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Value distribution of higher order differential-difference polynomial of an entire function

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Abstract: This paper deals with value distribution of higher order differential-difference operators of an entire function and obtained results improve some classical results on differential polynomials and deduce results of K. Liu, T.B. Cao and X.L. Liu [17] as particular case of our results.

Keywords: Nevanlinna theory, differential-difference polynomial of a function, difference operator etc.

1 Introduction and main results

We adopt fundamental results and the standard notations of the Neavanlinna theory of meromorphic functions as explained in ([7], [11] and [20]). A meromorphic function f means meromorphic in the whole complex plane. If no poles occur, then f reduces to an entire function. Given a meromorphic function $f(z)$, recall that $\alpha(z) \not\equiv 0, \infty$ is a small function with respect to $f(z)$ if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$ and $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. The order $\rho(f)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

A polynomial $p(z)$ is called a Borel exceptional polynomial of $f(z)$ whenever

$$\lambda(f(z)-p(z)) = \limsup_{r\to\infty} \frac{\log^+ N\left(r, \frac{1}{f(z)-p(z)}\right)}{\log r} < \rho(f),$$

where $\lambda(f(z)-p(z))$ is the exponent of convergence of zeros of $f(z)-p(z)$. In this paper, we assume that c is a nonzero complex constant, n is a positive integer and k is a nonnegative integer, unless otherwise specified.

Recently the topic of distribution of values of differential polynomials of different types in the complex plane $\mathbb{C}$ has attracted many mathematicians, a number of papers have focused on the zeros of $f(z)$ and its derivatives can be found in ([10],[11] and [21]).

In 2017, K. Liu, T.B. Cao and X.L. Liu [17] investigated some classical results on the distribution of zeros for differential polynomials and differential-difference polynomials and obtained the following results.

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Theorem 1. *Let $f(z)$ be a transcendental entire function of finite order. If $n \geq 1$, $k \geq 0$ and*

$$N\left(r, \frac{1}{f}\right) = S(r, f),$$

then $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.

Theorem 2. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c. If $n \geq 1$, $k \geq 0$ and $N\left(r, \frac{1}{f}\right) = S(r, f)$, then $[f(z)^n \Delta_c f]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 3. *Let f(z) be a transcendental entire function of finite order. If $n \geq \frac{k}{2} + 1$, $k \geq 0$ and f has infinitely many multiorder zeros, then $[f(z)^n f(z+c)]^{(k)} - p(z)$, has infinitely many zeros, where $p(z)$ is a nonzero polynomial.*

Theorem 4. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c. If $n \geq \frac{k}{2} + 1$, $k \geq 0$ and f has infinitely many multiorder zeros, then $[f(z)^n (f(z+c) - f(z))]^{(k)} - p(z)$, has infinitely many zeros.*

Theorem 5. *Let $f(z)$ be a transcendental entire function of finite order, let $p(z)$ be a nonzero polynomial, and let $n \geq 1$, $k \geq 0$. If f has a Borel exceptional polynomial $q(z)$, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros except $f(z) = q(z) + Aq(z)e^{\alpha z}$, n=1, and $p(z) = [q(z)q(z+c)]^{(k)}$, where $e^{\alpha c} = -1$ and A is a nonzero constant.*

Theorem 6. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c and let $n \geq 1$, $k \geq 0$. If f has a Borel exceptional polynomial $q(z)$, then $[f(z)^n (f(z+c) - f(z))]^{(k)} - p(z)$ has infinitely many zeros, except the cases $f(z) = q(z) + he^{\alpha z}$, $n = 1$, and $p(z) = [q(z)(q(z+c) - q(z))]^{(k)}$, where $e^{\alpha c} = 1$.*

In this paper, above theorems are generalized for higher order differential-difference operators as follows.

Theorem 7. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c. If n and m are positive integers with $n \geq 1$, $k \geq 0$ and $N\left(r, \frac{1}{f}\right) = S(r, f)$, then $[f(z)^n \Delta_c^m f]^{(k)} - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

Remark. If $m = 1$ in Theorem 7, then Theorem 7 reduces to Theorem 2.

Theorem 8. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c. If n and m are positive integers with $n \geq \frac{k}{2} + 1$, $k \geq 0$ and f has infinitely many multiorder zeros, then $[f(z)^n \Delta_c^m f]^{(k)} - p(z)$, has infinitely many zeros, where $p(z)$ is a nonzero polynomial.*

Remark. If $m = 1$ in Theorem 8, then Theorem 8 reduces to Theorem 4.

