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Coefficient bounds for certain subclasses of *m*-fold symmetric bi-univalent functions

Fethiye Müge Sakar and Naci Taşar

Batman University, Batman, Turkey

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Abstract: We consider two new subclasses $S_{\sum_m}(\tau,\lambda,\alpha)$ and $S_{\sum_m}(\tau,\lambda,\beta)$ of $\sum_m$ consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U. Furthermore, we establish bounds for the coefficients of functions in these subclasses and several related classes are also considered. In addition to these, connections to earlier known results are presented.

Keywords: Analytic, bi-univalent, m-fold symmetric, coefficient bounds.

1 Introduction

Let A denote the class of functions of the from

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

which are analytic in the open unit disk $U=\{z:|z|<\ 1\}$, and Let S be the subclass of A consisting of from (1) which is also univalent in U (for details, see [6]).

The Koebe one-quarter theorem [6] states that the image of U under every function f from S contains a disk of Radius $^1/_4$. Thus, every such univalent function has inverse f^{-1} which satisfies

$$f^{-1}(f(z))=z\,(z\in U)\,,f^{-1}(f(w))=w,\left(|w|<\ r_0(f),r_0\geq\frac{1}{4}\right), \tag{2}$$

where

$$f^{-1}(w)=w-a_2w^2+\left(2a_2{}^2-a_3\right)w^3-\left(5a_2{}^3-5a_2a_3+a_4\right)w^4+..... \tag{3}$$

A Function $f\in A$is said to be bi univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in unit disk U. For a brief history and interesting examples in class Σ, see [17]. Examples of functions in the class Σ are

$$\frac{z}{1-z},-\log(1-z)\,,\frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \tag{4}$$

and so on. However, the familiar Koebe functions is not a member of Σ. Other common examples of functions in such as

$$z-\frac{z^2}{2},\frac{z}{1-z^2}, \tag{5}$$

* Corresponding author e-mail: mugesakar@hotmail.com

are also not members of Σ(see [17]). For each function $f \in S$, function

$$h(z) = \sqrt[m]{f(z^m)}\,(z \in U, m \in \mathbb{N}) \tag{6}$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see[11],[16]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}\,(z \in U, m \in \mathbb{N}). \tag{7}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (7). In fact, the functions in the class S are one-fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (7) and the series expansion for f^{-1} , which has been recently proven by Srivastava et al. [18], is given as follows:

$$\begin{aligned} g(w) =& w - a_{m+1} w^{m+1} + \left[(m+1) a^2{}_{m+1} - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2) a^3{}_{m+1} - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \dots \end{aligned} \tag{8}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{1/m}, \left[-\log(1-z^m)\right]^{1/m}, \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{1/m}\right]. \tag{9}$$

Lewin [12] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \le \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [15] showed that $\max|a_2| = 4/3$if $f(z) \in \sum$. Brannan and Taha[4] introduced certain subclasses of b-iunivalent function class $\sum$ similar to the familiar subclasses. $S^*(\beta)$and $K^*(\beta)$ are of starlike and convex function order β $(0 \le \beta < 1)$, respectively (see[15]).

The classes $S^*_{\sum}(\alpha)$ and $K_{\sum}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of function classes $S^*{}_{\sum}(\alpha)$ and $K_{\sum}(\alpha)$, they found nonsharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [17] essentially revived the investigation of various subclasses of bi-univalent function class $\sum$ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see,[1],[2],[7],[8],[13], [17],[19]). Not much is known about the bounds on general coefficient $|a_n|$ for $n \ge 4$. In the literature, there are only a few works to determine general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see [5],[9],[10]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}/\{1,2\}; \mathbb{N} = \{1,2,3,...\})$ is still an open problem.

The aim of the this paper is to introduce two new subclasses of the function class $\sum_m$ and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. We have to remember the following lemma here so as to derive our basic results.

