New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2019.330

An approximate solution of equations characterizing spacelike curves of constant breadth in minkowski 3-space

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Received: 30 March 2018, Accepted: 3 November 2018
Published online: 25 December 2018.

Abstract: In this study, we first show that the system of Frenet-like differential equation characterizing spacelike curves of constant breadth is equivalent to a third order, linear, differential equation with variable coefficients. Then, by using a rational approximation based on Bernstein polynomials, we obtain the set of solution of the mentioned differential equation under the given initial conditions. Furthermore, we discuss that the obtained results are useable to determine spacelike curves of constant breadth in Minkowski 3-space E_1^3.

Keywords: Spacelike curve of constant breadth; Bernstein polynomials; Linear differential equations, Minkowski 3-space.

1 Introduction

The curves of constant breadth firstly were introduced by Euler in 1778 [5]. Reuleaux gave some methods of obtaining curves of constant breadth in 1971, which led to the use of these curves in kinematic of machinery [15]. So far, many scientists working on geometry have obtained only the geometrical properties of curves of constant breadth in plane, but little work has been done on space curves of constant breadth. [1, 2, 11, 12]. A number of interesting properties of these curves in plane are included in the works of Mellish [13]. Fujivara obtains a problem to determine whether there exist “space curve of constant breadth” or not and he defines “breadth” for space curves and obtains these curves on a surface of constant breadth [6]. By using the basic concepts concerned with the space curves of constant breadth [11], an integral characterization of these curves [4, 17] is obtained and a criterion for these curves is determined [16]. The curves of constant breadth are extended to E^4- space and some characterizations are obtained [12]. In addition, Akdoğan and Mağden [1] extend to E^n-space this kind of curves and they obtain some characterizations. Also Aydın [2] obtains differential equation characterizing curves of constant breadth in E^n and then she obtains approximate solutions of this equation using Taylor matrix collocation method. Studies in different spaces [14, 18] on these curves are going on nowadays, currently. These curves are used in the kinematics of machinary, engineering and com design.

In this study, our first aim is to establish differential equation discribing a spacelike curve of constant breadth in

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Minkowski 3-space. The second aim is to find an approximate solution based on Bernstein polynomials of this differential equation [3, 9]. In this study we also analyze the role of the obtained solution in determining these curves.

2 Preliminaries

Bernstein polynomials of nth-degree are defined by

$$B_{k,n}(x) = \binom{n}{k} \frac{x^k (R-x)^{n-k}}{R^n},\ k = 0,1,\ldots,n$$

where R is the maximum range of the interval $[0,R]$ over which the polynomials are defined to form a complete basis [3].

$$(R-x)^{n-k} = \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i R^{n-k-i} x^i$$

If the above expression is used in the definition, the following equation occurs.

$$B_{k,n}(x) = \sum_{i=0}^{n-k} \binom{n}{k} \binom{n-k}{i} \frac{(-1)^i}{R^{k-i}} x^{k+i}$$

The Minkowski 3-space is real vector space R^3provided with the standart flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where $(x_1, x_2,\ x_3)$ is a rectangular coordinate system of Minkowski 3-space E_1^3. An arbitrary vector $\overrightarrow{v} = (v_1, v_2,\ v_3)$ in E_1^3 can be spacelike if $g(\overrightarrow{v}, \overrightarrow{v}) > 0$ or $\overrightarrow{v} = 0$. Similarly, an arbitrary curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ locally be spacelike if all of its velocity $\overrightarrow{\alpha}'(s)$ are spacelike [14].

Furthermore, for an arbitrary spacelike curve $\overrightarrow{\alpha}(s)$ in space E_1^3, the following Frenet formulas are given

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

where $g\left(\overrightarrow{T}, \overrightarrow{T}\right) = 1$, $g\left(\overrightarrow{N}, \overrightarrow{N}\right) = \varepsilon = \pm 1$, $g\left(\overrightarrow{B}, \overrightarrow{B}\right) = -\varepsilon$, $g\left(\overrightarrow{T}, \overrightarrow{N}\right) = g\left(\overrightarrow{T}, \overrightarrow{B}\right) = g\left(\overrightarrow{N}, \overrightarrow{B}\right) = 0$ and k_1 and k_2 are the curvature and torsion of a spacelike curve $\overrightarrow{\alpha}$, respectively [18].

3 Differential equations characterizing spacelike curves of constant breadth in E_1^3

In this section, we establish differential equations characterizing the spacelike curves of constant breadth. This study is based on the concepts presented by Köse [10, 11] and Sezer [16, 17] for space curves of constant breadth.

