is said to be in the class $C_\beta(\alpha, p)$ if it is satisfies the inequality

$$\left|\frac{f^{(p)}(z)}{e^{i\beta}\left(zf^{(p)}(z)\right)'} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, (z \in U)$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the pth derivative of $f(z)$. The classes $S_\beta(\alpha, p)$ and $C_\beta(\alpha, p)$is introduced and studied by N.Khan et al., [1]. They given the following Theorem 2 and Theorem 3.

Theorem 2. [1]. *If $f(z) \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}\left\{(n+1) + \left|(n+1) - 2\alpha e^{-i\beta}\right|\right\}\left|a_{n+p}\right| \le 1 - \left|1 - 2\alpha e^{-i\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in S_\beta(\alpha, p)$.

Theorem 3. [1]. *If $f(z) \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty}\frac{(p+n)!}{n!}\left\{(n+1) + \left|(n+1) - 2\alpha e^{-i\beta}\right|\right\}\left|a_{n+p}\right| \le 1 - \left|1 - 2\alpha e^{-i\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in C_\beta(\alpha, p)$.

2 Some definitions and results for a new class of $p-$valent $\beta-$ spirallike functions of order $\frac{\alpha p}{p+\lambda}$

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k}z^{p+k}\left(n, p \in \mathrm{N} = \{1, 2, \ldots\}\right)$$

which are analytic and $p-$valent in the open unit disc U. We write $A(1,1)=A$. Now, we give the following equalities for the functions $f(z)$ belonging to the class $A(p,n)$:

$$\begin{aligned}
&D^0 f(z) = f(z),\\
&D^1 f(z) = Df(z) = z\left(D^0 f(z)\right)' = pz^p + \sum_{k=n}^{\infty}(p+k)\,a_{p+k}z^{p+k},\\
&D^2 f(z) = D(Df(z)) = z\left(D^1 f(z)\right)' = p^2z^p + \sum_{k=n}^{\infty}(p+k)^2 a_{p+k}z^{p+k},\\
&\vdots\\
&D^{\Omega} f(z) = D\left(D^{\Omega-1} f(z)\right) == p^{\Omega}z^p + \sum_{k=n}^{\infty}(p+k)^{\Omega} a_{p+k}z^{p+k}.
\end{aligned}$$

We define $G_{(\Omega,\lambda,p)}f(z): A(p,n)\to A(p,n)$ such that

$$G_{(\Omega,\lambda,p)}f(z) = \left(\frac{1}{p^{\Omega}}-\lambda\right)D^{\Omega}f(z)+\frac{\lambda}{p}z\left(D^{\Omega}f(z)\right)'\left(0\le\lambda\le\frac{1}{p^{\Omega}},\Omega\in\mathrm{N}\bigcup\{0\}\right). \tag{2}$$

Definition 1. *A function $f(z)\in A(p,n)$ belongs to the class $S_{\beta}(\alpha,p,\lambda)$ if it satisfies the inequality*

$$\left|\frac{G_{(\Omega,\lambda,p)}f(z)}{e^{i\beta}z(G_{(\Omega,\lambda,p)}f(z))'}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha} \tag{3}$$

for some real β and $0<\alpha<1$, $0\le\lambda\le\frac{1}{p^{\Omega}}$, $\Omega\in\mathrm{N}\cup\{0\}$and for all $z\in U$.

Theorem 4. *A function $f(z)\in S_{\beta}(\alpha,p,\lambda)$ iff $Re\left(e^{i\beta}\frac{zG'_{(\Omega,\lambda,p)}f(z)}{G_{(\Omega,\lambda,p)}f(z)}\right)>\frac{\alpha p}{p+\lambda}$.*

Proof. Let $H(z)=\frac{zG'_{(\Omega,\lambda,p)}f(z)}{G_{(\Omega,\lambda,p)}f(z)}$ for $f(z)\in A(p,n)$. If $f(z)\in S_{\beta}(\alpha,p,\lambda)$, we can write

$$\left|\frac{1}{e^{i\beta}H(z)}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha}\,(z\in U).$$

