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Codes associated to Schubert varieties in $G(2,5)$ over $\mathbb{F}_2$

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Abstract: Denote by $G(\ell,m)$, the Grassmannian of ℓ-dimensional subspaces of a m-dimensional vector space $\mathbb{F}_q{}^m$ over the finite field $\mathbb{F}_q$ and $\Omega_\alpha(\ell,m)$, the Schubert subvarieties of $G(\ell,m)$. A linear $[n,k]_q$- code is a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$. In this paper, we consider the problem of determining generalized spectrum of linear codes associated to Schubert subvarieties of Grassmannians. We make a small begining here by detremining the seond generalized spectrum (i.e. second weight distribution) of Schubert codes associated to Schubert subvarieties of $G(\ell,m)$ over $\mathbb{F}_2$ in case of $\ell = 2$ and $m = 5$.

Keywords: Grassmanian varieties, Schubert varieties, linear Codes.

1 Introduction

In [13], Victor Wei introduced the notion of weight hierarchy of a linear code, motivated by applications in type II wire-tap channel in cryptography. Wei defined the r-th generalized Hamming weight of a linear code as the minimum support weight of any of its r-dimensional subcode. For a class of Algebraic-geometric codes the generalized Hamming weights were investigated by a number of researchers such as Tsfasmann-Vlăduţ [12], Nogin [8], Ghorpade-Lachaud [1], Ghorpade-Tsfasman [2], Hirshfeld-Tsfasman-Vlăduţ [7], Ghorpade-Patil-Pillai [3].

Generalized Hamming weights proved to be of great applications in coding theory to study the structure of a code. It is therefore natural toconsider an extension of the notion of generalized weights-the generalized spectra of linear codes. The problem of determining the generalized spectra of a linear $[n,k]_q$ code is first studied by Kløve in [4] and [5]. In [4], he gave a MacWilliams identity for the support weight distribution of linear codes called the generalized MacWilliams identity. In [5], he determined the weight enumerator polynomial (also called support weight distribution function) for irreducible cyclic codes.

In [3], the problem of determining generalized spectrum for another class of linear codes arising from higher dimensional projective varieties namely Grassmannians varieties is studied.

In this paper, we investigate the problem of determining the generalized spectrum of code associated with Schubert subvarieties of Grassmannians $G(2,5)$ over $\mathbb{F}_2$.

1.1 Outline of the paper

This paper is organized as follows. In section 2, we recall the basic definitions and properties of the linear code which are useful for the rest of the work. In section 3, we define the projective system and give the correspondence between codes and projective system. we briefly describe the codes associated with Grassmannians and Schubert subvarieties of Grassmannians. Finally, in section 4, we determine the generator matrix for the code associated with every Scubert subvariety of $G(2,5)$ over $\mathbb{F}_2$ and give the generalized spectrum of these codes.

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2 Linear Codes

2.1 Basic definitions

Let $\mathbb{F}_q$ denote the finite field with q elements, $q = p^h$, p a prime and denote by ${\mathbb{F}_q}^n$ the n-dimensional vector space over $\mathbb{F}_q$. For any $x \in {\mathbb{F}_q}^n$, the *support* of x, $supp(x)$, is the set of nonzero entries in $x = (x_1, x_2, \cdots, x_n)$. The *support weight* (or *Hamming norm*) of x is defined by,

$$||x|| = |supp(x)|.$$

More generally, if D is a subspace of ${\mathbb{F}_q}^n$, the *support* of D, $Supp(D)$ is the set of positions where not all the vectors in D are zero and the *support weight* (or *Hamming norm*) of D is defined by,

$$||D|| = |supp(D)|.$$

A *linear* $[n,k]_q$*-code* is a k-dimensional subspace of $\mathbb{F}_q^n$. The parameters n and k are referred to as the *length* and *dimension* of the corresponding code. The *minimum distance* $d = d(C)$ of C is defined by

$$d = d(C) = \min\{||x|| : x \in C, x \neq 0\}$$

More generally, given any positive integer r, the *rth higher weight* $d_r = d_r(C)$ is defined by