Let (C) be a unit speed spacelike curve of the class C^3 having parallel tangents T and T^* in opposite directions at the opposite points α and α^* of the curve. If the chord joining the opposite points of (C) is a double-normal, then (C) has constant breadth, and conversely, if (C) is a spacelike curve of constant breadth, then every normal of (C) is a double-normal. A simple closed spacelike curve (C) of constant breadth having parallel tangents in opposite directions at opposite points may be represented by the equation

$$\overrightarrow{\alpha}^*(s) = \overrightarrow{\alpha}(s) + m_1(s)\overrightarrow{T}(s) + m_2(s)\overrightarrow{N}(s) + m_3(s)\overrightarrow{B}(s) \tag{1}$$

where α and α^* are opposite points, and $\overrightarrow{T}$, $\overrightarrow{N}$, $\overrightarrow{B}$ denote the unite tangent, principal normal, binormal at a generic point α, respectively. s denotes the arc length of (C) and $m_i(s)$, $(1 \leq i \leq 3)$ are the differentiable functions of s. If Eq.(1) is derived according to the s parameter and Frenet formulas defined for spacelike curves are used in this derivative, the following equation is obtained.

$$\frac{d\overrightarrow{\alpha}^*}{ds} = \overrightarrow{T}^*\frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} - \varepsilon k_1 m_2\right)\overrightarrow{T} + \left(m_1 k_1 + \frac{dm_2}{ds} + m_3 k_2\right)\overrightarrow{N} + (m_2 k_2 + \frac{dm_3}{ds})\overrightarrow{B}$$

Since $\overrightarrow{T} = -\overrightarrow{T}^*$ at corresponding points of (C), the following system is obtained.

$$\begin{aligned} &1 + \frac{dm_1}{ds} - \varepsilon k_1 m_2 = -\frac{ds^*}{ds} \\ &m_1 k_1 + \frac{dm_2}{ds} + m_3 k_2 = 0 \\ &m_2 k_2 + \frac{dm_3}{ds} = 0. \end{aligned}$$

The first curvature of the spacelike curves is defined as follows.

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \frac{d\varphi}{ds} = k_1(s)$$

where $\Delta\varphi$ is the angle of contengency. φ denotes the angle between tangent of the curve (C) at the point $\alpha(s)$ and a given fixed direction. Also it is clear that

$$\varphi(s) = \int_0^s k_1(s)ds.$$

The distance d between the opposite points $\alpha^*(s)$ and $\alpha(s)$ of the curve is the breadth of the curve and it is constant, that is

$$d^2 = \|d\|^2 = \|\alpha^* - \alpha\|^2 = m_1^2 + \varepsilon m_2^2 - \varepsilon m_3^2 = const.$$

On the other hand, the coefficients m_1, m_2 and m_3 may be obtained by the system (2)

$$\begin{aligned} m_1{}' &= \varepsilon m_2 - f(\varphi) \\ m_2{}' &= -\rho k_2 m_3 - m_1, \\ m_3{}' &= -\rho k_2 m_2 \end{aligned} \tag{2}$$

which is the system describing the spacelike curves of constant breadth. $f(\varphi)=\rho+\rho^{*}$ and,

$$\rho=\frac{1}{k_1} \text{ and } \rho^{*}=\frac{1}{k_2^{*}}$$

denote the radii of curvatures $\alpha(s)$ and $\alpha^{*}(s)$, respectively. (‘) denotes the derivative according to φ. Also, the vector $\overrightarrow{d}$ is the double normal of the curve (C) of constant breadth. First, it is clear that

$$m_2=\frac{1}{\varepsilon}m_1^{\prime}+\frac{1}{\varepsilon}f(\varphi) \tag{3}$$

On the other hand, by using the second equation of the system (2) the following differential equation is obtained;

$$m_3=\frac{1}{\rho k_2}m_2^{\prime}+\frac{1}{\rho k_2}m_1 \tag{4}$$

and by using the derivative of the equation (3), the following differential equation is obtained:

$$m_3=\frac{1}{\varepsilon\rho k_2}m_1^{\prime\prime}+\frac{1}{\rho k_2}m_1+\frac{1}{\varepsilon\rho k_2}f^{\prime}(\varphi) \tag{5}$$

In addition, the following is clear from the third equation of the system (2).

$$m_2=-\frac{1}{\rho k_2}m_3{}^{\prime} \tag{6}$$

By using the equality of the equations (3) and (6), the following equation is obtained

$$m_1^{\prime}+\frac{\varepsilon}{\rho k_2}m_3^{\prime}+f(\varphi)=0 \tag{7}$$

On the other hand, by using derivative of the equation (5), F is obtained as follows.

$$G=-(\rho k_2)f^{\prime\prime}+(\rho k_2)^{\prime}f^{\prime}-(\rho k_2)^3 f$$

Finally, the third order, linear, differential equation with variable coefficient is obtained as follows.

$$(\rho k_2)m_1^{\prime\prime\prime}-(\rho k_2)^{\prime}m_1^{\prime\prime}+(\rho k_2)\left((\rho k_2)^2-\varepsilon\right)m_1^{\prime}+(\varepsilon\rho k_2)^{\prime}m_1= G \tag{8}$$

As a result, it is clearly seen the system (2) characterizing the spacelike curves of constant breadth can be reduced to the linear differential equation (8). Furthermore, we can write this equation in the general form

$$\sum_{k=0}^{m}P_k(s)y^{(k)}(s)=G(s),\ m=2,\ 3,\ldots \tag{9}$$

where $P_k(s)$ are continuous functions of the expression $(\rho(s)k_2(s))$.

