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Hom-Lie triple systems with involution

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Abstract: In this work we we prove that all involutive Hom-Lie triple systems are whether simple or semi-simple. Moreover, we prove that an involutive simple Lie triple system give a rise of Involutive Hom-Lie triple system.

Keywords: Jordan triple system, Lie triple system, Casimir operator, quadratic Lie algebra, TKK construction.

1 Introduction

The classification of semisimple Lie algebras with involutions can be found in [5]. The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in [6] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. The Killing form K of g is nondegenerate and Î¸ is symmetric with respect to K. In [1], the author studied Hom-Lie triple system using the double extension and gives an inductive description of quadratic Hom-Lie triple system. In this work we recall the definition of involutive Hom-Lie triple systems and some related structure and we prove that all involutive Hom-Lie triple systems are whether simple or semi-simple. Moreover, we prove that an involutive simple Lie triple system give a rise of Involutive Hom-Lie triple system.

Definition 1. *A Hom-Lie triple system is a triple* $(\boldsymbol{L},[-,-,-],\alpha)$ *consisting of a linear space* $\boldsymbol{L}$, *a trilinear map* $[-,-,-]:\boldsymbol{L}\times\boldsymbol{L}\times\boldsymbol{L}\to\boldsymbol{L}$ *and a linear map* $\alpha:\boldsymbol{L}\to\boldsymbol{L}$ *such that*

$$
\begin{aligned}
&[x,x,z]=0, \quad \textit{(skewsymmetry)}\\
&[x,y,z]+[y,z,x]+[z,x,y]=0, \quad \textit{(ternary Jacobi identity)}\\
&[\alpha(u),\alpha(v),[x,y,z]]=[[u,v,x],\alpha(y),\alpha(z)]+[\alpha(x),[u,v,y],\alpha(z)]+[\alpha(x),\alpha(y),[u,v,z]],
\end{aligned}
$$

for all $x,y,z,u,v\in\boldsymbol{L}$. *If Moreover* α *satisfies* $\alpha([x,y,z])=[\alpha(x),\alpha(y),\alpha(z)]$ *(resp.* $\alpha^2=id_{\boldsymbol{L}}$*) for all* $x,y,z\in\boldsymbol{L}$, *we say that* $(\boldsymbol{L},[-,-,-],\alpha)$ *is a multiplicative (resp. involutive) Hom-Lie triple system. A Hom-Lie triple system* $(\boldsymbol{L},[-,-,-],\alpha)$ *is said to be regular if* α *is an automorhism of* $\boldsymbol{L}$.

When the twisting map α is equal to the identity map, we recover the usual notion of Lie triple system [4,3]. So, Lie triple systems are examples of Hom-Lie triple systems. If we introduce the right multiplication R defined for all $x,y\in\mathbf{L}$ by $R(x,y)(z):=[x,y,z]$, then the conditions above can be written as follow:

$$
\begin{aligned}
&R(x,y)=-R(y,x),\\
&R(x,y)z+R(y,z)x+R(z,x)y=0,\\
&R(\alpha(u),\alpha(v))[x,y,z]=[R(u,v)x,\alpha(y),\alpha(z)]+[\alpha(x),R(u,v)y,\alpha(z)]+[\alpha(x),\alpha(y),R(u,v)z].
\end{aligned}
$$

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We can also introduce the middle (resp. left) multiplication operator $M(x,z)y := [x,y,z]$ (resp. $L(y,z)x := [x,y,z]$) for all $x,y,z \in \mathbf{L}$. The equations above can be written in operator form respectively as follows:

$$M(x,y) = -L(x,y) \tag{1}$$
$$M(x,y) - M(y,x) = R(x,y) \text{ for all } x,y \in \mathbf{L}. \tag{2}$$

We can write the equation above as one of the equivalent identities of operators:

$$R(\alpha(u),\alpha(v))R(x,y) - R(\alpha(x),\alpha(y))R(u,v) = (R(R(u,v)x,\alpha(y)) + R(\alpha(x),R(u,v)y))\alpha.$$
$$R(\alpha(u),\alpha(v))M(x,z) - M(\alpha(x),\alpha(z))R(u,v) = (M(R(u,v)x,\alpha(z)) + M(\alpha(x),R(u,v)z))\alpha.$$

Definition 2. *Let $(\boldsymbol{L},[-,-,-],\alpha)$ and $(\boldsymbol{L}',[-,-,-]',\alpha')$ be two two Hom-Lie triple systems. A linear map $f : \boldsymbol{L} \to \boldsymbol{L}'$ is a morphism of Hom-Lie triple systems if*

$$f([x,y,z]) = [f(x), f(y), f(z)]' \text{ and } f \circ \alpha = \alpha' \circ f.$$

In particular, if f is invertible, then $\boldsymbol{L}$ and $\boldsymbol{L}'$ are said to be isomorphic.