Theorem 9. *Let $f(z)$ be a transcendental entire function of finite order, which is not a periodic function with period c and If n and m are positive integers with $n \geq 1$, $k \geq 0$. If f has a Borel exceptional polynomial $q(z)$, then $[f(z)^n \Delta_c^m f]^{(k)} - p(z)$ has infinitely many zeros, except the cases $f(z) = q(z) + h(z)e^{\alpha z}$, $n = 1$, and $p(z) = [q(z)(q(z+c) - q(z))]^{(k)}$, where $e^{\alpha c} = 1$, α is a nonzero constant and $h(z)$ is a nonzero entire function with $\rho(h) < \rho(f)$.*

Remark. If $m = 1$ in Theorem 9, then Theorem 9, reduces to Theorem 6.

2 Some lemmas

We need the following Lemmas to prove our results.

Lemma 1. *Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f).$$

Lemma 2. [6] *Let $f(z)$ be a transcendental meromorphic function of finite order. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 3. [20] *Let $f(z)$ be a transcendental meromorphic function and let n be a positive integer. Then*

$$T(r, f^{(n)}) \leq T(r,f) + n\overline{N}(r,f) + S(r,f).$$

Lemma 4. *Let $f(z)$ be a transcendental meromorphic function of finite order with $N(r,f) + N\left(r, \frac{1}{f}\right) = S(r,f)$ and $F(z) = f(z)^n \Delta_c^m f$, where n and m are positive integer. Then*

$$(n+1)T(r,f) + S(r,f) \leq T(r,F).$$

Proof. From the first fundamental theorem, Lemma 1 and the assumption, we obtain

$$(n+1)T(r,f(z)) = \quad T(r, f(z)^{n+1}) = T\left(r, \frac{f(z)F(z)}{\Delta_c^m f}\right) \leq T(r,F(z)) + T\left(r, \frac{f(z)}{\Delta_c^m f}\right) + S(r,f) \leq T(r,F(z)) + S(r,f).$$

Lemma 5. [2] *Let $g(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1, h > 0$. Then there exists an ε-set E such that*

$$\frac{g^{'}(z+c)}{g(z+c)} \to 0$$

and

$$\frac{g(z+c)}{g(z)} \to 1$$

as $z \to \infty$ in $\mathbf{C} \backslash E$, uniformly in c for $|c| \leq h$. Further, E can be chosen so that, for large $z \notin E$, the function g has no zeros or poles in $|\zeta - z| \leq h$.

Lemma 6. [21], *Lemma 1 Let f be a nonconstant meromorphic function and let $\alpha(z)$ be a small function of f such that $\alpha(z) \neq 0, \infty$. Then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - \alpha}\right) - N\left(r, \frac{1}{\left(\frac{f^{(k)}}{\alpha}\right)^{'}}\right) + S(r,f).$$

Lemma 7. [20], *Theorem 1.62 Let $f_j(z)$ be a meromorphic functions and let $f_k(z)$, $k= 1,2,...,n-1$, be not constant satisfying the relation*

$$\sum_{j=1}^{n} f_j = 1$$

with $n \geq 3$. *If* $f_n(z) \not\equiv 0$ *and*

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$, $k = 1, 2, ..., n-1$, *then* $f_n(z) \equiv 1$.

Lemma 8. [20], *Theorem* 1.51 *Let* $f_j(z)$, *j=1,2,...,n,* $n \geq 2$, *be meromorphic functions and let* $g_j(z)$ *j=1,2,...,n, be entire functions satisfying:*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0,$

(2) $g_j(z) - g_k(z)$ *is not constant for* $1 \leq j < k \leq n$,

(3) *for* $1 \leq j \leq n$, $1 \leq h < k \leq n, T(r, f_j) = o(T(r, e^{g_h - g_k}))(r \to \infty, r \notin E)$, *where* $E \subset (1, \infty)$ *is of finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0, j = 1, 2, ..., n$.