$$d_r = d_r(C) = \min\{||D|| : D \text{ is a subspace of C with } dimD = r\}.$$

Note that $d_1(C) = d(C)$. It also follows that $d_i \leq d_j$ when $i \leq j$ and that $d_k = |supp(C)|$, where k is dimension of code C. Thus we have $1 \leq d_1 = d < d_2 < \cdots < d_{k-1} < d_k = n$. The first weight d_1 is equal to the minimum distance and the last weight is equal to the length of the code. An $[n,k]_q$-code is said to be *nondegenerate* if it is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$. Two $[n,k]_q$-codes are said to be *equivalent* if one can be obtained from another by permuting coordinates and multying them by nonzero elements of $\mathbb{F}_q$. It is clear that this gives a natural equivalence relation on the set of $[n,k]_q$-codes. The *(usual) spectrum (or weight distribution)* of a code $C \subseteq \mathbb{F}_q^n$ is the sequence $\{A_0, A_1, \cdots, A_n\}$ defined by

$$A_i = A_i(C) = |\{c \in C : ||c|| \neq 0\}|.$$

More generally,the *rth higher weight spectrum (or rth support weight distribution)* of a code C is the sequence $\{A_0^r, A_1^r, \cdots, A_n^r\}$ defined by

$$A_i^r = |\{D \subseteq C : dimD = r, ||D|| = i\}| \tag{1}$$

This naturally allows us to define *rth support weight distribution function (or rth weight enumerator)* as

$$A^r(Z) = A_0^r + A_r^1 Z + \cdots + A_r^n Z^n \tag{2}$$

Hence for each $0 \leq r \leq k$, we have a weight enumerator. We can also define the *rth higher weight* as

$$d_r(C) = \min\{\mathrm{i} : \mathrm{A_i^r} \neq 0\}.$$

Note that $A^0(Z) = 1$. Also note that if $\bar{x} \in \mathbb{F}_q^n$, then

$$||x|| = ||\{\bar{x}\}|| = ||\{\lambda\bar{x} : \lambda \in \mathbb{F}_q\}||.$$

Lemma 1. *If C is a code with dimension k over $\mathbb{F}_2$ then for $Z = 1$*

$$A^r(1) = \begin{bmatrix} k \\ r \end{bmatrix}_2 \tag{3}$$

where $\begin{bmatrix} k \\ r \end{bmatrix}_2 = \frac{(2^k-1)(2^k-2)\cdots(2^k-2^{r-1})}{(2^r-1)(2^r-2)\cdots(2^r-2^{r-1})}$*, which is the number of subspaces of dimension r in a k dimensional space.*

2.2 Dual codes

The standard *inner product* on $\mathbb{F}_q^n$ is defined by $< x,y >:= \sum_{i=1}^n x_i y_i$.

Definition 1. *The Dual of a code* $C \subseteq \mathbb{F}_q^n$ *is the code*

$$C^\perp := \left\{ x \in \mathbb{F}_q^n :< x,c >= 0 \text{ for all } c \in \mathbb{F}_q^n \right\}.$$

Let $B^r(Z)$ be the rth support weight distribution function of the dual code $C^\perp$. In [4] Kløve gave the MacWilliams identity for the generalized spectrum of code C and its dual $C^\perp$,

Theorem 1. *[Generalized MacWilliams Identity] For all* $m \geq 0$ *we have*

$$\sum_{r=0}^{m} [m]_r B^r(Z) = q^{-mk}[1+(q^m-1)Z]^n \left\{ \sum_{r=0}^{m} [m]_r A^r \left(\frac{1-Z}{1+(q^m-1)Z} \right) \right\},$$

where $[m]_r = (q^m-1)(q^m-q)(q^m-q^2)\cdots(q^m-q^{r-1})$.

The number $[m]_r$ is known as the number of the ordered linear independent r-elements in the m-dimensional space. For $r = 1$, we can write the MacWilliams identity for usual spectrum in the following theorem.