4 Bernstein series method

In this section, we will explain Bernstein series solution method for the solution of the differential equations defined as follows.

$$\sum_{k=0}^{m} P_k(s) y^{(k)}(s) = G(s), 0 \neq s \neq R \tag{10}$$

Let f be a solution of Eq.(10). We wish to approximate f by

$$p_n(s) = \sum_{k=0}^{n} a_k B_{k,n}(s),\ n \geq 1 \tag{11}$$

such that $p_n(s)$ satisfies Eq.(10) on the nodes $0 < s_i < s_{i+1} < \cdots < s_{i+d} < R$. Putting $p_n(s)$ into Eq.(10), we get the system of linear equations depending on $a_0,\ a_1,\ \ldots,\ a_n$.

Assume Eq.(10) has a solution f. Let us consider Eq.(10) and find the matrix forms of each term in the equation. First we can convert the Bernstein series solution $y = p_n(s)$ defined by (11) and its derivatives $y^{(k)}(s)$ to matrix forms

$$y(s) = B_n(s)A \ \ and \ \ y^{(k)}(s) = B_n^{(k)}(s)A \tag{12}$$

where

$$B_n(s) = \left[B_{0,n}(s)\ B_{1,n}(s)\ \ldots\ B_{n,n}(s) \right],\ \ A = \left[a_0\ a_1\ \ldots\ a_n \right]^T.$$

On the other hand, it can be written $[B_n(s)]^T$ as $[B_n(s)]^T = D(S(s))^T$ or

$$B_n(s) = S(s) D^T \tag{13}$$

where

$$D = \begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0n} \\ d_{10} & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \cdots & d_{nn} \end{bmatrix} \ \ and\ d_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^j} \binom{n}{1} \binom{n-i}{j-i}, & i \leq j \\ 0,\ i > j \end{cases}$$

It is clearly seen that the relation between the matrix $S(s)$ and its derivative $S'(s)$ is $S'(s) = S(s)B$, where

$$B = \begin{bmatrix} 0 & 1 & 0 & \ldots 0 \\ 0 & 0 & 2 & \ldots 0 \\ \vdots & \vdots & \vdots & \ddots\ 0 \\ 0 & 0 & 0 & \ldots \mathrm{n} \\ 0 & 0 & 0 & \ldots 0 \end{bmatrix}.$$

To obtain the matrix $S^{(k)}(s)$ in terms of the matrix $S(s)$, we can use the following procedure:

$$S''(s) = S'(s)B = S(s)B^2 \; nonumber \tag{14}$$

$$\begin{aligned} &\vdots \\ &S^{(k)}(s) = S^{(k-1)}(s)B = \cdots = S(s)B^k. \end{aligned} \tag{15}$$

Consequently, we have the following matrix relationship, substituting the matrix forms (13) and (15) in expression (12).

$$y^{(k)}(s) = S(s)B^kD^TA \tag{16}$$

Then, using the matrix relation (16) in Eq.(10), we can easily obtain the following matrix equation.

$$\sum_{k=0}^{m} P_k(s)S(s)B^kD^TA = G(s) \tag{17}$$

Using the collocation points defined as $\{s_i\ (i = 0,1,\ldots,n); 0 \le s_0 < s_1 < \cdots < s_n \le R\}$ in (17), we get the following system of matrix equations.

$$\sum_{k=0}^{m} P_k(s_j)S(s_j)B^kD^TA = G(s_j), \; j = 0,1,\ldots,n,$$

Hence, the fundamental matrix equation can be expressed as follows.

$$\sum_{k=0}^{m} P_kSB^kD^TA = G \tag{18}$$

where

$$P_k = \begin{bmatrix} P_k(s_0) & 0 & \cdots & 0 \\ 0 & P_k(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(s_n) \end{bmatrix}, G = \begin{bmatrix} g(s_0) \\ g(s_1) \\ \vdots \\ g(s_n) \end{bmatrix},$$

$$S = \begin{bmatrix} S(s_0) \\ S(s_1) \\ \vdots \\ S(s_n) \end{bmatrix} = \begin{bmatrix} 1 & s_0 & s_0^2 & \ldots & s_0^n \\ 1 & s_1 & s_1^2 & \ldots & s_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \ldots & s_n^n \end{bmatrix}.$$

Hence, the fundamental matrix Eq.(18) can be written as follow

$$WA = G \;\; or \;\; [W;G] = A, \;\; W = [W_{kh}], \quad k,h = 0,1,\ldots,n \tag{19}$$

where

$$W = \sum_{k=0}^{m} P_kSB^kD^T$$

and expression (19) corresponds to a system of (n+1) linear algebraic equations with unknown coefficients $a_0,\ a_1,\ldots,a_n$.