Lemma 9. [6], *Theorem* 9.2 *Let* $A_0(z), ..., A_n(z)$ *be entire functions for which there exists an integer* l, $0 \leq l \leq n$, *such that*

$$\rho(A_l(z)) > \max_{0 \leq l \leq n, j \neq l} \rho(A_j(z)).$$

If $f(z)$ *is a meromorphic solution of*

$$A_n(z)y(z + c_n) + ... + A_1(z)y(z + c_1) + A_0(z)y(z) = 0,$$

then

$$\rho(f) \geq \rho(A_l(z)) + 1.$$

Lemma 10. [5], *Theorem* 1.2 *Let* $P_0(z), ..., P_n(z)$ *be polynomials such that* $P_n(z)P_0(z) \not\equiv 0$ *satisfying the relation*

$$deg(P_n(z) + ... + P_0(z)) = \max\{degP_j(z) : j = 0, ..., n\} \geq 1.$$

Then every finite order meromorphic solution $f(z)(\not\equiv 0)$ *of*

$$P_n(z)f(z + c_n) + ... + P_1(z)f(z + c_1) + P_0(z)f(z) = 0$$

satisfies the inequality $\rho(f) \geq 1$.

Proof. **(Proof of theorem 7)** Let $F(z) = f(z)^n \Delta_c^m f$, that is $F(z) = f(z)^n \Delta_c^{m-1}(\Delta_c f)$. Since by hypothesis

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f).$$

We conclude that

$$N(r, F) + N\left(r, \frac{1}{F}\right) \leq N\left(r, \frac{1}{f(z)^n \Delta_c^m f}\right) \leq N\left(r, \frac{1}{f(z)^n}\right) + N\left(r, \frac{1}{\Delta_c^m f}\right) \leq T(r, \Delta_c^m f) + S(r, f)$$
$$\leq m(r, \Delta_c^m f) + N(r, \Delta_c^m f) + S(r, f) \leq m\left(r, \frac{\Delta_c^m f}{f}\right) + m(r, f) + S(r, f) \leq T(r, f) + S(r, f). \quad (1)$$

We have

$$\begin{aligned} T(r,F) &= T(r,f(z)^n\Delta_c^m f) \le T(r,f(z)^n) + T(r,\Delta_c^m f) + S(r,f) \le nT(r,f) + m(r,\Delta_c^m f) + N(r,\Delta_c^m f) + S(r,f) \\ &\le nT(r,f) + m\left(r,\frac{\Delta_c^m f}{f}\right) + m(r,f) + S(r,f) \le nT(r,f) + m(r,f) + S(r,f) \\ &\le nT(r,f) + T(r,f) + S(r,f) \le (n+1)T(r,f) + S(r,f). \end{aligned}$$

From the above inequality and Lemma 4, we get

$$(n+1)T(r,f) + S(r,f) = T(r,F).$$

Using, this with Lemma 6 and (1)

$$\begin{aligned} (n+1)T(r,f) + S(r,f) &\le \overline{N}(r,F) + N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{F^{(k)}-\alpha}\right) + S(r,F) \\ &\le N(r,F) + N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{F^{(k)}-\alpha}\right) + S(r,F) \\ &\le T(r,f) + N\left(r,\frac{1}{F^{(k)}-\alpha}\right) + S(r,f) \\ nT(r,f) &\le N\left(r,\frac{1}{F^{(k)}-\alpha}\right) + S(r,f). \end{aligned}$$

Since $n \ge 1$ we conclude that $[f(z)^n\Delta_c^m f]^{(k)} - \alpha(z)$ has infinitely many zeros.

Proof. (**Proof of theorem 8**) Let

$$F(z) = f(z)^n\Delta_c^m f.$$

Assume that $F^{(k)}(z) - p(z)$ has finitely many zeros. From Hadamard factorization theorem, we have

$$F^{(k)}(z) - p(z) = h(z)e^{q(z)}, \tag{2}$$

where $h(z)$ is a nonzero polynomial and $q(z)$ is a nonconstant polynomial, otherwise if $q(z) = A$, where A is a constant, then $F^{(k)}(z) - p(z) = h(z)e^A$. This implies that $F(z) = f(z)^n\Delta_c^m f$ is also a polynomial, which contradicts $f(z)$ is transcendental entire function. Differentiating (2), we get

$$F^{(k+1)}(z) - p'(z) = [h'(z) + h(z)q'(z)]e^{q(z)}. \tag{3}$$

Combining (2) with (3) and eliminating $e^{q(z)}$, we obtain

$$\frac{F^{(k+1)}(z)}{F^{(k)}(z)} = \frac{h'(z)+h(z)q'(z)}{h(z)} + \left[p'(z) - \frac{h'(z)+h(z)q'(z)}{h(z)}p(z)\right]\frac{1}{F^{(k)}(z)}. \tag{4}$$