Theorem 2.

$$1+(q-1)B^1(Z) = q^{-k}(1+(q-1)Z)^n \left\{ 1+(q-1)A^1 \left(\frac{1-Z}{1+(q-1)Z} \right) \right\}$$

3 Projective systems

An alternative way to describe codes is via the language of projective systems introduced in [12]. Let $\mathbb{P}^{k-1}$ be a projective space of dimension $k-1$ over $\mathbb{F}_q$. A $[n,k]_q$-*projective system* is a (multi)set X of n points in the projective space $\mathbb{P}^{k-1}$ over $\mathbb{F}_q$. We call X *nondegenerate* if these n points are not contained in any hyperplane of $\mathbb{P}^{k-1}$. Two $[n,k]_q$-projective systems are said to be equivalent if one can be obtained from another by a projective transformation. For any positive integer r, the *rth higher weight* of a projective system X is defined by

$$d_r = d_r(X) = n - \max\left\{ |X \cap \Pi| : \Pi \text{ is a subspace of } \mathbb{P}^{\mathrm{k}-1} \text{ of codimension r} \right\}.$$

The *generalized spectrum* of a projective system X is defined by,

$$A_i^r = A_i^r(X) = |\left\{ \Pi \subseteq \mathbb{P}^{k-1} : |X \cap \Pi| = n-i,\ codim\Pi = r \right\}|$$

for all $i = 1,2,\cdots,n$, $r = 1,2,\cdots k-2$. It can be proved that $A_i^r = A_i^r(C) = A_i^r(X)$.

For any $[n,k]_q$-linear code C, one can construct corresponding $[n,k]_q$-projective system in the following way: Consider coordinate forms $x_i : C \to \mathbb{F}_q$ such that

$$x_i : (v_1,\cdots,v_n) \mapsto v_i.$$

These forms can be considered as n points of the space C^* of linear functions on C (the dual linear space). If C is nondegenerate, that is, all forms x_i are nonzero as functions on C, then they define n points in $\mathbb{P}^{k-1} = p(C^*)$, or a projective system.

A subcode $D \subset C$ of dimension r correspond to the set of elements of C^* vanishing on D, that is, to the subspace $D^* \subset C^*$ of codimension r and, therefore, to a subspace of codimension r in $\mathbb{P}^{k-1}$. The weight of a subcode D equals to the number of coordinate forms not vanishing on it, that is, the number of points of X not lying on this subspace of codimension r. On the other hand, now we show how one can construct a linear code for a nondegenerate projective system. Given a projective system $X = \{P_1, P_2, \cdots, p_n\} \subset \mathbb{P}^{k-1} = \mathbb{P}(V)$, we lift it to a system $\{y_1, y_2, \cdots, y_n\}$ of vectors in V. Any y_i defines a mapping $V^* \to \mathbb{F}_q$, and the set $(y_1, y_2, \cdots, y_n)$ defines the mapping $V^* \to {\mathbb{F}_q}^n$, given by $(v_1, v_2, \cdots, v_n) \mapsto (y_1(v), y_2(v), \cdots, y_n(v))$ whose image is some linear code. Moreover it is an $[n,k]_q$-code if the projective system is nondegenerate.

The above correlation provides the proof for the following theorem (see [12]).

Theorem 3. *There is a one-to-one correspondence between the set of the equivalence classes of nondegenerate $[n,k]_q$-projective systems and the set of the equivalence classes of nondegenerate linear $[n,k]_q$-codes. This correspondence preserves the parameters n,k and the higher weights $d-1, d_2, \cdots, d_k$.*

The above correspondence in terms of generator matrix can be viewed as follows: Let G is a generator matrix for a $[n,k]_q$-linear code C, and let $g_1, g_2, \cdots, g_n \in {\mathbb{F}_q}^k$ be the columns of G. Suppose that none of the g_i's is the zero vector. then each g_i determines a point $[g_i]$ in the projective space $\mathbb{P}^{k-1} = \mathbb{P}\left({\mathbb{F}_q}^k\right)$. If these g_i are pairwise independent, then $X := \{[g_1], [g_2], \cdots, [g_n]\}$ is a set of n points in $\mathbb{P}^{k-1}$. This will be the corresponding projective system. Thus the n columns of G determines a projective system X. Vice versa, If X is a projective system, then a generator matrix for C is the $k \times n$ matrix whose columns are the representatives of points in projective system X.