Definition 3. *Let $(\boldsymbol{L},[-,-,-],\alpha)$ be a Hom-Lie triple system and $\boldsymbol{I}$ be a subspace of $\boldsymbol{L}$. We say that $\boldsymbol{I}$ is an ideal of L if $[\boldsymbol{I},\boldsymbol{L},\boldsymbol{L}] \subset \boldsymbol{I}$ and $\alpha(\boldsymbol{I}) \subset \boldsymbol{I}$.*

Definition 4. *A Hom-Lie triple system $\boldsymbol{L}$ is said to be simple (resp. semisimple) if it contains no nontrivial ideal (resp. $Rad(\boldsymbol{L}) = \{0\}$).*

According to a result in [2], if A is a Malcev algebra, then $(A,[-,-,-])$ is a Lie triple system with triple product

$$[x,y,z] = 2(xy)z - (zx)y - (yz)x. \tag{3}$$

Thus, if A is a Malcev algebra and $\alpha : A \to A$ is an algebra morphism, then, $A_\alpha = (A,[-,-,-]_\alpha = \alpha \circ [-,-,-],\alpha)$ is a multiplicative Hom-Lie triple system, where $[-,-,-]$ is the triple product in (3).

Proposition 1. *Let $\boldsymbol{L}$ be a Lie triple system and α be an automorphism of $\boldsymbol{L}$. If $\boldsymbol{L}$ is simple, the $\boldsymbol{L}$ is also simple.*

Since $\mathbf{L}$ is not abelian, then $\mathbf{L}_\alpha$ is also not abelian. Moreover, let $\mathbf{I}$ be an ideal of $\mathbf{L}_\alpha$. For all $x,y \in \mathbf{L}$ and $a \in \mathbf{I}$ we have,

$$[a,x,y]_\alpha \in \mathbf{I}.$$

That is,

$$[\alpha(a),\alpha(x),\alpha(y)] \in \mathbf{I}.$$

Consequently, $\mathbf{I}$ is an ideal of $\mathbf{L}$ because α is an automorphism. Thus, $\mathbf{I} = \{0\}$.

Theorem 1. *Let $(\boldsymbol{L},[.,.,.],\theta)$ be an involutive Hom-Lie triple system. Then, $(\boldsymbol{L},[.,.,.]_\theta,\theta)$ is simple or semi-simple. Moreover, in the second case $\boldsymbol{L}$ can be wrien as $\boldsymbol{L} := \boldsymbol{L}_\theta = \mathscr{S} \oplus \theta(\mathscr{S})$ where $\mathscr{S}$ is a simple ideal of $\boldsymbol{L}$. Conversely, If $(\boldsymbol{L},[.,.,.],\theta)$ is an involutive simple Lie triple system, then $(\boldsymbol{L},[.,.,.]_\theta,\theta)$ is an involutive Hom-Lie triple system.*

Suppose that $\mathbf{L}_\theta$ is not simple and put $\mathscr{S}$ a minimal ideal of $\mathbf{L}_\theta$. We get $[\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}]_\theta$ is an ideal of $\mathbf{L}_\theta$ which is contained on $\mathscr{S}$. Thus,

$$[\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}]_\theta = \{0\} \text{ or } [\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}]_\theta = \mathscr{S}.$$

Now, firstly, if $[\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}]_\theta = \{0\}$, then $\theta([\mathbf{L}_\theta),\theta(\mathbf{L}_\theta),\theta(\mathscr{S})] = \{0\}$. That is, $[\mathbf{L},\mathbf{L},\theta(\mathscr{S})] = \{0\}$, because θ is a bijective linear map. which mean that $\theta(\mathscr{S}) \subset Z(\mathbf{L}) = \{0\}$. Thus, $[\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}]_\theta = \mathscr{S}$. Hence, $[\mathbf{L},\mathbf{L},\theta(\mathscr{S})] = \mathscr{S}$. Which implies that $\theta([\mathbf{L},\mathbf{L},\theta(\mathscr{S})]) = [\theta(\mathbf{L}),\theta(\mathbf{L}),\theta^2(\mathscr{S})] = \theta(\mathscr{S})$. Consequently,

$$[\mathbf{L},\mathbf{L},\mathscr{S}+\theta(\mathscr{S})] \subset \mathscr{S}+\theta(\mathscr{S}).$$

Furthermore,

$$\theta(\mathscr{S}+\theta(\mathscr{S})) = \theta(\mathscr{S})+\theta^2(\mathscr{S}) = \theta(\mathscr{S})+\mathscr{S}.$$

Thus, $\mathscr{S}+\theta(\mathscr{S})$ is an ideal of $(\mathbf{L},[.,.,.],\theta)$. Since $\mathscr{S}+\theta(\mathscr{S}) \neq \{0\}$, then $\mathbf{L} = \mathscr{S}+\theta(\mathscr{S})$.