We note that poles of $\dfrac{F^{(k+1)}(z)}{F^{(k)}(z)}$ on the left hand side of (4) must be simple. If f has infinitely many multiorder zeros and

$$n \ge \frac{k}{2} + 1,$$

then we can find z_0 which is a zero of f and not a zero of $h(z)$ and $p'(z)-\frac{h'(z)+h(z)q'(z)}{h(z)}p(z)$. Thus, the poles of right hand side of (4) must be multiorder, a contradiction.

Proof. (**Proof of theorem 9**) Assume that $\rho(f)=s$, where s is a positive integer. Then the transcendental entire function $f(z)$ can be represented as

$$f(z)=q(z)+h(z)e^{\alpha z^s} \tag{5}$$

where α is a nonzero constant and $h(z)$ is a nonzero entire function with

$$\lambda(h)\leq\rho(h)<\rho(f)=s.$$

It follows from (5) that

$$f(z+c)=q(z+c)+h(z+c)e^{\alpha(z+c)^s}=q(z+c)+h_1(z)e^{\alpha z^s},$$

where

$$h_1(z)=h(z+c)e^{\alpha(C_s^1z^{s-1}c+C_s^2z^{s-2}c^2+...+C_s^{s-1}zc^{s-1}+c^s)}. \tag{6}$$

Thus

$$f(z+c)-f(z)=q(z+c)-q(z)+(h_1(z)-h(z))e^{\alpha z^s}=q_1(z)+h_2(z)e^{\alpha z^s}.$$

We have

$$\Delta_c^m f= \quad \Delta_c^{m-1}(f(z+c)-f(z))=\Delta_c^{m-1}[q(z+c)-q(z)+(h_1(z)-h(z))e^{\alpha z^s}]=\Delta_c^{m-1}[q_1(z)+h_2(z)e^{\alpha z^s}], \tag{7}$$

where $q_1(z)=q(z+c)-q(z)$ and $h_2(z)=h_1(z)-h(z)$. Suppose that $[f(z)^n\Delta_c^m f]^{(k)}-p(z)$ has finitely many zeros. Thus, from the Hadamard factorization theorem, we obtain

$$[f(z)^n\Delta_c^m f]^{(k)}-p(z)=C(z)e^{\gamma z^s}.$$

This implies

$$[f(z)^n\Delta_c^{m-1}(f(z+c)-f(z))]^{(k)}-p(z)=C(z)e^{\gamma z^s}, \tag{8}$$

where $C(z)$ is an entire function with finitely many zeros of order $\rho(C)<s$ and γ is a nonzero constant.

Case 1. Let $k=0$ and $n=1$ in (8), we get

$$[f(z)\Delta_c^{m-1}(f(z+c)-f(z))]-p(z)=C(z)e^{\gamma z^s}. \tag{9}$$

Substituting $f(z)=q(z)+h(z)e^{\alpha z^s}$ in (9) we get

$$[q(z)+h(z)e^{\alpha z^s}][\Delta_c^{m-1}(q(z+c)+h(z+c)e^{\alpha(z+c)^s}-q(z)-h(z)e^{\alpha z^s})]-p(z)=C(z)e^{\gamma z^s}.$$

We get

$$h(z)[\Delta_c^{m-1}(h_1(z)-h(z))]e^{2\alpha z^s}+[h(z)(\Delta_c^{m-1}(q(z+c)-q(z)))+q(z)(\Delta_c^{m-1}(h_1(z)-h(z)))]e^{\alpha z^s}$$

$$=p(z)-q(z)[\Delta_c^{m-1}(q(z+c)-q(z))]+C(z)e^{\gamma z^s}. \tag{10}$$

Subcase 1.1. If $p(z)-q(z)[\Delta_c^{m-1}(q(z+c)-q(z))]\equiv 0$. If $h_1(z)=h(z)$, then $\alpha=\gamma$, follows from the equation presented above. It follows from (10) and Lemma 9 that $s=1$. Therefore $h(z)=h(z+c)e^{\alpha c}$. By using Lemma 5 we conclude that

$h(z)$ is a constant and $e^{\alpha c} = 1$. Thus, $f(z) = q(z) + he^{\alpha z}$, where $e^{\alpha c} = 1$.