Then, we can obtain

$$\begin{aligned}
&\left|\frac{1}{e^{i\beta}H(z)}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha}\Leftrightarrow\left|\frac{2p\alpha-(p+\lambda)e^{i\beta}H(z)}{2p\alpha e^{i\beta}H(z)}\right|<\frac{p+\lambda}{2p\alpha}\Leftrightarrow\\
&\left|2p\alpha-(p+\lambda)e^{i\beta}H(z)\right|^2<(p+\lambda)^2\left|e^{i\beta}H(z)\right|^2\Leftrightarrow\\
&\left[2p\alpha-(p+\lambda)e^{i\beta}H(z)\right]\left[2p\alpha-(p+\lambda)e^{-i\beta}\overline{H(z)}\right]<(p+\lambda)^2\left(e^{i\beta}H(z)\right)\left(e^{-i\beta}\overline{H(z)}\right)\Leftrightarrow\\
&4(p\alpha)^2-2\alpha p(p+\lambda).2Re\left\{e^{i\beta}H(z)\right\}+(p+\lambda)^2|H(z)|^2<(p+\lambda)^2|H(z)|^2\Leftrightarrow\\
&2\alpha p(p+\lambda).2Re\left\{e^{i\beta}H(z)\right\}>4(p\alpha)^2\Leftrightarrow Re\left\{e^{i\beta}H(z)\right\}>\frac{p\alpha}{p+\lambda}\Leftrightarrow(2.3)Re\left(e^{i\beta}\frac{zG'_{(\Omega,\lambda,p)}f(z)}{G_{(\Omega,\lambda,p)}f(z)}\right)>\frac{\alpha p}{p+\lambda}.
\end{aligned} \tag{4}$$

This complete the proof of Theorem 4.

When $p=1$, $\lambda=0$ and $\Omega=0$, we have the following known result proved by Owa et al., [4].

Corollary 1. $f\in S_{\beta}(\alpha,1,0)=S_{\beta}(\alpha)$ *iff* $Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right)>\alpha.$

Theorem 5. *If* $f\in A(p,n)$ *satisfies*

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left\{(p+\lambda)(k+p)+\left|(p+\lambda)(k+p)-2p\alpha e^{-i\beta}\right|\right\}\left|a_{p+k}\right|\leq p^{\Omega+1}\left\{(p+\lambda)-\left|(p+\lambda)-2\alpha e^{-i\beta}\right|\right\} \quad (5)$$

for some $|\beta|<\frac{\pi}{2}$ *and* $0<\frac{\alpha}{p+\lambda}<\cos\beta$*, then* $f(z)\in S_{\beta}(\alpha,p,\lambda)$.

Proof. It sufficient to show that

$$\left|\frac{2p\alpha e^{-i\beta}G_{(\Omega,\lambda,p)}f(z)-(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}{(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}\right|<1$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{2\alpha}{p+\lambda}<\cos\beta$, where

$$G_{(\Omega,\lambda,p)}f(z)=\left(\frac{1}{p^{\Omega}}-\lambda\right)D^{\Omega}f(z)+\frac{\lambda}{p}z\left(D^{\Omega}f(z)\right)'.$$

Note that

$$\begin{aligned}\left|\frac{2p\alpha e^{-i\beta}G_{(\Omega,\lambda,p)}f(z)-(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}{(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}\right|&=\left|\frac{2p\alpha e^{-i\beta}\{z^{p}+C\}-(p+\lambda)\{pz^{p}+B\}}{(p+\lambda)\{pz^{p}+B\}}\right|\\ &=\left|\frac{2p\alpha e^{-i\beta}-p(p+\lambda)+\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left[2p\alpha e^{-i\beta}-(p+k)(p+\lambda)\right]a_{p+k}z^{k}}{p(p+\lambda)+\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)(p+\lambda)(1+\lambda kp^{\Omega-1})a_{p+k}z^{k}}\right|\\ &\leq\frac{\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|+\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left|(k+p)(p+\lambda)-2p\alpha e^{-i\beta}\right|\left|a_{p+k}\right||z|^{k}}{p(p+\lambda)-\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(1+\lambda kp^{\Omega-1})(k+p)(p+\lambda)\left|a_{p+k}\right||z|^{k}}\\ &<\frac{\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|+\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left|(k+p)(p+\lambda)-2p\alpha e^{-i\beta}\right|\left|a_{p+k}\right|}{p(p+\lambda)-\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(1+\lambda kp^{\Omega-1})(k+p)(p+\lambda)\left|a_{p+k}\right|}.\end{aligned} \quad (6)$$