Now let us obtain the matrix equation of the conditions by means of the relation (16), as follows

$$\sum_{k=0}^{m-1} (a_{jk})y^{(k)} + (b_{jk})y^{(k)} = [\alpha_j]\,,\ j = 0,1,\ldots,m-1$$

On the other hand, the matrix forms for the conditions can be written as

$$U_jA = [\alpha_j] \ \ or \ \ [U_j\,;\alpha_j\,] = A, j = 0,1,\ldots,m-1 \tag{20}$$

where

$$U_j = S(0)B^kD^T = \begin{bmatrix} u_{j0}\ u_{j1}\ \cdots\ u_{jn} \end{bmatrix}$$

$$V_j = S(R)B^kD^T = \begin{bmatrix} u_{j0}\ u_{j1}\ \cdots\ u_{jn} \end{bmatrix}$$

Replacing the row matrices (20) by any m rows of the matrix (19), we get the augmented matrix as

$$[\tilde{W};\tilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n} & ; & g(s_0) \\ w_{10} & w_{11} & \cdots & w_{1n} & ; & g(s_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{(n-m)0} & w_{(n-m)1} & \cdots & w_{(n-m)n} & ; & g(s_{n-m}) \\ u_{00} & u_{01} & \cdots & u_{0n} & ; & \alpha_0 \\ u_{10} & u_{11} & \cdots & u_{1n} & ; & \alpha_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \cdots & u_{(m-1)n} & ; & \alpha_{m-1} \end{bmatrix}.$$

Note that $rank\tilde{W} = rank\left[\tilde{W};\tilde{G}\right] = n+1$ in the case of the exact solution $f \in C^{n+1}(0,R)$. As a result we can write $A = (\tilde{W})^{-1}\tilde{G}$ and hence the elements $a_0,\ a_1,\ldots,a_n$ of A are uniquely determined.

5 The solution of differential equations characterizing the spacelike curves of constant breadth in E_1^3

$$(\rho(s)k_2(s)) = t(s)$$

$P_0 = \varepsilon t',\ P_1 = -\varepsilon t + t^3,\ P_2 = -t',\ P_3 = t$ and $y = m_1$.

Using the above equations we can rewrite the Eq.(8) characterizing the spacelike curves of constant breadth as fallows;

$$\sum_{k=0}^{m} P_k(s)y^{(k)}(s) = G(s)\,,\ m = 3,\ 0 \le s \le 2\pi \tag{21}$$

Let f be a solution of Eq.(21). We wish to approximate f by

$$p_n(s) = \sum_{k=0}^{n} a_kB_{k,n}(s)\,,\ n = 4 \tag{22}$$

such that $p_n(s)$ satisfies Eq.(21) on the nodes $0 \leq s_0 < s_1 < \cdots < s_4 \leq 2\pi$. We take n = 4 for simplicity. Putting $p_n(s)$ into Eq.(21), we get the system of linear equations depending on $a_0,\ a_1,\ \ldots,\ a_4$. Let us consider the Eq.(21) and find the matrix forms of each term in the equation. First we can convert the Bernstein series solution $y = p_n(s)$ defined by (22) and its derivatives $y^{(k)}(s)$ to matrix forms, for $n = 4$ and $k = 0, 1, 2, 3$

$$y(s) = B_4(s)A \text{ and } y^{(k)}(s) = B_4^{(k)}(s)A \tag{23}$$

where

$$B_4(s) = \begin{bmatrix} B_{0,0}(s) & B_{1,0}(s) & \cdots & B_{4,0}(s) \\ B_{0,1}(s) & B_{1,1}(s) & \cdots & B_{4,1}(s) \\ \vdots & \vdots & \ddots & \vdots \\ B_{0,4}(s) & B_{1,4}(s) & \cdots & B_{4,4}(s) \end{bmatrix},\ A = \begin{bmatrix} a_0 & a_1 & \ldots & a_4 \end{bmatrix}^T$$

