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On characterizations of s-topological vector spaces

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Abstract: In the paper [4], M. Khan et. al. introduced the s-topological vector spaces and studied several of their properties. In this paper, we continue their work and set forth some new properties of s-topological vector spaces.

Keywords: Semi-open sets, semi-closed sets, s-topological vector spaces.

1 Introduction

The notion of topological vector spaces is one of the most important tool in mathematics and due to nice properties, it earns a great importance in various branches of mathematics like fixed point theory, operator theory, variational inequalities, etc. In 2015, M. Khan et. al. [4] introduced the concept of s-topological vector spaces which is basically a generalization of topological vector spaces [6]. The main purpose of the present paper is to give some new properties and characterizations of s-topological vector spaces.

In 1963, N. Levine [7] introduced the notion of semi-open sets in topological spaces. He defines a set S in a topological space X to be semi-open if there exists an open set U in X such that $U \subseteq S \subseteq Cl(U)$; or equivalently, a subset S of X is semi-open if $S \subseteq Cl(Int(S))$, where $Int(S)$ and $Cl(S)$ denote the interior of S and the closure of S respectively. In [1], the authors define a set S in a topological space X is semi-closed if and only if its complement is semi-open; or equivalently, S is semi-closed in X if $Int(Cl(S)) \subseteq S$. The semi-closure of a subset S of X, denoted by $sCl(S)$, is the intersection of all semi-closed subsets of X containing S. In other words, the semi-closure of a subset S of X is the smallest semi-closed subset of X containing S. It is known that a set S in X is semi-closed if and only if $sCl(S) = S$. In [1], it is proved that $x \in sCl(S)$ if and only if $S \cap U \neq \emptyset$ for any semi-open set U in X containing x. A point $x \in X$ is called a semi-interior point of S if there exists a semi-open set U in X such that $x \in U \subseteq S$. The set of all semi-interior points of S, denoted by $sInt(S)$, is called semi-interior of A. Equivalently, $sInt(A)$ is the largest semi-open subset of A in X. The family of all semi-open (resp. semi-closed) sets in X is denoted by $SO(X)$ (resp. $SC(X)$). We represent the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. The notations ε and δ denote negligibly small positive real numbers.

Also, we recall some more definitions that will be used in the sequel.

Definition 1. *A subset A of a topological space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set U in X such that $x \in U \subseteq A$*

Definition 2. *A collection $\{U_\alpha : \alpha \in \Lambda\}$ of semi-open sets in a topological space X is called a semi-open cover of a subset B of X if $B \subseteq \cup\{U_\alpha : \alpha \in \Lambda\}$ holds.*

Definition 3. *A topological space X is called semi-compact [2] if every semi-open cover of X has a finite subcover. A subset B of a topological space X is said to be semi-compact relative to X if, for every collection $\{U_\alpha : \alpha \in \Lambda\}$ of semi-*

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open subsets of X such that $B \subseteq \cup\{U_\alpha : \alpha \in \Lambda\}$, there exists a finite subset Λ_0 of Λ such that $B \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. A subset B of a topological space X is said to be semi-compact if B is semi-compact as a subspace of X.

Definition 4. *A mapping $f : X \to Y$ from a topological space X to a topological space Y is called semi-continuous [7] if $f^{-1}(V)$ is semi-open in X, for each open set V in Y. In other words, f is semi-continuous if for each $x \in X$ and for each open neighborhood V of $f(x)$ in Y, there exists a semi-open neighborhood U of x in X such that $f(U) \subseteq V$.*

Definition 5. [4] *Let L be a vector space over the topological field K, where $K = \mathbb{R}$ or $\mathbb{C}$ with standard topology. Let τ be a topology on L such that the following two conditions are satisfied:*

(1) *For each $x, y \in L$ and each open neighborhood W of $x+y$ in L, there exist semi-open neighborhoods U and V of x and y respectively in L such that $U+V \subseteq W$ and*
(2) *For each $x \in L$, $\lambda \in K$ and each open neighborhood W of λx in L, there exist semi-open neighborhoods U of λ in K and V of x in L such that $U.V \subseteq W$. Then the pair $(L_{(K)}, \tau)$ is called an s-topological vector space.*

Now we present an example of s-topological vector spaces. Actually, this is an improvement of example 3.2 in the paper [4].