If $h_1(z) \not\equiv h(z)$, then $\gamma = 2\alpha$. From Lemma 8, we obtain $[h(z)(\Delta_c^{m-1}(q(z+c) - q(z))) + q(z)(\Delta_c^{m-1}(h(z+c)e^{\alpha}(C_s^1 z^{s-1}c + ... + C_s^{s-1}zc^{s-1} + c^s) - h(z)))]e^{\alpha z^s} = 0$. By Lemma 9, we get $s = 1$. therefore,

$$[h(z)(\Delta_c^{m-1}(q(z+c) - q(z))) + q(z)(\Delta_c^{m-1}(h(z+c)e^{\alpha c} - h(z)))] = 0. \tag{11}$$

Since $\rho(h) < 1$ in Lemma 10, we see that either $q(z)$ must be constant or $e^{\alpha c} = 1$ if $q(z)$ is not a constant. If q(z) is a constant from Lemma 5 then $h(z)$ reduces a constant. Hence, $f(z)$ is a periodic function with period c, a contradiction.

If $e^{\alpha c} = 1$ and $q(z)$ is not a constant, in view of (10), we get $h(z)[\Delta_c^{m-1}(h(z+c) - h(z))] = C(z)$. Combining this with the inequality $\rho(h) < 1$ and the fact that $h(z)$ is an entire function, we obtain $\rho(h(z+c) - h(z)) < 1$. Therefore $C(z)$ must have infinitely many zeros and we arrive at the contradiction.

Subcase 1.2. If $p(z) - q(z)[\Delta_c^{m-1}(q(z+c) - q(z))] \not\equiv 0$, then

$$\begin{aligned}[h(z)(\Delta_c^{m-1}(h_1(z) - h(z))) - C(z)]e^{2\alpha z^s} + [h(z)(\Delta_c^{m-1}(q(z+c) - q(z))) + q(z)(\Delta_c^{m-1}(h_1(z) - h(z)))]e^{\alpha z^s} \\ = p(z) - q(z)[\Delta_c^{m-1}(q(z+c) - q(z))].\end{aligned} \tag{12}$$

Let

$$f_1(z) = \frac{[h(z)(\Delta_c^{m-1}(h_1(z) - h(z))) - C(z)]}{p(z) - q(z)[\Delta_c^{m-1}(q(z+c) - q(z))]} e^{2\alpha z^s}$$

and

$$f_2(z) = \frac{[h(z)(\Delta_c^{m-1}(q(z+c) - q(z))) + q(z)(\Delta_c^{m-1}(h_1(z) - h(z)))]}{p(z) - q(z)[\Delta_c^{m-1}(q(z+c) - q(z))]} e^{\alpha z^s}.$$

Thus, $f_1(z) + f_2(z) = 1$. It follows from the second main theorem that

$$T(r, f_1) \le N(r, f_1) + N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_1 - 1}\right) + S(r, f_1) \le N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_2}\right) + S(r, f_1) \le O(r^{s-1+\varepsilon}) + S(r, f_1),$$

which is contradiction with $\rho(f_1) = s$. Thus, $[f(z)^n \Delta_c^m f]^{(k)} - p(z)$ has infinitely many zeros.

Case 2. If $k \ge 1$, then it follows from (7) and (8) that

$$[(q(z) + h(z)e^{\alpha z^s})^n (\Delta_c^{m-1}(q_1(z) + h_2(z)e^{\alpha z^s}))]^{(k)} - C(z)e^{\gamma z^s} = p(z).$$

This yields

$$\begin{aligned}[q(z)^n \Delta_c^{m-1}(q_1(z)) + \Delta_c^{m-1}(D_1(z)e^{\alpha z^s} + ... + D_j(z)e^{j\alpha z^s} + ... + D_n(z)e^{n\alpha z^s} \\ + h(z)h_2(z)e^{(n+1)\alpha z^s})]^{(k)} - C(z)e^{\gamma z^s} = p(z),\end{aligned} \tag{13}$$

where

$$D_j(z) = C_n^j q(z)^{n-j} q(z+c)h(z)^j + C_n^{j-1} q(z)^{n-j+1} h_1(z)h(z)^{j-1}$$