On the other hand, it can be written $[B_4(s)]^T$ as $[B_4(s)]^T = D(S(s))^T$ or

$$B_4(s) = S(s)D^T \tag{24}$$

The matrix D is calculated as follows

$$D = \begin{bmatrix} 1 & -2/\pi & 3/2\pi^2 & -1/2\pi^3 & 1/16\pi^4 \\ 0 & 2/\pi & -3/\pi^2 & 3/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 3/2\pi^2 & -3/2\pi^3 & 3/8\pi^4 \\ 0 & 0 & 0 & 1/2\pi^3 & -1/4\pi^4 \\ 0 & 0 & 0 & 0 & 1/16\pi^4 \end{bmatrix}$$

To obtain the matrix $S^{(k)}(s)$ $(k = 0, 1, 2)$ in terms of the matrix $S(s) = \begin{bmatrix} 1 \text{ s } \text{s}^2 & \text{s}^3 \text{ s}^4 \end{bmatrix}$, we can use the following procedure:

$$\begin{aligned} S'(s) &= S(s)B \\ S''(s) &= S'(s)B = S(s)B^2 \\ S'''(s) &= S''(s)B = \cdots = S(s)B^3 \end{aligned} \tag{25}$$

where

$$B^2 = \begin{bmatrix} 0\ 0\ 2 & 0\ 0 \\ 0\ 0\ 0 & 6\ 0 \\ 0\ 0\ 0 & 0\ 12 \\ 0\ 0\ 0 & 0\ 0 \\ 0\ 0\ 0 & 0\ 0 \end{bmatrix}$$

and

$$B^3 = \begin{bmatrix} 0\ 0\ 0 & 6\ 0 \\ 0\ 0\ 0 & 0\ 24 \\ 0\ 0\ 0 & 0\ 0 \\ 0\ 0\ 0 & 0\ 0 \\ 0\ 0\ 0 & 0\ 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (24) and (25) into (23), we have the matrix relation.

$$\begin{aligned} y(s) &= S(s)D^T A \\ y'(s) &= S(s)BD^T A \\ y''(s) &= S(s)B^2D^T A \\ y'''(s) &= S(s)B^3D^T A \end{aligned} \tag{26}$$

Substituting the matrix relation (26) into (21) and then simplifying, we obtain the matrix equation

$$\sum_{k=0}^{m} P_k(s)S(s)B^kD^T A = G(s) \tag{27}$$

Using the nodes $\{s_i; i = 0,1,\ldots,4; 0 \le s_0 < s_1 < \cdots < s_4 \le 2\pi\}$ in (27) we get the system of matrix equations

$$\sum_{k=0}^{m=3} P_k(s_i)S(s_i)B^kD^T A = G(s_i),\ i = 0,1,\ldots,\ 4$$

where $s_0 = 0,\ s_1 = \frac{\pi}{2},\ s_2 = \pi,\ s_3 = \frac{3\pi}{2},\ s_4 = 2\pi$ and

$$P_k(s_i) = \begin{bmatrix} P_k(0) & 0 & 0 & 0 & 0 \\ 0 & P_k(\pi/2) & 0 & 0 & 0 \\ 0 & 0 & P_k(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_k(3\pi/2) & 0 \\ 0 & 0 & 0 & 0 & P_k(2\pi) \end{bmatrix}$$

$$S(s_i) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & (\pi/2) & (\pi/2)^2 & (\pi/2)^3 & (\pi/2)^4 \\ 1 & (\pi) & (\pi)^2 & (\pi)^3 & (\pi)^4 \\ 1 & (3\pi/2) & (3\pi/2)^2 & (3\pi/2)^3 & (3\pi/2)^4 \\ 1 & (2\pi) & (2\pi)^2 & (2\pi)^3 & (2\pi)^4 \end{bmatrix}, G(s_i) = \begin{bmatrix} g(0) \\ g(\pi/2) \\ g(\pi) \\ g(\pi/2) \\ g(2\pi) \end{bmatrix}$$

The fundamental matrix equation can be written briefly as

$$\sum_{k=0}^{m} P_k S B^k D^T A = G \tag{28}$$

Hence, the fundamental matrix Eq.(28) corresponding to (22) can be written in the form

$$WA = G\ or\ [W;G] = A,\ W = [W_{kh}],\ k,h = 0,1,\ldots,4 \tag{29}$$

The Eq. (29) corresponds to a matrix of type (5x5). Now let us obtain the matrix equation of the conditions by means of the relation (26). Firstly, the matrix forms for the conditions can be written as

$$U_k A = [\alpha_k] \ or \ [U_k\, ; \alpha_k\,] = A, \ k = 0, 1, 2 \tag{30}$$

where for

$$U_0 = \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} -2/\pi \ 2/\pi \ \ 0 \ 0 \ 0 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 12/\pi^2 \ -6/\pi^2 \ \ 3/\pi^2 \ 0 \ 0 \end{bmatrix}$$