where $C=\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)a_{p+k}z^{p+k}$, $B=\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)a_{p+k}z^{p+k}$. Therefore, if

$$\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left\{(p+\lambda)(k+p)+\left|(p+\lambda)(k+p)-2p\alpha e^{-i\beta}\right|\right\}\left|a_{p+k}\right|\leq p(p+\lambda)-\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$, then

$$\begin{aligned}\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left|(p+\lambda)(k+p)-2p\alpha e^{-i\beta}\right|\left|a_{p+k}\right|&\leq p(p+\lambda)-\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|\\ &-\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(k+p)(p+\lambda)\left|a_{p+k}\right|.\end{aligned}$$

Using this inequality in (6), we obtain

$$\left|\frac{2p\alpha e^{-i\beta}G_{(\Omega,\lambda,p)}f(z)-(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}{(p+\lambda)zG'_{(\Omega,\lambda,p)}f(z)}\right|<\frac{\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|-\left|p(p+\lambda)-2p\alpha e^{-i\beta}\right|+AW}{p(p+\lambda)-AW}=1,$$

where $AW=p(p+\lambda)-\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(k+p)(p+\lambda)\left|a_{p+k}\right|$. Therefore, $f(z)\in S_{\beta}(\alpha,p,\lambda)$ for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$.

Taking $\beta=\frac{\pi}{4}$ in Theorem 5, we have the following Corollary 3.

Corollary 2. *$f(z)\in A(p,n)$ satisfies*

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left\{(p+\lambda)(k+p)+\sqrt{(p+\lambda)^2(k+p)^2-2\sqrt{2}p\alpha(p+\lambda)(k+p)+4\alpha^2p^2}\right\}\left|a_{p+k}\right|$$
$$\leq p^{\Omega+1}\left\{(p+\lambda)-\sqrt{(p+\lambda)^2-2\sqrt{2}\alpha(p+\lambda)+4\alpha^2}\right\}$$

for some $0<\frac{\alpha}{p+\lambda}<\frac{\sqrt{2}}{2}$, then $f(z)\in S_{\frac{\pi}{4}}(\alpha,p,\lambda)$. Putting $\beta=0$ in Theorem 5, we have the following Corollary 2.

Corollary 3. *Let $0<\alpha<p+\lambda$. $f(z)\in A(p,n)$ satisfies the following coefficient inequality*

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\{(p+\lambda)(k+p)-p\alpha\}\left|a_{p+k}\right|\leq\begin{cases}p^{\Omega+1}\alpha; & 0<\alpha\leq\frac{p+\lambda}{2}\\ p^{\Omega+1}(p+\lambda-\alpha); & \frac{p+\lambda}{2}\leq\alpha<p\end{cases}$$

then $f(z)\in S_0(\alpha,p,\lambda)$. Taking $\Omega=0,\lambda=0,p=1$ in Corollary 3, we write the Corollary 4 given by Owa et al.[5].

Corollary 4. [5] *Let $0<\alpha<1$. If $f(z)\in A$ satisfies the following coefficient inequality*

$$\sum_{k=1}^{\infty}(k+1-\alpha)\left|a_{k+1}\right|\leq\begin{cases}\alpha; & 0<\alpha\leq\frac{1}{2}\\ 1-\alpha; & \frac{1}{2}\leq\alpha<1\end{cases}$$

then $f(z)\in M(\alpha)$. To prove Theorem 6, first we give the following Lemma 1 due to Miller and Mocanu [3].

Lemma 1. *Let $\phi(u,v)$ be a complex –valued function such that*

$$\phi:D\to\mathrm{C},D\subset\mathrm{C}\times\mathrm{C}$$

C being (as usual) the complex plane, and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that the function $\phi(u,v)$ satisfies each of the following conditions:

(1) *$\phi(u,v)$ is continuous in D;*
(2) *$(1,0)\in D$ and $Re\{\phi(1,0)\}>0$;*
(3) *$Re\{\phi(iu_2,v_1)\}\leq0$ for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{(1+u_2^2)}{2}$.*