3.1 Codes from Grassmannians

The Grassmannians $G(\ell,m)$ is the set of ℓ-dimensional subspaces of an m-dimensional vector space V over $\mathbb{F}_q$. We have the well-known Plücker embedding of the Grassmannian into a projective space (cf.[1]), and this embedding is known to be nondegenerate. Considering the $\mathbb{F}_q$-rational points of $G(\ell,m)$ as a projective system, we obtain a q-ary linear code, called the *Grassmann code*, which we denote by $C(\ell,m)$. These codes were first studied by Ryan [10, 11] in the binary case and by Nogin [8] and Ghorpade and Lachaud [1] in the q-ary case. It is clear that the length n and the dimension k of $C(\ell,m)$ are given by,

$$n = \begin{bmatrix} m \\ l \end{bmatrix}_q = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})} \quad \text{and} \quad k = \binom{m}{\ell} \tag{1}$$

The higher weights of $(2,m)$ is given by the following elegant formula due to Hansen-Johnsen-Ranestad [6] and Ghorpade-Patil-Pillai [3].

Theorem 4. *For* $\mu = \max(\ell, m-\ell)+1$,

$$d_{\mu+1}(C(2,m)) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-\mu+1}. \tag{2}$$

and

$$d_{k-\mu+1}(C(2,m)) = n - (1+q+\cdots+q^{\mu-1}+q^2. \tag{3}$$

3.2 Codes from Schubert varieties

Ghorpade and Lachaud in [1] proposed the generalization of Grassmann codes as Schubert codes. The Schubert codes are indexed by the elements of the set

$$I(\ell,m) := \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell) \in \mathbb{Z} : 1 \le \alpha_1 < \cdots < \alpha_\ell \le m\}.$$

Given any $\alpha \in I(\ell,m)$, the corresponding *Schubert code* is denoted by $C_\alpha(\ell,m)$, and it is the code obtained from the projective system defined by the Schubert variety Ω_α in $G(\ell,m)$ with a nondegenerate embedding induced by the Plücker embedding. We define Ω_α as

$$\Omega_\alpha = \{W \in G(\ell,m) : \dim(W \cap A_{\alpha_i}) \geq i \text{ for } i = 1,2,\cdots,\ell\},$$

where A_j denotes the span of the first j vectors in a fixed basis of V, for $1 \leq j \leq m$. Ghorpade and Tsfasman in [2], determined the length n_α and dimension k_α of $C_\alpha(\ell,m)$. It was conjectured by Ghorpade in [1], that

$$d(C_\alpha(\ell,m)) = q^{\delta_\alpha} \tag{4}$$

where $\delta_\alpha := \sum_{i=1}^{\ell}(\alpha_i - i) = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell - \frac{\ell(\ell+1)}{2}$.

The complete weight hierarchy and second support weight distribution of codes associated with all Schubert subvarieties of $G(2,4)$ is known due to Patil ([9]). In Next section, we give the second support weight distribution of all the codes associated with Schubert subvarieties of Grassmannians $G(2,5)$ over $\mathbb{F}_2$.

4 Codes associated with Schubert varieties in $G(2,5)$ over $\mathbb{F}_2$

Let $I(2,5)$ be an indexing set defined by,

$$I(2,5) := \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$$

Now by definition given any $\alpha \in I(\ell,m)$, the Schubert variety is defined by,

$$\Omega_\alpha := \left\{P \in G(\ell,m) : p_\beta = 0 \quad \forall \quad \beta \neq \alpha\right\}.$$

We consider Schubert varieties for each α above and the codes associated with them.