Lemma 1. [16]. *If $p \in P$, then*

$$|p_n| \le 2, \ \ (n \in \mathbb{N} = \{1,2,...\}) \ \ \text{and} \ \ \left|p_2 - \frac{p_1{}^2}{2}\right| \le 2 - \frac{|p_1|^2}{2}, \tag{10}$$

where the Carathèodary class P is the family of all functions p analytic in U for which

$$Re\{p(z)\} > 0, p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., (z \in U).$$

2 Coefficient bounds for function class $S_{\sum_m}(\lambda,\tau,\alpha)$

Definition 1. *A function $f \in \sum_m$ is said to be in the class $S_{\sum_m}(\tau,\lambda,\alpha)$, $(\tau \in C/\{0\},\ 0 < \alpha \leq 1,\ \ 0 \leq \lambda < 1)$ if the following conditions are satisfied:*

$$\left|\arg\left\{1+\frac{1}{\tau}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right\}\right| < \frac{\alpha\pi}{2}, z \in U \tag{11}$$

$$\left|\arg\left\{1+\frac{1}{\tau}\left(\frac{wg'(w)+\lambda w^2 g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right)\right\}\right| < \frac{\alpha\pi}{2}, w \in U \tag{12}$$

where the function $g = f^{-1}$.

Theorem 1. *Let fgiven by (7) be in the class $S_{\sum_m}(\tau,\lambda,\alpha)$, $0 < \alpha \leq 1$. Then,*

$$|a_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{2m(m+2m^2\lambda - m^2\lambda^2)\alpha\tau - (\alpha-1)m^2(1+m\lambda)^2}}, \tag{13}$$

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2\tau^2}{m^2(1+m\lambda)^2} + \frac{\alpha|\tau|}{m(1+2m\lambda)}. \tag{14}$$

Proof. Let $f \in S_{\sum_m}(\tau,\lambda,\alpha)$. Then we can write

$$1+\frac{1}{\tau}\left(\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right) = [p(z)]^{\alpha}, \tag{15}$$

$$1+\frac{1}{\tau}\left(\frac{wg'(w)+\lambda w^2 g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right) = [q(w)]^{\alpha}, \tag{16}$$

where $g = f^{-1}$ and p, qin P have the following forms:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + ..., q(w) = 1 + q_m w^m + q_{2m} w^{2m} + ... \tag{17}$$

Now , equating the coefficients (15) and (16) we get

$$\frac{1}{\tau}m(1+m\lambda)a_{m+1} = \alpha p_m, \tag{18}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda)a_{2m+1} - m(1+m\lambda)^2 a_{m+1}^2\right] = ap_{2m} + \frac{a(a-1)}{2}p^2{}_m, \tag{19}$$

$$-\frac{1}{\tau}m(1+m\lambda)a_{m+1} = \alpha q_m, \tag{20}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda)\left[(m+1)a_{m+1}^2 - a_{2m+1}\right] - m(1+m\lambda)^2 a_{m+1}^2\right] = aq_{2m} + \frac{a(a-1)}{2}q^2{}_m. \tag{21}$$

Form (18) and (20), we obtain

$$p_m = -q_m, \tag{22}$$

$$\frac{2}{\tau^2}m^2(1+m\lambda)^2 a_{m+1}^2=\alpha^2\left({p_m}^2+{q_m}^2\right). \tag{23}$$

Also from (19), (21) and (23) we have

$$\frac{1}{\tau}\left[a_{m+1}^2 2m\left[(1+2m\lambda)(m+1)-(1+m\lambda)^2\right]\right]=\alpha\left(p_{2m}+q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left({p_m}^2+{q_m}^2\right), \tag{24}$$

$$a_{m+1}^2=\frac{\alpha^2\tau^2\left(p_{2m}+q_{2m}\right)}{2m\left(m+2m^2\lambda-m^2\lambda^2\right)\alpha\tau-(\alpha-1)m^2(1+m\lambda)^2}. \tag{25}$$

Applying Lemma 1 for coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\frac{2\alpha|\tau|}{\sqrt{2m\left(m+2m^2\lambda-m^2\lambda^2\right)\alpha\tau-(\alpha-1)m^2(1+m\lambda)^2}}. \tag{26}$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (21) from (19), we obtain

$$\frac{1}{\tau}\left[2m(1+2m\lambda)\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)\right]=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left({p_m}^2-{q_m}^2\right). \tag{27}$$

Then, in view of (22) and (23) and applying Lemma 1 for coefficients p_m, p_{2m} and q_m, q_{2m} we have

$$|a_{2m+1}|\leq\frac{2(m+1)\alpha^2\tau^2}{m^2(1+m\lambda)^2}+\frac{\alpha|\tau|}{m(1+2m\lambda)}. \tag{28}$$