Replacing the row matrices (30) by any m rows of the matrix (29), we get the augmented matrix as

$$[\tilde{W}; \tilde{G}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} & ; g(0) \\ w_{10} & w_{11} & w_{12} & w_{13} & w_{14} & ; g(\pi/2) \\ u_{00} & u_{01} & u_{02} & u_{03} & u_{04} & ; \ \alpha_0 \\ u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & ; \ \alpha_1 \\ u_{20} & u_{21} & u_{22} & u_{23} & u_{24} & ; \ \alpha_2 \end{bmatrix}$$

where, w_{ij} $(i = 0, 1 \ \ j = 0, 1, \ldots, 4)$ obtained as follows;

$$w_{00} = \varepsilon t^{'}(0) - \frac{2}{\pi}[-\varepsilon t(0) + t^3(0)] + \frac{3}{\pi^2}[-t^{'}(0)] - \frac{3}{\pi^3} t(0)$$

$$w_{01} = \frac{2}{\pi}\left[-\varepsilon t(0) + t^3(0)\right] - \frac{6}{\pi^2}\left[-t^{'}(0)\right] + \frac{9}{\pi^3} t(0)$$

$$w_{02} = \frac{3}{\pi^2}\left[-t^{'}(0)\right] - \frac{9}{\pi^3} t(0), \ w_{03} = \frac{3}{\pi^3} t(0), \ \ w_{04} = 0,$$

$$w_{10} = \frac{81}{256}\varepsilon t^{'}\left(\frac{\pi}{2}\right) - \frac{27}{32\pi}\left[-\varepsilon t\left(\frac{\pi}{2}\right) + t^3\left(\frac{\pi}{2}\right)\right] + \frac{27}{16\pi^2}[-t^{'}\left(\frac{\pi}{2}\right)] - \frac{9}{4\pi^3} t\left(\frac{\pi}{2}\right)$$

$$w_{11} = \frac{27}{64}\varepsilon t^{'}\left(\frac{\pi}{2}\right) - \frac{9}{4\pi^2}[-t^{'}\left(\frac{\pi}{2}\right)] + \frac{6}{\pi^3} t\left(\frac{\pi}{2}\right)$$

$$w_{12} = \frac{27}{128}\varepsilon t^{'}\left(\frac{\pi}{2}\right) + \frac{9}{16\pi}\left[-\varepsilon t\left(\frac{\pi}{2}\right) + t^3\left(\frac{\pi}{2}\right)\right] - \frac{3}{8\pi^2}[-t^{'}\left(\frac{\pi}{2}\right)] - \frac{9}{2\pi^3} t\left(\frac{\pi}{2}\right)$$

$$w_{13} = \frac{3}{64}\varepsilon t^{'}\left(\frac{\pi}{2}\right) + \frac{1}{4\pi}\left[-\varepsilon t\left(\frac{\pi}{2}\right) + t^3\left(\frac{\pi}{2}\right)\right] + \frac{3}{4\pi^2}[-t^{'}\left(\frac{\pi}{2}\right)]$$

$$w_{14}=\frac{1}{256}\varepsilon t'\left(\frac{\pi}{2}\right)+\frac{1}{32\pi}\left[-\varepsilon t\left(\frac{\pi}{2}\right)+t^{3}\left(\frac{\pi}{2}\right)\right]+\frac{3}{16\pi^{2}}[-t'\left(\frac{\pi}{2}\right)]+\frac{3}{4\pi^{3}}t\left(\frac{\pi}{2}\right)$$

As a result we can write

$$A=(\tilde{W})^{-1}\tilde{G}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \pi/2 & 0\\ 0 & 0 & -2 & \pi & \pi^{2}/3\\ R & 0 & T & K & V\\ Y & Z & Q & L & C\end{bmatrix}\begin{bmatrix} g(0)\\ g(\pi/2)\\ \alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix}$$

where

$$\begin{aligned}
&R=1/w_{03},T=2w_{02}-w_{00}-w_{01}/w_{03},K=-\pi(w_{01}+2w_{02})/2w_{03}\\
&V=-\pi^{2}w_{02}/3w_{03},Y=-w_{13}/w_{03}w_{14},Z=1/w_{14}\\
&Q=-w_{13}\left(w_{10}+w_{11}-2w_{12}\right)+w_{13}(w_{00}+w_{01}-2w_{02})/w_{03}w_{14}\\
&L=-\pi w_{03}\left(w_{11}+w_{12}\right)+\pi w_{13}(w_{01}+2w_{02})/2w_{03}w_{14}\\
&C=\pi^{2}\left(w_{13}w_{02}-w_{03}w_{12}\right)/w_{03}w_{14}
\end{aligned}$$

and hence the elements $a_0,\ a_1,\dots,a_4$ of A are uniquely determined as follow

$$\begin{aligned}
&a_0=\alpha_0,a_1=\alpha_0+\frac{\pi}{2}\alpha_1,a_2=-2\alpha_0+\pi\alpha_1+\frac{\pi^{2}}{3}\alpha_2\\
&a_3=Rg\left(0\right)+T\alpha_0+K\alpha_1+V\alpha_2,\\
&a_4=Yg\left(0\right)+Zg\left(\frac{\pi}{2}\right)+Q\alpha_0+L\alpha_1+C\alpha_2.
\end{aligned}$$

If we put this a_4 unknowns in Eq.(22), we obtain Bernstein series solution $y=p_n\left(s\right)=m_1$ of the Eq.(21).