(I) Code associated with the Schubert variety for $\alpha = (2,4)$:

By definition,

$$\Omega_{(2,4)} = \{P \in G(2,5) : p_{15} = p_{24} = p_{25} = p_{34} = p_{35} = p_{45} = 0\}.$$

Dimension of $\Omega_{(2,4)} = \delta_{24} = 8$, where $\delta_{24} = 2+4-3 = 3$. Now

$$P \in G(2,5) \Rightarrow P = (p_{12}, p_{13}, p_{14}, p_{15}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{45}) \in G(2,5) \hookrightarrow \mathbb{P}^9.$$

So,

$$P \in \Omega_{(2,4)} \Rightarrow P = (p_{12}, p_{13}, p_{14}, 0, p_{23}, p_{24}, 0, 0, 0, 0).$$

The projective system consists of $\mathbb{F}_2$-rational points of $\Omega_{(2,4)}$. The number of rational points on $\Omega_{(2,4)}$is given by,

$$n = \sum_{\beta \leq \alpha} q^{\delta_\beta} = 2^{1+2-3} + 2^{1+3-3} + 2^{1+4-3} + 2^{2+3-3} + 2^{2+4-3} = 1+2+4+4+8 = 19.$$

These points are listed below:

$$\begin{array}{lll}
P_1=(1,0,0,0,0,0,0,0,0,0) & P_8=(0,1,0,0,1,0,0,0,0,0) & P_{15}=(1,0,1,0,0,1,0,0,0,0)\\
P_2=(0,1,0,0,0,0,0,0,0,0) & P_9=(0,1,1,0,0,0,0,0,0,0) & P_{16}=(1,0,0,0,1,1,0,0,0,0)\\
P_3=(0,0,1,0,0,0,0,0,0,0) & P_{10}=(0,1,1,0,1,1,0,0,0,0) & P_{17}=(1,0,0,0,0,1,0,0,0,0)\\
P_4=(0,0,0,0,1,0,0,0,0,0) & P_{11}=(1,1,1,0,0,0,0,0,0,0) & P_{18}=(1,0,0,0,1,0,0,0,0,0)\\
P_5=(0,0,0,0,0,1,0,0,0,0) & P_{12}=(1,1,1,0,1,1,0,0,0,0) & P_{19}=(1,0,1,0,0,0,0,0,0,0)\\
P_6=(0,0,0,0,1,1,0,0,0,0) & P_{13}=(1,1,0,0,1,0,0,0,0,0) & \\
P_7=(0,0,1,0,0,1,0,0,0,0) & P_{14}=(1,1,0,0,0,0,0,0,0,0) &
\end{array}$$

$\dim(C)=\#\{\beta:\beta\le(2,4)\}=\#\{(1,2),(1,3),(1,4),(2,3),(2,4)\}=5\text{length}(C)=\text{Number of } \mathbb{F}_2-\text{rational points}=19.$

Thus the generator matrix for C is of order 5×19 which is given by,

$$\begin{matrix}
1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&1&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&0&0&0&1&0&1&1&1&1&0&0&1&0&0&0&1\\
0&0&0&1&0&1&0&1&0&1&0&1&1&0&0&1&0&1&0\\
0&0&0&0&1&1&1&0&0&1&0&1&0&0&1&1&1&0&0
\end{matrix}$$

We have the following theorem on second higher spectrum for these codes.

Theorem 5. $A_{12}^2=6;\quad A_{14}^2=90,\quad A_{16}^2=57;\quad A_{18}^2=2,\quad A_i^2=0,\ (\text{otherwise.})$

we have also verified these calculations with the following formula,

$$\sum_{i=0}^{n}A_i^2=\begin{bmatrix}5\\2\end{bmatrix}_2=\frac{31\times30}{3\times2}=155.$$

(II) Code associated with the Schubert variety for $\alpha=(2,5)$:
By definition,

$$\Omega_{(2,5)}=\{P\in G(2,5):p_{34}=p_{35}=p_{45}=0\}.$$

Dimension of $\Omega_{(2,5)}=\delta_{25}=4$, where $\delta_{25}=2+5-3=4$.
Thus,

$$P\in\Omega_{(2,5)}\Rightarrow P=(p_{12},p_{13},p_{14},p_{15},p_{23},p_{24},p_{25},0,0,0).$$