Now, we have to prove that the summation is direct. In fact, since θ is an automorphism of $\mathbf{L}_\theta$, then $\theta(\mathscr{S})$ is an ideal of $\mathbf{L}_\theta$. Thus, $\mathscr{S}\cap\theta(\mathscr{S}) = \mathscr{S}$ or $\mathscr{S}\cap\theta(\mathscr{S}) = \{0\}$ because $\mathscr{S}$ is minimal. Suppose that $\mathscr{S}\cap\theta(\mathscr{S}) = \mathscr{S}$, then $\mathscr{S} = \theta(\mathscr{S})$ because θ is bijective. On the other hand,

$$[\mathbf{L},\mathbf{L},\mathscr{S}] = \theta([\theta(\mathbf{L}),\theta(\mathbf{L}),\theta(\mathscr{S})]) = \theta([\mathbf{L},\mathbf{L},\mathscr{S}]_\theta) \subset \theta(\mathscr{S}) = \mathscr{S}.$$

Thus, $\mathscr{S}$ is an ideal of $(\mathbf{L},[.,.,.],\theta)$ and $\mathscr{S} = \mathbf{L}$ because $(\mathbf{L},[.,.,.])$ which contradict the fact that $\mathscr{S} \neq \mathbf{L}$ and $\mathscr{S} \neq \{0\}$. Consequently, $\mathscr{S}\cap\theta(\mathscr{S}) = \{0\}$ and $\mathbf{L} = \mathscr{S}\oplus\theta(\mathscr{S})$.

Let us prove that $\mathscr{S}$ is a simple ideal of $(\mathbf{L}_\theta,[.,.,.]_\theta)$. In fact, $\mathbf{L} = \mathbf{L}_\theta = \mathscr{S}\oplus\theta(\mathscr{S})$. Since θ is an automorphism of $\mathbf{L}$ then θ is an automorphism of $\mathbf{L}_\theta$.

$$[\theta(\mathscr{S}),\mathbf{L},\mathbf{L}] = \theta([\theta(\mathscr{S}),\mathbf{L},\mathbf{L}] = \theta([\mathscr{S},\theta(\mathbf{L}),\theta(\mathbf{L})] = \theta([\mathscr{S},\mathbf{L},\mathbf{L}]) \subset \theta(\mathscr{S}).$$

Thus, $\theta(\mathscr{S})$ is an ideal of $\mathbf{L}_\theta$. Furthermore,

$$[\mathscr{S},\mathscr{S},\mathscr{S}]_\theta = [\mathscr{S}\oplus\theta(\mathscr{S}),\mathscr{S}\oplus\theta(\mathscr{S}),\mathscr{S}]_\theta = [\mathbf{L}_\theta,\mathbf{L}_\theta,\mathscr{S}] = \mathscr{S}.$$

Thus, $\mathscr{S}$ is a simple ideal of $\mathbf{L}_\theta$ because it is simple with $[\mathscr{S},\mathscr{S}]_t heta = \mathscr{S}$. Consequently, $\mathbf{L}_\theta$ is semi-simple.

Corollary 1. *Let $(\boldsymbol{L},[.,.,.])$ be a Lie triple system with involution θ. such that , $\boldsymbol{L} = \mathscr{S}\oplus\theta(\mathscr{S})$ where $\mathscr{S}$ is a simple ideal of $(\boldsymbol{L},[.,.,.])$. Then the Hom-Lie triple system $(\boldsymbol{L}_\theta,[.,.,.]_\theta,\theta)$ is simple.*

Let $\mathscr{I}$ be an ideal of $\mathbf{L}_\theta$ such that $\mathscr{I} \neq \{0\}$. We have

$$[\mathbf{L},\mathbf{L},\theta(\mathscr{I})] = [\theta(\mathbf{L}),\theta(\mathbf{L}),\theta(\mathscr{I})] = [\mathbf{L},\mathbf{L},\mathscr{I}]$$

because $\mathbf{L} = \theta(\mathbf{L})$ and $\mathscr{I}$ is an ideal of $\mathbf{L}_\theta$. Moreover,

$$[\mathbf{L},\mathbf{L},\mathscr{I}] = \theta([\theta(\mathbf{L}),\theta(\mathbf{L}),\theta(\mathscr{I})]) = \theta([\mathbf{L},\mathbf{L},\mathscr{I}]_t heta) \subset \theta(\mathscr{I}) = \mathscr{I},$$

because $\mathscr{I}$ is stable under θ since it is an ideal of the Hom-Lie triple system of $\mathbf{L}_\theta$. Consequently, $\mathscr{I}$ is an ideal of $\mathbf{L}$. Thus, $\mathscr{I} = \mathscr{S}$ or $\mathscr{I} = \theta(\mathscr{S})$ or $\mathscr{I} = \mathbf{L}$. Since $\theta(\mathscr{I}) \subset \mathscr{I}$, then $\mathscr{I} \neq \mathscr{S}$ and $\mathscr{I} \neq \theta(\mathscr{S})$. Thus, $\mathscr{I} =\neq \mathscr{S}\oplus\theta(\mathscr{S}) = \mathbf{L}$. Moreover, since $[\mathbf{L},\mathbf{L},\mathbf{L}] = \mathbf{L}$, then $[\mathbf{L}_\theta,\mathbf{L}_\theta,\mathbf{L}_\theta] = \mathbf{L}_\theta$. Thus, $(\mathbf{L}_\theta,[.,.,.]_\theta,\theta)$ is a simple Hom-Lie triple system.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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