3 Coefficient bounds for function class $S_{\Sigma_m}(\lambda,\tau,\beta)$

Definition 2. *A function $f\in\Sigma_m$ given by (7) is said to be in class $S_{\Sigma_m}(\lambda,\tau,\beta)$, $(\tau\in C/\{0\},\ 0<\beta\leq1,\ 0\leq\lambda<1)$ if the following conditions are satisfied:*

$$Re\left(1+\frac{1}{\tau}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right)>\beta, z\in U, \tag{29}$$

$$Re\left(1+\frac{1}{\tau}\left(\frac{wg'(w)+\lambda w^2g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right)\right)>\beta, w\in U, \tag{30}$$

where the function $g=f^{-1}$.

Theorem 2. *Let given by (7) be in class textbf$S_{\Sigma_m}(\lambda,\tau,\beta)$, $0\leq\beta<1$. Then,*

$$|a_{m+1}|\leq\sqrt{\frac{2|\tau|(1-\beta)}{m\left(m+2m^2\lambda-m^2\lambda^2\right)}}, \tag{31}$$

$$|a_{2m+1}|\leq\frac{2(m+1)\tau^2(1-\beta)^2}{m^2(1+m\lambda)^2}+\frac{|\tau|(1-\beta)}{m(1+2m\lambda)}. \tag{32}$$

Proof. Let $f\in S_{\Sigma_m}(\lambda,\tau,\beta)$. Then we can write

$$1+\frac{1}{\tau}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)=\beta+(1-\beta)p(z), \tag{33}$$

$$1+\frac{1}{\tau}\left(\frac{wg'(w)+\lambda w^2g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right)=\beta+(1-\beta)q(w), \tag{34}$$

where $p,q\in P$ and $g=f^{-1}$. It follows from (33) , (34) that

$$\frac{1}{\tau}m(1+m\lambda)a_{m+1}=(1-\beta)p_m, \tag{35}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda)a_{2m+1}-m(1+m\lambda)^2a_{m+1}^2\right]=(1-\beta)p_{2m}, \tag{36}$$

$$-\frac{1}{\tau}m(1+m\lambda)a_{m+1}=(1-\beta)q_m, \tag{37}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]-m(1+m\lambda)^2a_{m+1}^2\right]=(1-\beta)q_{2m}. \tag{38}$$

From (35) and (37), we obtain

$$p_m=-q_m, \tag{39}$$

$$\frac{2}{\tau^2}m^2(1+m\lambda)^2a_{m+1}^2=(1-\beta)^2\left(p_m{}^2+q_m{}^2\right). \tag{40}$$

Adding (36) and (38), we have

$$\frac{1}{\tau}\left[2m(1+2m\lambda)(m+1)-2m(1+m\lambda)^2\right]a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{41}$$

Therefore, we obtain

$$a_{m+1}^2=\frac{\tau(1-\beta)(p_{2m}+q_{2m})}{2m(m+2m^2\lambda-m^2\lambda^2)}. \tag{42}$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\sqrt{\frac{2|\tau|(1-\beta)}{m(m+2m^2\lambda-m^2\lambda^2)}}. \tag{43}$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracintig (38) from (36), we obtain

$$\frac{1}{\tau}\left[4m(1+2m\lambda)a_{2m+1}-2m(1+2m\lambda)(m+1)a_{m+1}^2\right]=(1-\beta)(p_{2m}-q_{2m}). \tag{44}$$

Then , in view of (39) and (40), applying Lemma 1 for coefficients $p_m,\ \ p_{2m}$ and $q_m,\ \ q_{2m}$ we have

$$|a_{2m+1}|\leq\frac{2(m+1)\tau^2(1-\beta)^2}{m^2(1+m\lambda)^2}+\frac{|\tau|(1-\beta)}{m(1+2m\lambda)}. \tag{45}$$

This completes the proof of Theorem 2.