The projective system of $\Omega_{(2,5)}$ consists of $\mathbb{F}_2$-rational points on $\Omega_{(2,5)}$. The number of these rational points are given by,

$$n=\sum_{\beta\le\alpha}q^{\delta_\beta}=2^{1+2-3}+2^{1+3-3}+2^{1+4-3}+2^{2+3-3}+2^{2+4-3}=1+2+4+4+8=19.$$

These points are listed in the following matrix in columns.

$$\begin{matrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1&1&1&1&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&1&0&1&1&0&1&1&0&0&1&1&0&1&0&1&0&1&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&1&1&0&0&0&0&1&0&0&0&1&1&1&1&0&1&0&1&1&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0\\
0&0&0&0&0&1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&1&1&0&1&0&0&0&0&0&1&1&0&1&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&1&1&1&0&1&1&0&0&1&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&1&1&1&1&1&0&0&0&1&0&0&0&0
\end{matrix}$$

Note that this is the generator matrix for code associated with $\Omega_{(2,5)}$. Its dimension is 7 and length is 43. Thus, we have the following theorem on second higher spectra of these codes.

Theorem 6.

$$A_{24}^2=28;\quad A_{28}^2=126,\quad A_{30}^2=672;\quad A_{32}^2=315,\quad A_{34}^2=1344;\quad A_{36}^2=182,\quad A_i^2=0;(\text{otherwise.})$$

we have also verified these calculations with the following formula,

$$\sum_{i=0}^{n} A_i^2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}_2 = \frac{127 \times 126}{3 \times 2} = 2667.$$

$$\begin{aligned} \text{That is, } A_{24}^2 + A_{28}^2 + A_{30}^2 + A_{32}^2 + A_{34}^2 + A_{36}^2 =& 2667 \\ 28 + 126 + 672 + 315 + 1344 + 182 =& 2667 \\ 2667 =& 2667. \end{aligned}$$

(III) Code associated with the Schubert variety for $\alpha = (3,4)$:
By definition,

$$\Omega_{(3,4)} = \{P \in G(2,5) : p_{35} = p_{45} = 0\}.$$

Dimension of $\Omega_{(3,4)} = \delta_{34} = 4$, where $\delta_{34} = 3+4-3 = 4$. Thus,

$$P \in \Omega_{(3,4)} \Rightarrow P = (p_{12}, p_{13}, p_{14}, 0, p_{23}, p_{24}, 0, p_{34}, 0, 0).$$

The projective system of $\Omega_{(3,4)}$ consists of $\mathbb{F}_2$-rational points on $\Omega_{(3,4)}$. The number of these rational points are given by,

$$n = \sum_{\beta \leq \alpha} q^{\delta_\beta} = 35.$$

These points are listed in the following matrix in columns.

$$\begin{array}{ccccccccccccccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&1&1&0&1&0&1&0&1&1&1&1&1&1&0&0&0&0&1&1&1&0&0&0&1\\
0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&1&1&1&0&0&1&1&1&0&1&0&1&1&0&1&0&1&0\\
0&0&0&0&1&0&1&1&0&1&0&1&1&0&0&0&0&0&1&1&0&1&0&1&1&1&0&0&1&0&1&1&1&0&0\\
0&0&0&0&0&1&1&0&1&1&1&0&1&0&0&1&1&1&0&1&0&1&1&0&1&1&0&0&1&1&0&0&0&0&0
\end{array}$$

Note that this is the generator matrix for code associated with $\Omega_{(3,4)}$. Its dimension is 6 and length is 35. Hence, we have the following theorem.