Example 1. Let $L = \mathbb{R}$ be the vector space of real numbers over the field K, where $K = \mathbb{R}$ with standard topology and the topology τ on L be generated by the base $\mathscr{B} = \{(a,b), [c,d) : a, b, c$ and d are real numbers with $0 < c < d\}$. We show that $(L_{(K)}, \tau)$ is an s-topological vector space. For which we have to verify the following two conditions:

(1) Let $x, y \in L$. Then, for open neighborhood $W = [x+y, x+y+\varepsilon)$ (resp. $(x+y-\varepsilon, x+y+\varepsilon)$) of $x+y$ in L, we can opt for semi-open neighborhoods $U = [x, x+\delta)$ (resp. $(x-\delta, x+\delta)$) and $V = [y, y+\delta)$ (resp. $(y-\delta, y+\delta)$) of x and y respectively in L such that $U+V \subseteq W$ for each $\delta < \frac{\varepsilon}{2}$.
(2) Let $x \in L$ and $\lambda \in K$. We have following cases:
Case (1). If $\lambda > 0$ and $x > 0$, then clearly $\lambda x > 0$. Consider an open neighborhood $W = [\lambda x, \lambda x + \varepsilon)$ (resp. $(\lambda x - \varepsilon, \lambda x + \varepsilon)$) of λx in L. We can choose semi-open neighborhoods $U = [\lambda, \lambda + \delta)$ (resp. $(\lambda - \delta, \lambda + \delta)$) of λ in K and $V = [x, x+\delta)$ (resp. $(x-\delta, x+\delta)$) of x in L such that $U.V \subseteq W$ for each $\delta < \frac{\varepsilon}{\lambda+x+1}$.
Case (II). If $\lambda < 0$ and $x < 0$, then $\lambda x > 0$. So, for open neighborhood $W = [\lambda x, \lambda x + \varepsilon)$ (resp. $(\lambda x - \varepsilon, \lambda x + \varepsilon)$) of λx in L, we can choose semi-open neighborhoods $U = (\lambda - \delta, \lambda]$ (resp. $(\lambda - \delta, \lambda + \delta)$) of λ in K and $V = (x - \delta, x]$ (resp. $(x-\delta, x+\delta)$) of x in L such that $U.V \subseteq W$ for sufficiently appropriate $\delta \le \frac{-\varepsilon}{\lambda+x-1}$.
Case (III). If $\lambda = 0$ and $x > 0$ (resp. $\lambda > 0$ and $x = 0$). Then $\lambda x = 0$. Therefore, for any open neighborhood $W = (-\varepsilon, \varepsilon)$ of 0 in L, we can opt for semi-open neighborhoods $U = (-\delta, \delta)$ (resp. $U = (\lambda - \delta, \lambda + \delta)$) of λ in $\mathbb{R}$ and $V = (x-\delta, x+\delta)$ (resp. $V = (-\delta, \delta)$) of x in L such that $U.V \subseteq W$ for each $\delta < \frac{\varepsilon}{x+1}$ (resp. $\delta < \frac{\varepsilon}{\lambda+1}$).
Case (IV). If $\lambda = 0$ and $x < 0$ (resp. $\lambda < 0$ and $x = 0$). Then, for the selection of semi-open neighborhoods $U = (-\delta, \delta)$ (resp. $U = (\lambda - \delta, \lambda + \delta)$) of λ in $\mathbb{R}$ and $V = (x-\delta, x+\delta)$ (resp. $V = (-\delta, \delta)$) of x in L, we have $U.V \subseteq W = (-\varepsilon, \varepsilon)$ for every $\delta < \frac{\varepsilon}{1-x}$ (resp. $(\delta < \frac{\varepsilon}{1-\lambda})$).
Case (V). If $\lambda = 0$ and $x = 0$. Consider any open neighborhood $W = (-\varepsilon, \varepsilon)$ of 0 in L, we can find semi-open neighborhoods $U = (-\delta, \delta)$ of λ in $\mathbb{R}$ and $V = (-\delta, \delta)$ of x in L, we have $U.V \subseteq W$ for each $\delta < \sqrt{\varepsilon}$.