4 Coefficient bounds for function class $S_{\Sigma_m}(\lambda,\tau,\beta)$

Definition 3. [3] *Let $p_n(\beta)$ with $n \geq 2$ and $0 \leq \beta < 1$ denote the class of univalent analytic function p, normalized with $p(0) = 1$ and satisfying*

$$\int_0^{2\pi} \left| \frac{Re\, p(z) - \beta}{1-\beta} \right| d\theta \leq k\pi,$$

where $z = re^{i\theta}$. For $\beta = 0$, we denote, $p_n = p_n(0)$ hence the class p_n represents the class of functions $p(z)$, analytic in U, normalized with $p(0) = 1$ and having the representation

$$p(z) = \int_0^{2\pi} \frac{1-ze^{it}}{1+ze^{it}} du(t),$$

where u is α real valued function with bounded variation which satisfies

$$\int_0^{2\pi} du(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |du(t)| \leq n, \;\; n \geq 2.$$

Note that $p = p_2$ is the well known class of Caratheodory function (the normalized functions with positive real part in the open unit disk U).

Definition 4. *For $0 \leq \lambda \leq 1$ and $0 \leq \beta \leq 1$, a function $f \in \Sigma_m$ given by (1) is said to be in the class $S_{\Sigma_m}(\lambda,\tau,\beta)$, if the following two conditions are satisfied:*

$$1 + \frac{1}{\tau}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f''(z) + (1-\lambda) f(z)} - 1 \right) \in p_n(\beta), \tag{46}$$

$$1 + \frac{1}{\tau}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{\lambda w g'(w) + (1-\lambda) g(w)} - 1 \right) \in p_n(\beta), \tag{47}$$

where, $\tau \in \mathrm{C}/\{0\}$ the function $g = f^{-1}$ is given by (3) , and $z, w \in U$. In order to derive Theorem 3, we shall need the following lemma:

Lemma 2. [20]. *Let the function $\phi(z) = 1 + h_1 z + h_2 z^2 + \dots$; $z \in U$ such that $\phi \in p_n(\beta)$ then, $|h_k| \leq n(1-\beta)$; $k \geq 1$.*

Theorem 3. *If $f \in S_{\Sigma_m}(\lambda,\tau,\beta)$, then*

$$|a_{m+1}| \leq \min\left\{ \sqrt{\frac{n|\tau|(1-\beta)}{m(m+2m^2\lambda - m^2\lambda^2)}}, \frac{n|\tau|(1-\beta)}{m(1+m\lambda)} \right\}, \tag{48}$$

$$|a_{2m+1}| \leq \frac{(m+1)n|\tau|(1-\beta)}{2m(m+2m^2\lambda - m^2\lambda^2)}. \tag{49}$$

Proof. Since $f \in S_{\Sigma_m}(\lambda,\tau,\beta)$, from the definition relations (46) and (47) it follows that

$$\frac{1}{\tau} m(1+m\lambda) a_{m+1} = p_m, \tag{50}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda) a_{2m+1} - m(1+m\lambda)^2 a_{m+1}^2 \right] = p_{2m}, \tag{51}$$

$$-\frac{1}{\tau} m(1+m\lambda) a_{m+1} = q_m, \tag{52}$$

$$\frac{1}{\tau}\left[2m(1+2m\lambda)\left[(m+1)a_{m+1}^2-a_{2m+1}\right]-m(1+m\lambda)^2 a_{m+1}^2\right]=q_{2m}. \tag{53}$$

From (50) and (52), it follows that

$$a_{m+1}=\frac{\tau p_m}{m(1+m\lambda)}=\frac{-\tau q_m}{m(1+m\lambda)}, \tag{54}$$

and (51), (53) yields

$$a_{m+1}^2=\frac{\tau(p_{2m}+q_{2m})}{2m(m+2m^2\lambda-m^2\lambda^2)}. \tag{55}$$

Applying Lemma 2 for coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\min\left\{\sqrt{\frac{n|\tau|(1-\beta)}{m(m+2m^2\lambda-m^2\lambda^2)}},\frac{n|\tau|(1-\beta)}{m(1+m\lambda)}\right\}. \tag{56}$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (51) from (53), we obtain

$$\frac{1}{\tau}\left[2m(1+2m\lambda)\left(2a_{2m+1}-(m+1)a_{m+1}^2\right)\right]=p_{2m}-q_{2m}. \tag{57}$$

Then, in view of (54) and (55) and applying Lemma 2 for coefficients p_m, p_{2m} and q_m, q_{2m} we have

$$|a_{2m+1}|\leq\frac{(m+1)n|\tau|(1-\beta)}{2m(m+2m^2\lambda-m^2\lambda^2)}. \tag{58}$$

This completes the proof of Theorem 3.