Let $q(z)=1+q_1z+q_2z^2+q_3z^3+...$ be analytic (regular) in the unit disk U such that

$$\left(q(z),zq'(z)\right)\in D$$

for all $z\in U$. If

$$Re\left\{\phi\left(q(z),zq'(z)\right)\right\}>0\,(z\in U),$$

then

$$Re\left(q\left(z\right)\right)>0\left(z\in U\right).$$

Theorem 6. *Let the function $f\left(z\right)$defined by (1) be in the class $S_{\beta}\left(\alpha,p,\lambda\right)$ and let*

$$0<\xi\leq\frac{1}{2p\left(\cos\beta-\frac{\alpha}{p+\lambda}\right)}\quad and\quad 0<\frac{\alpha}{p+\lambda}\leq\cos\beta. \tag{7}$$

Then, we have

$$Re\left\{\left(\frac{G_{(\Omega,\lambda,p)}f\left(z\right)}{z^{p}}\right)^{\xi e^{i\beta}}\right\}>\frac{1}{2\xi\left(p\cos\beta-\frac{p\alpha}{p+\lambda}\right)+1}\left(z\in U\right).$$

Proof. If we put

$$A=\frac{1}{2\xi\left(p\cos\beta-\frac{p\alpha}{p+\lambda}\right)+1} \tag{8}$$

and

$$\left(\frac{G_{(\Omega,\lambda,p)}f\left(z\right)}{z^{p}}\right)^{\xi e^{i\beta}}=\left(1-A\right)q\left(z\right)+A \tag{9}$$

where ξ satisfies (7) then $q\left(z\right)$ is regular in the unit disk U and $q\left(z\right)=1+q_{1}z+\ldots$ From (9) after taking the logarithmical differentiation we have that

$$\xi e^{i\beta}\left[\frac{G'_{(\Omega,\lambda,p)}f\left(z\right)}{G'_{(\Omega,\lambda,p)}f\left(z\right)}-\frac{p}{z}\right]=\frac{\left(1-A\right)q'\left(z\right)}{\left(1-A\right)q\left(z\right)+A}\Rightarrow$$

$$e^{i\beta}\frac{zG'_{(\Omega,\lambda,p)}f\left(z\right)}{G_{(\Omega,\lambda,p)}f\left(z\right)}-pe^{i\beta}=\frac{\left(1-A\right)zq'\left(z\right)}{\xi\left\{\left(1-A\right)q\left(z\right)+A\right\}}$$

and from there

$$e^{i\beta}\frac{zG'_{(\Omega,\lambda,p)}f\left(z\right)}{G_{(\Omega,\lambda,p)}f\left(z\right)}-\frac{p\alpha}{p+\lambda}=pe^{i\beta}-\frac{p\alpha}{p+\lambda}+\left(1-A\right)\frac{zq'\left(z\right)}{\xi\left\{\left(1-A\right)q\left(z\right)+A\right\}}. \tag{10}$$

Since $f\left(z\right)\in S_{\beta}\left(\alpha,p,\lambda\right)$ then from (10) we get

$$Re\left\{pe^{i\beta}-\frac{p\alpha}{p+\lambda}+\left(1-A\right)\frac{zq'\left(z\right)}{\xi\left\{\left(1-A\right)q\left(z\right)+A\right\}}\right\}>0,\left(z\in U,0<\frac{\alpha}{p+\lambda}<\cos\beta\right).$$

Let consider the function $\phi\left(u,v\right)$ defined by

$$\phi\left(u,v\right)=pe^{i\beta}-\frac{p\alpha}{p+\lambda}+\left(1-A\right)\frac{v}{\xi\left\{\left(1-A\right)u+A\right\}},$$

where $u=q\left(z\right)$ and $v=zq'\left(z\right)$. Then $\phi\left(u,v\right)$ is continuous in $D=\left(\mathbb{C}-\frac{A}{A-1}\right)\times\mathbb{C}$.