Thus, $(L_{(\mathbb{R})}, \tau)$ is an s-topological vector space.

2 Characterizations

Henceforth, an s-topological vector space L means an s-topological vector space $(L_{(K)}, \tau)$ and by a scalar, we mean an element of K. Now we obtain some useful characterizations of s-topological vector spaces.

Theorem 1. *Let A be any subset of an s-topological vector space L. The following assertions are valid:*

(a) *$x + sCl(A) \subseteq Cl(x+A)$ for each $x \in L$.*
(b) *$sCl(x+A) \subseteq x + Cl(A)$ for each $x \in L$.*

(c) $x+Int(A) \subseteq sInt(x+A)$ *for each* $x \in L$.
(d) $Int(x+A) \subseteq x+sInt(A)$ *for each* $x \in L$.

Proof. (a) Let $z \in x+sCl(A)$. Then $z = x+y$ for some $y \in sCl(A)$. Let W be any open neighborhood of z. Then, by definition of s-topological vector spaces, there exist semi-open neighborhoods U of x and V of y in L such that $U+V \subseteq W$. Since $y \in sCl(A)$, there is $g \in A \cap V$. Now $x+g \in (x+A) \cap (U+V) \subseteq (x+A) \cap W \Rightarrow (x+A) \cap W \neq \emptyset$. Therefore, $z \in Cl(x+A)$. This proves that $x+sCl(A) \subseteq Cl(x+A)$.

(b) Assume $y \in sCl(x+A)$. Let $z = (-x)+y$ and let W be any open neighborhood of z in L. Since L is an s-topological vector space, there exist semi-open neighborhoods U and V of $-x$ and y respectively, in L such that $U+V \subseteq W$. Since $y \in sCl(x+A)$, $V \cap (x+A) \neq \emptyset$. So, there is $g \in V \cap (x+A)$. Now $-x+g \in (-x+x+A) \cap (U+V) = A \cap (U+V) \subseteq A \cap W$ showing that $A \cap W \neq \emptyset$. Hence $-x+y \in Cl(A)$; that is $y \in x+Cl(A)$. Therefore, $sCl(x+A) \subseteq x+Cl(A)$.

(c) Suppose that $y \in x+Int(A)$. Then $-x+y \in Int(A)$. Therefore, there exist semi-open sets U in L containing $-x$ and V in L containing y such that $U+V \subseteq Int(A)$. In particular, $-x+V \subseteq Int(A) \subseteq A$ and, as a consequence, $V \subseteq x+A$. Therefore, $y \in sInt(x+A)$. Hence $x+Int(A) \subseteq sInt(x+A)$.

(d) Suppose that $z \in Int(x+A)$. Then $z = x+y$ for some $y \in A$. By definition of s-topological vector spaces, there exist $U,V \in SO(L)$ such that $x \in U$, $y \in V$ and $U+V \subseteq Int(x+A)$. Consequently, $z = x+y \in x+V \subseteq x+sInt(A)$. Therefore, $Int(x+A) \subseteq x+sInt(A)$.

Remark. The reverse inclusions do not hold in any part of the theorem above. We account for part (a) of this theorem. Counterexamples for other parts follow analogously. Consider example 1, take $A = (0,1)$ and $x = 1$ in L. Then $Cl(x+A) = [1,2)$ but $x+sCl(A) = (1,2)$. Hence the inclusion $Cl(x+A) \subseteq x+sCl(A)$ fails to hold in L.

If we notice carefully, we observe that Theorem 1 is completely based on the first condition of the definition of s-topological vector spaces. The analog of this theorem which is based on the second condition of the same definition is the following result:

Theorem 2. *Let A be any subset of an s-topological vector space L. Then the following statements hold:*

(a) $\lambda sCl(A) \subseteq Cl(\lambda A)$ *for every non-zero scalar* λ.
(b) $sCl(\lambda A) \subseteq \lambda Cl(A)$ *for every non-zero scalar* λ.
(c) $Int(\lambda A) \subseteq \lambda sInt(A)$ *for every non-zero scalar* λ.
(d) $\lambda Int(A) \subseteq sInt(\lambda A)$ *for every non-zero scalar* λ.