6 Analyse of Differential Equations Characterizing Spacelike Curves of Constant Breadth in E_1^3

We found that the expression is $\mathrm{y}=\mathrm{m}_1$ coefficient which is determined the spacelike curves of constant breadth in E_1^3. Also m_2 coefficient is finded with method similar under the same initial conditions. For this first, it is clear that in the second equation of the system (2).

$$m_1=-\rho k_2 m_3-m_2{}' \tag{31}$$

We used where the first equation of the system (2), the derivative of the equation (31)

$$-m_2{}''-\left(\rho k_2\right)' m_3-\left(\rho k_2\right) m_3'=\varepsilon m_2-f \tag{32}$$

Also, it is clear that in the second equation of the system (2)

$$m_3 = \frac{-m_2{}' - m_1}{\rho k_2} \tag{33}$$

By using the third equation of the system (2) and the equation (33) in the equation (32), we obtain the following differential equation:

$$\rho k_2 m_2{}'' + (\rho k_2)' m_2{}' + [(\rho k_2)^3 - \varepsilon \rho k_2] m_2 + (\rho k_2)' m_1 + \rho k_2 f = 0$$

m_1 is conjugated and then by using derivative of the expression obtained, we obtain the following differential equation;

$$\begin{aligned} m_1{}' = &-\frac{\rho k_2}{(\rho k_2)'} m_2{}''' - \left[\left(\frac{\rho k_2}{(\rho k_2)'}\right)' + 1\right] m_2{}'' - \left[\frac{(\rho k_2)^3 - \varepsilon \rho k_2}{(\rho k_2)'}\right] m_2{}' \\ &- \left[\frac{(\rho k_2)^3 - \varepsilon \rho k_2}{(\rho k_2)'}\right]' m_2 - \left(\frac{\rho k_2}{(\rho k_2)'}\right)' f - \frac{\rho k_2}{(\rho k_2)'} f' \end{aligned} \tag{34}$$

By using the equation (34) and the first equation of the system (2) following equation is obtained

$$\begin{aligned} &\frac{\rho k_2}{(\rho k_2)'} m_2{}''' + \left[\left(\frac{\rho k_2}{(\rho k_2)'}\right)' + 1\right] m_2{}'' + \left[\frac{(\rho k_2)^3 - \varepsilon \rho k_2}{(\rho k_2)'}\right] m_2{}' + [(\frac{(\rho k_2)^3 - \varepsilon \rho k_2}{(\rho k_2)'})' + \varepsilon] m_2 \\ &= -[\left(\frac{\rho k_2}{(\rho k_2)'}\right)' + 1] f - \frac{\rho k_2}{(\rho k_2)'} f' \end{aligned} \tag{35}$$

Finally, while $(\rho(s) k_2(s)) = \mathrm{t(s)}$ and F as follows:

$$F = \left[\left(\frac{\rho k_2}{(\rho k_2)'}\right)' - 1\right] f + \frac{\rho k_2}{(\rho k_2)'} f'$$

we obtain the third order, linear, differential equation with variable coefficients as follows

$$\frac{\mathrm{t}}{\mathrm{t}'} m_2{}''' + \left[\left(\frac{\mathrm{t}}{\mathrm{t}'}\right)' + 1\right] m_2{}'' + \left[\frac{t^3 - \varepsilon \mathrm{t}}{\mathrm{t}'}\right] m_2{}' + \left[\left(\frac{t^3 - \varepsilon \mathrm{t}}{\mathrm{t}'}\right)' + \varepsilon\right] m_2 = F. \tag{36}$$

This equation is differential equation with unknown m_2 characterizing spacelike curves of constant breadth in E_1^3.