Also $(1,0) \in D$ and $Re\{\phi(1,0)\} = p\left(\cos\beta - \frac{\alpha}{p+\lambda}\right) > 0$. Furthermore, for all $\phi(iu_2, v_1) \in D$ such that

$$v_1 \leq -\frac{1+u_2^2}{2},$$

we have

$$\begin{aligned}
Re\{\phi(iu_2,v_1)\} &= p\cos\beta - \frac{p\alpha}{p+\lambda} + Re\left\{(1-A)\frac{v_1}{\xi\{(1-A)iu_2+A\}}\right\} \\
&\leq p\cos\beta - \frac{p\alpha}{p+\lambda} - Re\left\{(1-A)\frac{(1+u_2^2)}{2\xi\{(1-A)iu_2+A\}}\right\} \\
&= p\cos\beta - \frac{p\alpha}{p+\lambda} - Re\left\{(1-A)\frac{\{(1-A)(-iu_2)+A\}(1+u_2^2)}{2\xi\{(1-A)iu_2+A\}\{(1-A)(-iu_2)+A\}}\right\} \\
&= p\cos\beta - \frac{p\alpha}{p+\lambda} - (1-A)\frac{A(1+u_2^2)}{2\xi\left\{(1-A)^2u_2^2+A^2\right\}}. \qquad (11)
\end{aligned}$$

When $(1-A)$ is withdrawn from equality (8),we write

$$2\xi A\left(p\cos\beta - \frac{\alpha p}{p+\lambda}\right) = 1-A.$$

By using this last equality in (11), we obtain

$$\begin{aligned}
Re\{\phi(iu_2,v_1)\} &\leq p\cos\beta - \frac{p\alpha}{p+\lambda} - \frac{2\xi A\left(p\cos\beta - \frac{\alpha p}{p+\lambda}\right)A(1+u_2^2)}{2\xi\left\{(1-A)^2u_2^2+A^2\right\}} \\
&= \left\{p\cos\beta - \frac{p\alpha}{p+\lambda}\right\}\left\{A^2\frac{\left\{\left(\frac{1-A}{A}\right)^2-1\right\}u_2^2}{\left\{(1-A)^2u_2^2+A^2\right\}}\right\} \\
&= \left\{p\cos\beta - \frac{p\alpha}{p+\lambda}\right\}\left\{A^2\frac{\left\{\left[2\xi\left(p\cos\beta - \frac{\alpha p}{p+\lambda}\right)\right]^2-1\right\}u_2^2}{\left\{(1-A)^2u_2^2+A^2\right\}}\right\} \leq 0
\end{aligned}$$

because $0 < \frac{\alpha}{p+\lambda} < \cos\beta$ and

$$\left[2\xi\left(p\cos\beta - \frac{\alpha p}{p+\lambda}\right)\right]^2 - 1 \leq 0 \Rightarrow \xi \leq \frac{1}{2p\left(\cos\beta - \frac{\alpha}{p+\lambda}\right)}.$$

Therefore, the function $\phi(u,v)$ satisfies the conditions in Lemma 1. This proves that $Re(q(z)) > 0$ for $z \in U$, that is , that from (2.8)

$$Re\left\{\left(\frac{G_{(\Omega,\lambda,p)}f(z)}{z^p}\right)^{\xi e^{i\beta}}\right\} > A \Rightarrow Re\left\{\left(\frac{G_{(\Omega,\lambda,p)}f(z)}{z^p}\right)^{\xi e^{i\beta}}\right\} > \frac{1}{2p\xi\left(\cos\beta - \frac{\alpha}{p+\lambda}\right)+1}$$

which is equivalent to the statement Theorem 5. Taking $p=1$, $\Omega=0$and $\lambda=0$ in Theorem 6, we obtain the following Corollary 5.

Corollary 5. [4]. *Let the function $f(z)$ defined by (1) be in the class $S_\beta(\alpha,1,0)=S_\beta(\alpha)$ and let $0<\xi\le\frac{1}{2(\cos\beta-\alpha)}$ and $0<\alpha<\cos\beta$. Then, we have*

$$Re\left\{\left(\frac{f(z)}{z}\right)^{\xi e^{i\beta}}\right\}>\frac{1}{2\xi(\cos\beta-\alpha)+1}\,(z\in U).$$

Taking $p=1$, $\Omega=0,\lambda=0$and $\beta=0$ in Theorem 6, we obtain the following Corollary 6.