Theorem 3. *Let A be any open subset of an s-topological vector space L. Then* $x+A \subseteq Cl(Int(x+A))$ *for each* $x \in L$.

Proof. Straightforward.

Theorem 4. *Let F be any closed subset of an s-topological vector space L. Then* $Int(Cl(x+F)) \subseteq x+F$ *for each* $x \in L$.

Proof. Straightforward.

Definition 6. *A topological space X is said to be P-regular [5] if for each semi-closed set F and* $x \notin F$, *there exist disjoint open sets U and V such that* $x \in U$ *and* $F \subseteq V$.

Theorem 5. *Let L be a P-regular s-topological vector space. Then the algebraic sum of any semi-compact set and closed set in L is closed.*

Proof. let A be a semi-compact and F be a closed subset of L. We have to show that $A+F$ is closed set in L. For, let $x \notin A+F$. Then $\forall a \in A$, $x \notin a+F$. Since F is closed in L and translation of a closed set in s-topological vector spaces is semi-closed, we find that $a+F$ is semi-closed in L. As L is P-regular, there exist open sets U_a and V_a such that $x \in U_a$, $a+F \subseteq V_a$ and $U_a \cap V_a = \emptyset$. Therein we find that $a \in V_a - F$ and hence $A \subseteq \cup_{a \in A}(V_a - F)$. Since any union of semi-open sets is semi-open, by theorem 3.4 of [4], $V_a - F = \cup_{b \in F}(V_a - b)$ is semi-open set in L. Consequently, by semi-compactness of A, there exists a finite subset S of A such that $A \subseteq \cup_{x \in S}(V_x - F)$. Let $U = \cap_{x \in S} U_x$, then U is an open neighborhood of x such that $U \cap (A+F) = \emptyset$.

If $y \in U \cap (A+F)$, then $y \in U_x \cap V_x$ for some $x \in S$, which is a contradiction to the fact that $U_a \cap V_a = \emptyset$, for each $a \in A$. thereby we find that $x \notin Cl(A+F)$. This gives $Cl(A+F) = A+F$. Hence $A+F$ is closed in L.

Next, we investigate further properties of s-topological vector spaces by using their basic idea.

Theorem 6. *Let A be any subset of an s-topological vector space L. Then $sCl(x+sCl(A)) \subseteq x+Cl(A)$ for each $x \in L$.*

Proof. Let $y \in sCl(x+sCl(A))$. Consider $z = -x+y$ and let W be an open neighborhood of $-x+y$ in L. Then there exist semi-open sets U and V in L containing $-x$ and y respectively, such that $U+V \subseteq W$. Since $y \in sCl(x+sCl(A))$, $V \cap (x+sCl(A)) \neq \emptyset$. So, there is $g \in V \cap (x+sCl(A))$. Now $-x+g \in (-x+x+sCl(A)) \cap (U+V) \subseteq sCl(A) \cap W$ implies $sCl(A) \cap W \neq \emptyset$. Since W is semi-open, $A \cap W \neq \emptyset$. Consequently, $-x+y \in Cl(A)$, i.e. $y \in x+Cl(A)$. Hence $sCl(x+sCl(A)) \subseteq x+Cl(A)$.

Theorem 7. *Let A be any subset of an s-topological vector space L. Then $x+Int(A) \subseteq sInt(x+sInt(A))$ for each $x \in L$.*

Proof. Let $y \in x+Int(A)$. Then $-x+y \in Int(A)$ and consequently, there exist semi-open neighborhoods U and V of $-x$ and y respectively, in L such that $U+V \subseteq Int(A)$. Now $-x+V \subseteq U+V \subseteq Int(A) \subseteq sInt(A)$ implies that $V \subseteq x+sInt(A)$. Since V is semi-open in L, we have that $y \in V \subseteq sInt(x+sInt(A))$. Therefore, $x+Int(A) \subseteq sInt(x+sInt(A))$.

Comparing Theorem 6 with part (b) of Theorem 1 and Theorem 7 with part (c) of Theorem 1, we find that the former is a generalization and the later is an improvement of corresponding parts.