Also, m_3 coefficient is finded with method similar under the same initial conditions. First, it is clear that in the third equation of the system (2)

$$m_2 = -\frac{1}{\rho k_2} m_3{}' \tag{37}$$

We used where the second equation of the system (2), the derivative of the equation (37).

$$m_1 = \frac{1}{\rho k_2} m_3{}'' + \left(\frac{1}{\rho k_2}\right)' m_3{}' - (\rho k_2) m_3 \tag{38}$$

By using the derivative of the equation (38) in the first equation of the system (2), we obtain the following differential equation:

$$\frac{1}{\rho k_2} {m_3}''' + 2\left(\frac{1}{\rho k_2}\right)' {m_3}'' + \left[\left(-\frac{(\rho k_2)'}{(\rho k_2)^2}\right)' + \frac{\varepsilon}{\rho k_2} - \rho k_2\right] {m_3}' - (\rho k_2)' m_3 = -f \tag{39}$$

Finally, while $(\rho k_2) = \mathrm{t}$ we obtain the third order, linear, differential equation with variable coefficients as follows:

$$\frac{1}{\mathrm{t}} {m_3}''' + 2\left(\frac{1}{\mathrm{t}}\right)' {m_3}'' + \left[\left(-\frac{\mathrm{t}'}{\mathrm{t}^2}\right)' + \frac{\varepsilon}{\mathrm{t}} - \mathrm{t}\right] {m_3}' - \mathrm{t}' m_3 = -f \tag{40}$$

This equation is differential equation with unknown m_3 characterizing the spacelike curves of constant breadth in E_1^3.

7 Corollary

By using Bernstein series solution method, the solutions of the equations (36) and (40) are approximately obtained. If we use the coefficients m_i , $(\ i = 1,2,3)$, which we have calculated, in equation $d^2 = m_1^2 + \ \varepsilon m_2^2 - \varepsilon m_3^2$, we get the constant value of the breadth of the curve in E_1^3.

Thus, we obtain general expression connected with torsion and curvature of a spacelike curve of constant breadth in E_1^3.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Akdoğan, Z., Mağden, A. (2001) Some Characterization Of Curves Of Constant Breadth In E^n Space, *Turk J. Math.*, 25: 433-444.

[2] Aydın, T.A. (2014) *Differential Equations Characterizing Curves Of Constant Breadth And Spherical Curves In E^n–Space And Their Solutions*, Doctoral Thesis, Muğla Sitki Kocman Universty, Mugla.

[3] Bhatti, MI., Brocken, B. (2007) *Solutions Of Differential Equations İn A Bernstein Polynomial Basis*. Journal Of Computational And Applied Mathematics. 205: 272-280.

[4] Dannon, V. (1981) Integral Characterizations And The Theory Of Curves, *Proc. Amer. Math. Soc.*, 81, (4): 600-602.

[5] Euler L., De Curvis trangularibis, Acta Acad. Petropol., (3-30, (1778),(1780)).

[6] Fujivara, M. (1914) On Space Curves Of Constant Breadth, *Thoku Math. J.*, 5, 179-184.

[7] Gluck. H. (1966) Higher Curvatures Of Curves In Euclidean Space, *Proc. Amer. Math. Montly*, 73 : 699-704.

[8] Hacısalihoğlu, H.H. (1993). *Diferensiyel Geometri*, Ankara Üniversitesi Fen Fakültesi, Ankara, 269s.

[9] Işık, O.R., Sezer, M. and Güney, Z. (2011) A rational approximation based on Bernstein polynomials for high order initial and boundary values problems, *Applied Mathematics and Computation*, 217: 9438-9450.

[10] Köse, Ö., Düzlemde Ovaller ve Sabit Genişlikli Eğrilerin bazı özellikleri, Doğa Bilim Dergisi, Seri B, 8 (2), 119-126, (1984).

[11] Köse, Ö. (1986) On Space Curve Of Constant Breadth, *Doğa TU J. Math.*, 10, (1): 11-14.

[12] Mağden A., Köse Ö. (1997) On The Curves Of Constant Breadth In E^4 Space, *Turk J. Math.*, 21: 277-284.

[13] Mellish, A.P. (1931) *Notes On Differential Geometry*, Ann Of Math. (2) 32, no.1, 181-190.

[14] Önder, M., Kocayiğit, H., Candan, E. (2011) Differential Equations Characterizing Timelike and Spacelike Curves Of Constant Breadth İn Minkowski 3-Space E_1^3. *J. Korean Math. Soc.* 48, no.4, 849-866.

[15] Reuleaux, F. (1963) *The Kinematics Of Machinery*, Trans. By Kennedy A.B.W., Dover Pub., New York.

[16] Sezer, M. (1989) Differential Equations Characterizing Space Curves Of Constant Breadth And A Criterion For These Curves, *Doğa TU J. Math*., 13, (2): 70-78.

[17] Sezer, M. (1991) Integral Characterizations For A System Of Frenet Like Differential Equations and Applications, E. U. Faculity of Science, Series Of Scientific Meetings, no. 1, 435-444.

[18] Walrave, J. (1995) Curves and Surfaces İn Minkowski Space, *Doctoral Thesis*, K. U. Leuven, Faculty Of Sciences, Leuven.