Corollary 6. *Let the function $f(z)$ defined by (1) be in the class $S_0(\alpha,1,0)=S(\alpha)$ and let $0<\xi\le\frac{1}{2(1-\alpha)}$ and $0<\alpha<1$. Then, we have*

$$Re\left\{\left(\frac{f(z)}{z}\right)^{\xi}\right\}>\frac{1}{2\xi(1-\alpha)+1}\,(z\in U).$$

3 Some definitions and results for a new class of $p-$valent $\beta-$ convexlike functions of order $\frac{\alpha p}{p+\lambda}$

Definition 2. *Let $K(z)=1+\frac{zG''_{(\Omega,\lambda,p)}f(z)}{G'_{(\Omega,\lambda,p)}f(z)}$ for $f(z)\in A(p,n)$. A function $f(z)\in A(p,n)$ is a said to be in the class $K_\beta(\alpha,p,\lambda)$ if it is satisfies the inequality*

$$\left|\frac{1}{e^{i\beta}K(z)}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha} \tag{12}$$

for some real β and $0<\alpha<1$, $0\le\lambda\le\frac{1}{p^{\Omega}}$, $\Omega\in\mathrm{N}\cup\{0\}$and for all $z\in U$.

Theorem 7. *A function $f(z)\in K_\beta(\alpha,p,\lambda)$ iff $Re\left\{e^{i\beta}\left(1+\frac{zG''_{(\Omega,\lambda,p)}f(z)}{G'_{(\Omega,\lambda,p)}f(z)}\right)\right\}>\frac{\alpha p}{p+\lambda}$.*

Proof. Let $K(z)=1+\frac{zG''_{(\alpha,p,\lambda)}f(z)}{G'_{(\alpha,p,\lambda)}f(z)}$ for $f(z)\in A(p,n)$. If $f(z)\in K_\beta(\alpha,p,\lambda)$, we can write

$$\left|\frac{1}{e^{i\beta}K(z)}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha}\,(z\in U).$$

Then, we can obtain

$$\left|\frac{1}{e^{i\beta}K(z)}-\frac{p+\lambda}{2p\alpha}\right|<\frac{p+\lambda}{2p\alpha}\Leftrightarrow\left|\frac{2p\alpha-(p+\lambda)e^{i\beta}K(z)}{2p\alpha e^{i\beta}K(z)}\right|<\frac{p+\lambda}{2p\alpha}\Leftrightarrow$$
$$\left[2p\alpha-(p+\lambda)e^{i\beta}K(z)\right]\left[\overline{2p\alpha-(p+\lambda)e^{-i\beta}K(z)}\right]<(p+\lambda)^2\left(e^{i\beta}K(z)\right)\left(\overline{e^{i\beta}K(z)}\right)\Leftrightarrow$$
$$2\alpha p(p+\lambda).2Re\left\{e^{i\beta}K(z)\right\}>4(p\alpha)^2\Leftrightarrow Re\left\{e^{i\beta}K(z)\right\}>\frac{p\alpha}{p+\lambda}\Leftrightarrow Re\left\{e^{i\beta}\left(1+\frac{zG''_{(\Omega,\lambda,p)}f(z)}{G'_{(\Omega,\lambda,p)}f(z)}\right)\right\}>\frac{\alpha p}{p+\lambda}.$$

This complete the proof of Theorem 7.

Theorem 8. *If $f\in A(p,n)$ satisfies*

$$\sum_{k=n}^{\infty}(p+k)^{\Omega+1}\left(1+\lambda kp^{\Omega-1}\right)\left\{(p+\lambda)(p+k)+\left|(p+\lambda)(p+k)-2p\alpha e^{-i\beta}\right|\right\}\left|a_{p+k}\right|\le p^{\Omega+2}FG \tag{13}$$

where $FG=\left\{(p+\lambda)-\left|(p+\lambda)-2\alpha e^{-i\beta}\right|\right\}$, for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$, then $f(z)\in K_\beta(\alpha,p,\lambda)$.

Proof. It sufficient to show that

$$\left|\frac{2p\alpha e^{-i\beta}G'_{(\Omega,\lambda,p)}f(z)-(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}{(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}\right|<1$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$, where

$$G_{(\Omega,\lambda,p)}f(z)=\left(\frac{1}{p^{\Omega}}-\lambda\right)D^{\Omega}f(z)+\frac{\lambda}{p}z\left(D^{\Omega}f(z)\right)'.$$