Theorem 8. *Let C be any semi-compact subset of an s-topological vector space L. Then $x+C$ is compact, for each $x \in L$.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $x+C$. Then $x+C \subseteq \cup\{U_\alpha : \alpha \in \Lambda\} \Rightarrow C \subseteq \cup\{-x+U_\alpha : \alpha \in \Lambda\}$. By [4, Theorem 3.4], $-x+U_\alpha$ is semi-open in L, for each $\alpha \in \Lambda$. Consequently, by semi-compactness of C, there exists a finite subset Λ_0 of Λ such that $C \subseteq \cup\{-x+U_\alpha : \alpha \in \Lambda_0\}$. Thereby we find that $x+C \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. Hence the assertion follows.

Theorem 9. *Let L be an s-topological vector space. Then scalar multiple of a semi-compact set is compact.*

Proof. Let C be a semi-compact set in L and $\lambda \neq 0$ be any scalar. We have to show that λC is compact. For, let $\{U_\alpha : \alpha \in \Lambda$, where Λ is an indexed set$\}$ be an open cover of λC. Then $\lambda C \subseteq \cup\{U_\alpha : \alpha \in \Lambda\}$. Thereby $C \subseteq \frac{1}{\lambda}.\cup\{U_\alpha : \alpha \in \Lambda\}$. This implies that $C \subseteq \cup\{\frac{1}{\lambda}U_\alpha : \alpha \in \Lambda\}$. By [4, Theorem 3.4], $\frac{1}{\lambda}U_\alpha$ is semi-open. Consequently, $\{\frac{1}{\lambda}U_\alpha : \alpha \in \Lambda\}$ is a semi-open cover of C. But C is semi-compact. Therefore, there exists a finite subset Λ_0 of Λ such that $C \subseteq \cup\{\frac{1}{\lambda}U_\alpha : \alpha \in \Lambda_0\}$. This gives $\lambda C \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$ showing that λC is compact. This completes the proof.

Remark. Let L and T be s-topological vector spaces over the field K. A mapping $f : L \to T$ is said to be linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for all $x, y \in L$ and $\alpha, \beta \in K$.

A mapping $f : L \to K$ is called linear functional if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for all $x, y \in L$ and $\alpha, \beta \in K$. The kernel of f is defined by $Ker(f) = \{x \in L : f(x) = 0\}$.

Theorem 10. *Let $f : L \to M$ be a linear map, where L is an s-topological vector space and M is a topological vector space. If f is continuous at origin, then f is semi-continuous everywhere.*

Proof. Suppose that f is continuous at origin. Let $0 \neq x \in L$ and V be an open neighborhood of $f(x)$ in M. Then $V - f(x)$ is an open neighborhood of zero in M because M is a topological vector space. By continuity of f at origin, there exists an open neighborhood U of zero in L such that $f(U) \subseteq V - f(x)$. By linearity of f, $f(x+U) \subseteq V$. Since L is an s-topological vector space, $x+U$ is semi-open neighborhood of x in L. Consequently, f is semi-continuous at x in L. Since x is an arbitrary non-zero element of L, it follows that f is semi-continuous at every non-zero element of L. Further, continuity of f at origin implies that f is semi-continuous at zero. Hence f is semi-continuous everywhere.

Corollary 1. *Let L be an s-topological vector space. Assume* $0 \neq f : L \to K$ *be a linear functional. If* f *is continuous at origin, then* f *is semi-continuous.*

Theorem 11. *Let* $f : L \to K$ *be a linear functional, where L is an s-topological vector space. If* f *is continuous at origin, then* $Ker(f)$ *is semi-closed in L.*

Proof. Suppose that f is continuous at origin. If $f = 0$, we are done. Assume $f \neq 0$. By virtue of corollary 1, f is semi-continuous. Since K ($\mathbb{R}$ or $\mathbb{C}$) is endowed with standard topology which is Hausdorff. Therefore, $\{0\}$ is closed in K. By semi-continuity of f, $f^{-1}(\{0\})$ is semi-closed in L. But $f^{-1}(\{0\}) = \{x \in L : f(x) = 0\} = Ker(f)$. Hence the assertion follows.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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