Theorem 7.

$$A_{24}^2 = 105; \quad A_{26}^2 = 280, \quad A_{28}^2 = 210; \quad A_{30}^2 = 56, \quad A_i^2 = 0; \text{ (otherwise.)}$$

we have also verified these calculations with the following formula,

$$\sum_{i=0}^{n} A_i^2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}_2 = \frac{63 \times 62}{3 \times 2} = 651.$$

$$\begin{aligned} \text{That is, } A_{24}^2 + A_{26}^2 + A_{28}^2 + A_{30}^2 &= 651 \\ 105 + 280 + 210 + 56 &= 651 \\ 651 &= 651. \end{aligned}$$

(IV) Code associated with the Schubert variety for $\alpha = (3,5)$:
By definition,

$$\Omega_{(3,5)} = \{P \in G(2,5) : p_{45} = 0\}.$$

Dimension of $\Omega_{(3,4)} = \delta_{35} = 4$, where $\delta_{35} = 3+5-3 = 5$. Thus,

$$P \in \Omega_{(3,5)} \Rightarrow P = (p_{12}, p_{13}, p_{14}, p_{15}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, 0).$$

The projective system of $\Omega_{(3,5)}$ consists of $\mathbb{F}_2$-rational points on $\Omega_{(3,5)}$. The number of these rational points are given by,

$$n = \sum_{\beta \leq \alpha} q^{\delta_\beta} = 91.$$

Note that the generator matrix for code associated with $\Omega_{(3,5)}$ is of order 9×91. Its dimension is 9 and length is 91. Hence, we have the following theorem for these codes.

Theorem 8.

$$A^2_{48} = 28; \quad A^2_{56} = 630, \quad A^2_{60} = 1792; \quad A^2_{64} = 7539, \quad A^2_{68} = 16128; \quad A^2_{72} = 17318, \quad A^2_i = 0; \text{ (otherwise.)}$$

we have also verified these calculations with the following formula,

$$\sum_{i=0}^{n} A_i^2 = \begin{bmatrix} 9 \\ 2 \end{bmatrix}_2 = \frac{511 \times 510}{3 \times 2} = 43435.$$

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Ghorpade, S. R., Lachaud, G., Higher weights of Grassmann codes, Coding Theory, Cryptography and Related Areas (J. Buchmann, T. Hoeholdt, H. Stictenoth, and H. Tapia - Resillas, Eds.) Springer-Verlag, Heidelberg, G Germany, 120-131, (2000).
[2] Ghorpade, S. R., Tsfasman, M. A., Schubert varieties, linear codes and enumerative combinatorics, Finite Fields and Their Applications. **vol. 11, No.4**, pp.684-699,(2005).
[3] Ghorpade, S. R., Patil A. R., Pillai H. Decomposable subspaces, linear sections of Grassmann varieties, and higher weights of Grassmann codes, Finite Fields and Their Applications **vol. 15, No. 1**, pp.54-68,(2009).
[4] Kløve, Torleiv, Support weight distribution of linear codes, Discrete Mathematics **106/107**, 311-316, (1992).
[5] Kløve, Torleiv, The weight distribution of linear codes over $GF(q^\ell)$ having generator matrix over $GF(q^*)$, Discrete Mathematics **23**, 159-168, (1978).
[6] Hirschfeld, J. W. P., Tsfasman, M.A., Vlăduţ, S. G., The weight hierarchy of higher-dimensional Hermitian codes, IEEE Trans. Inform. Theory **40**, 275-278, (1994).
[7] J. P. Hansen, T. Johnsen, K. Ranestad, Schubert unions in Grassmann varieties, Finite Fields Appl.**13**, 738–750 (2007).
[8] Nogin, D. Yu., Codes associated to Grassmannians, Arithmetic, Geometry and Coding Theory R. Pellikan, M. Perret, S.G. Vlăduţ. Eds., 145-154, Walter de Gruyter, Berlin/ New York,(1996).
[9] Patil, A. R., Generalized Spectrum of Grassmann Codes, Proceedings of Asian Mathematical Conference 2005 (AMC 2005) held at NUS, Singapore.
[10] Ryan, C. T. An application of Grassmannian varieties to coding theory, Congr. Numer. **57**, 257-271, (1987).
[11] Ryan, C. T. Projective codes based on Grassmann varieties, Congr. Numer. **57**, 273-279, (1987).
[12] Tsfasman, M. A., Vlăduţ, S. G. Algebraic Geometric Codes, Kluwer, Amsterdam, (1991).
[13] Wei, V. K. Generalized Hamming weights for linear codes, IEEE Trans. Inform. Theory **37**, 1412-1418, (1991).