Note that

$$\begin{aligned}&\left|\frac{2p\alpha e^{-i\beta}G'_{(\Omega,\lambda,p)}f(z)-(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}{(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}\right|\\ &=\left|\frac{2p^2\alpha e^{-i\beta}-p^2(p+\lambda)+\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}(p+k)\left(1+\lambda kp^{\Omega-1}\right)\left[2p\alpha e^{-i\beta}-(p+k)(p+\lambda)\right]a_{p+k}z^k}{[p+\lambda)\left\{p^2+\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}(p+k)^2\left(1+\lambda kp^{\Omega-1}\right)a_{p+k}z^k\right\}}\right|\\ &<\frac{p^2\left|(p+\lambda)-2\alpha e^{-i\beta}\right|+\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}(p+k)\left(1+\lambda kp^{\Omega-1}\right)\left|(p+k)(p+\lambda)-2p\alpha e^{-i\beta}\right|\left|a_{k+p}\right|}{p^2(p+\lambda)-\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(p+k)^2(p+\lambda)\left|a_{k+p}\right|}.\end{aligned}\tag{14}$$

Therefore, if

$$\sum_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}(p+k)\left(1+\lambda kp^{\Omega-1}\right)\{\Gamma\}\left|a_{p+k}\right|\le p^2\left\{(p+\lambda)-\left|(p+\lambda)-2\alpha e^{-i\beta}\right|\right\}$$

where $\Gamma=\left\{(p+\lambda)(p+k)+\left|(p+\lambda)(p+k)-2p\alpha e^{-i\beta}\right|\right\}$, for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$, then

$$\begin{aligned}\sum_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(p+k)\left|(p+\lambda)(p+k)-2p\alpha e^{-i\beta}\right|\left|a_{p+k}\right|\le p^2\left\{(p+\lambda)-\left|(p+\lambda)-2\alpha e^{-i\beta}\right|\right\}\\ -\sum_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(p+k)^2(p+\lambda)\left|a_{p+k}\right|\end{aligned}$$

Using this inequality in (14), we obtain

$$\left|\frac{2p\alpha e^{-i\beta}G'_{(\Omega,\lambda,p)}f(z)-(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}{(p+\lambda)\left[G'_{(\Omega,\lambda,p)}f(z)+zG''_{(\Omega,\lambda,p)}f(z)\right]}\right|<\frac{\Upsilon-\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(p+k)^2(p+\lambda)\left|a_{p+k}\right|}{p^2(p+\lambda)-\sum\limits_{k=n}^{\infty}\left(\frac{p+k}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)(p+k)^2(p+\lambda)\left|a_{p+k}\right|}=1$$

where $\Upsilon=p^2\left|(p+\lambda)-2\alpha e^{-i\beta}\right|+p^2\left\{(p+\lambda)-\left|(p+\lambda)-2\alpha e^{-i\beta}\right|\right\}$. Therefore, $f(z)\in K_{\beta}(\alpha,p,\lambda)$ for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{p+\lambda}<\cos\beta$. Putting $\beta=\frac{\pi}{4}$ in Theorem 8, we have the following Corollary 7.

Corollary 7. *$f\in A(p,n)$ satisfies*

$$\sum_{k=n}^{\infty}(k+p)^{\Omega+1}\left(1+\lambda kp^{\Omega-1}\right)\{\Lambda\}\left|a_{p+k}\right|\le p^{\Omega+2}\left\{(p+\lambda)-\sqrt{(p+\lambda)^2-2\sqrt{2}\alpha(p+\lambda)+4\alpha^2}\right\}$$

where $\Lambda = \left\{ (p+\lambda)(k+p) + \sqrt{(p+\lambda)^2 (k+p)^2 - 2\sqrt{2}p\alpha (p+\lambda)(k+p) + 4\alpha^2 p^2} \right\}$, *for some* $0 < \alpha < \frac{\sqrt{2}}{2}$, *then* $f(z) \in K_{\frac{\pi}{4}}(\alpha, p, \lambda)$. *Taking* $p = 1$, $\Omega = 0$,$\lambda = 0$*and* $\beta = 0$ *in Theorem 8, we obtain the following Corollary 8.*

Corollary 8. [5] *If* $f \in A$ *satisfies*

$$\sum_{k=1}^{\infty} (k+1) \{(k+1) + |(k+1) - 2\alpha|\} |a_{k+1}| \leq 1 - |1 - 2\alpha|$$

for some $0 < \alpha < 1$, *then* $f(z) \in N(\alpha)$.

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