5 Conclusions

If we set $\lambda=0$ and $\tau=1$ in Theorems 1 and 2, then the classes $S_{\Sigma_m}(\tau,\lambda,\alpha)$ and $S_{\Sigma_m}(\tau,\lambda,\beta)$ reduce to the classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ respectively. Thus we obtain the following corollaries.

Corollary 1. [2]. *Let f given by (7) be in the class $S_{\Sigma_m}^{\alpha}$ $(0<\alpha\leq 1)$. Then ,*

$$|a_{m+1}|\leq\frac{2\alpha}{m\sqrt{\alpha+1}}\quad and\quad |a_{2m+1}|\leq\frac{\alpha}{m}+\frac{2(m+1)a^2}{m^2}. \tag{59}$$

Corollary 2. [2]. *Let f given by (7) be in the class $S_{\Sigma_m}^{\beta}$ $(0\leq\beta<1)$. Then,*

$$|a_{m+1}|\leq\frac{\sqrt{2(1-\beta)}}{m}\quad and\quad |a_{2m+1}|\leq\frac{2(m+1)(1-\beta)^2}{m^2}+\frac{1-\beta}{m}. \tag{60}$$

Classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ are, respectively, defined as follows.

Definition 5. [2]. *A function $f\in S_{\Sigma_m}$ given by (7) is said to be in class $S_{\Sigma_m}^{\alpha}$ if the following conditions are satisfied:*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}, f\in\Sigma, (0<\alpha\leq 1, z\in U), \tag{61}$$

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}, g\in\Sigma, (0<\alpha\leq 1, w\in U), \tag{62}$$

where the function $g = f^{-1}$.

Definition 6. [2]. *A function* $f \in S_{\Sigma_m}$ *given by* (7) *is said to be in class* $S^{\beta}_{\Sigma_m}$ *if the following conditions are satisfied:*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, f \in \sum, \quad (0 \leq \beta < 1, \ z \in U), \tag{63}$$

$$Re\left(\frac{wg'(w)}{g(w)}\right) > \beta, g \in \sum, \quad (0 \leq \beta < 1, \quad w \in U), \tag{64}$$

where the function $g = f^{-1}$.

For one-fold symmetric bi-univalent functions and $\lambda = 0$, Theorems 2 and 3 reduce to Corollaries 6 and 7, respectively, which were proven earlier by Murugusundaramoorty et. al. [14].

Corollary 3. [14]. *Let* f *given by* (7) *b in class* $S^{*}_{\Sigma}(\alpha)$ $(0 < \alpha \leq 1)$. *Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}} \ \text{and} \ |a_3| \leq 4\alpha^2 + \alpha. \tag{65}$$

Corollary 4. [14]. *Let* f *given by* (7) *be in the class* $S^{*}_{\Sigma}(\beta)$ $(0 \leq \alpha < 1)$. *Then,*

$$|a_2| \leq \sqrt{2(1-\beta)} \ \text{and} \ |a_3| \leq 4(1-\beta)^2 + (1-\beta). \tag{66}$$

If we set $\lambda = 0$, $\lambda = 1$ *and* $\tau = 1$ *in Theorem 1, then the classes* $S_{\Sigma_m}(\tau, \lambda, \beta)$ *reduce to the class* $S^{\beta}_{\Sigma_m}$. *Thus, we obtain the following corollaries.*

Corollary 5. [20]. *If* $1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \in p_n(\beta)$ *and* $1 + \frac{1}{\tau}\left[\frac{wg'(w)}{g(w)} - 1\right] \in p_n(\beta)$ *then,*

$$|a_2| \leq \min\left\{\sqrt{n|\lambda|(1-\beta)}, \quad n|\tau|(1-\beta)\right\} \ \text{and} \ a_3 \leq n|\tau|(1-\beta).$$

Corollary 6. [20]. *If* $1 + \frac{1}{\tau}\left[\frac{zf''(z)}{f'(z)}\right] \in p_n(\beta)$ *and* $1 + \frac{1}{\tau}\left[\frac{wg''(w)}{g'(w)}\right] \in p_n(\beta)$ *then,*

$$|a_2| \leq \min\left\{\sqrt{\frac{n|\tau|(1-\beta)}{2}}, \quad \frac{n|\tau|(1-\beta)}{2}\right\} \ \text{and} \ |a_3| \leq \frac{n|\tau|(1-\beta)}{2}.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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