and $\rho(D_j(z)) < s, j = 1,...,n$. For any integer k, from (13), we obtain

$$F_1(z)e^{\alpha z^s} + ... + F_j(z)e^{j\alpha z^s} + ... + F_n(z)e^{n\alpha z^s} + F_{n+1}(z)e^{(n+1)\alpha z^s} - C(z)e^{\gamma z^s} = p(z) - [q(z)^n \Delta_c^{m-1}(q_1(z))]^{(k)}, \quad (14)$$

where $F_j(z)$ are differential polynomials of $h(z), h_1(z), q(z)$ and $q(z+c)$, and their powers of derivatives and in addition, $\rho(F_j(z)) < s, j = 1,...,n+1$. In what follows we consider two cases.

Subcase 2.1. If $p(z) - [q(z)^n \Delta_c^{m-1}(q_1(z))]^{(k)} \equiv 0$ in Lemma 8, then all $F_j(z) \equiv 0, j = 1,2,...n$, and $F_{n+1}(z) - C(z) \equiv 0$. We state that $n = 1$; otherwise, let $n \geq 2$. If $k = 1$ in $F_1(z) \equiv 0$, then we get

$$D_1^{'}(z) + \alpha s z^{s-1} D_1(z) = 0,$$

which gives the nontrivial solution $D(z)$ of the first-order differential equation presented above satisfying the equality $\rho(D(z)) = s$ in contradiction with the condition $\rho(D(z)) < s$. Thus, $D_1(z) \equiv 0$.
For $k = 2$, let

$$g(z) = D_1^{'}(z) + \alpha s z^{s-1} D_1(z) = 0.$$

Thus we have $g^{'}(z) + \alpha s z^{s-1} g(z) = 0$, which also implies that $\rho(g(z)) = s$ in contradiction with the condition $\rho(g(z)) = \rho(D(z)) < s$. By using this method for any positive integer k, we conclude that $D_1(z) \equiv 0$, that is,

$$C_n^1 q(z)^{n-1} q(z+c) h(z) + q(z)^n h(z+c) e^{\alpha(C_s^1 z^{s-1} c + ... + C_s^{s-1} z c^{s-1} + c^s)} \equiv 0.$$

From Lemma 9, we get $s = 1$, which implies that

$$C_n^1 q(z)^{n-1} q(z+c) h(z) + q(z)^n h(z+c) e^{\alpha c} \equiv 0. \quad (15)$$

In view of Lemma 10, since $\rho(h(z)) < 1$, the degree of $C_n^1 q(z)^{n-1} q(z+c) + q(z)^n e^{\alpha c}$ must be smaller than the degree of $q(z)^n$. Hence, we arrive at the equality

$$e^{\alpha c} = -C_n^1 \quad (16)$$

provided that $q(z)$ is not a constant. By using the same arguments as above, we also get $D_2(z) \equiv 0$, that is

$$C_n^2 q(z)^{n-2} q(z+c) h(z)^2 + C_n^1 q(z)^{n-1} h(z+c) e^{\alpha c} h(z) \equiv 0.$$

As above, if $q(z)$ is not a constant, then we find

$$C_n^1 e^{\alpha c} = -C_n^2. \quad (17)$$

It follows from (16) and (17) that $n = 0$; a contradiction. Thus, we have $n = 1$. By using (13) and Lemmas 8 and 7, we obtain $D_1(z) \equiv 0$. Hence (15) is equivalent to (11). Therefore, we get

$$f(z) = q(z) + h e^{\alpha z},$$

where $e^{\alpha c} = 1$.

Subcase 2.2. If $p(z) - [q(z)^n \Delta_c^{m-1}(q_1(z))]^{(k)} \not\equiv 0$, then we combine $\rho(F_j(z)) < s$ and $\rho(C(z)) < s$ with (14) and Lemma 7 we obtain

$$[F_{n+1}(z) - C(z)] e^{(n+1)\alpha z^s} = p(z) - [q(z)^n \Delta_c^{m-1}(q_1(z))]^{(k)}$$

or

$$F_j(z) e^{j\alpha z^s} = p(z) - [q(z)^n \Delta_c^{m-1}(q_1(z))]^{(k)}$$

which is impossible. Thus, $[f(z)^n \Delta_c^m f]^{(k)} - p(z)$ has infinitely many zeros.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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