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Contact Pseudo-slant submanifolds of a cosymplectic manifold

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Abstract: This paper is concerned with the study of the contact pseudo-slant submanifolds of a cosymplectic manifold. We derive the integrability conditions of involved distributions in the definition of a pseudo-slant submanifold. The notion contact parallel and contact pseudo-slant product is defined and the necessary and sufficient conditions for a submanifold to be contact parallel and contact pseudo-slant product are given. Also, an non-trivial example is used to demonstrate that the method presented in this paper is effective.

Keywords: Cosymplectic manifold, cosymplectic space form, contact slant submanifold, contact pseudo-slant submanifold.

1 Introduction

B-Y. Chen introduced the concept of slant submanifold through differential points of view as a generalization of complex and totally real submanifold of an almost Hermitian manifold[2]. After then, Papaghuic initiated the notion of semi-slant submanifolds as a generalization of slant submanifolds and CR-submanifolds[3].

Furthermore, Carriazo defined pseudo-slant submanifold with the name anti-slant submanifolds as a special class of bi-slant submanifolds[4]. Also pseudo-slant submanifolds have been studied by Khan et. al. in [5]. Later, U. C. De et. al. studied and characterized pseudo-slant submanifolds of trans-Sasakian Manifolds[6]

Recently, M. Atceken and S. Dirik also have investigated contact pseudo-slant submanifolds in cosymplectic space forms and gave some results om mixed-geodesic, totally geodesic and the induced tensor fields to be parallel[7].

2 Preliminaries

An odd-dimensional counterpart of a Kaehler manifold is given by a cosymplectic manifold, which is locally a product of a Kaehler manifold with a circle or a line.

A $(2n+1)$-dimensional C^{∞}-manifold $\bar{M}$ is said to be have an almost contact structure if there exist on $\bar{M}$ a tensor field φ of type $(1,1)$, a vector field ξ and 1-form η satisfying;

$$\varphi^2 = -I + \eta \oplus \xi, \quad \varphi\xi = 0, \quad \eta o \varphi = 0, \quad \eta(\xi) = 1. \tag{1}$$

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There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying the following compatibility condition

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\ \ \eta(X)=g(X,\xi), \tag{2}$$

where X and Y are vector fields on $\bar{M}$.

An almost contact structure (φ,ξ,η) is said to be normal the almost complex J on the product manifold $\bar{M}x\mathbb{R}$ given by

$$J(X,f\frac{d}{dt})=(\varphi X-f\xi,\eta(X)\frac{d}{dt}), \tag{3}$$

where f is a differentiable function on $\bar{M}x\mathbb{R}$, has no torsion, i.e., J is integrable. The condition for normality in terms of φ, ξ, η is $[\varphi,\varphi]+2d\eta\oplus\xi=0$ on $\bar{M}$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ. Finally the fundamental 2-form Φ is defined by $g(X,\varphi Y)=\Phi(X,Y)$. Al almost contact metric structure (φ,ξ,η,g) is said to be cosymplectic structure if it is normal and Φ and η are closed, that is,

$$(\bar{\nabla}_X\varphi)Y=0, \tag{4}$$

for any $X,Y\in\Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the set of the differentiable vector fields on $\bar{M}$ and $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$[8].

A plane section π in $T_{\bar{M}}(p)$ of an almost contact metric manifold $\bar{M}$ is called a φ-section if $\pi\perp\xi$ and $\varphi(\pi)=\pi$. M is of constant φ-sectional curvature if sectional curvature $K(\pi)$ does not depend on the choice of the φ-section π of $T_{\bar{M}}(p)$ and the choice of a point $p\in\bar{M}$. A cosymplectic manifold $\bar{M}$ is said to be a cosymplectic space form if the φ sectional curvature is constant c along $\bar{M}$. A cosymplectic space form will be denoted by $\bar{M}(c)$. Then the Riemannian curvature tensor $\bar{R}$ on $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y,Z,W)=\frac{c}{4}\{&g(X,W)g(Y,Z)-g(X,Z)g(Y,W)+g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)-2g(X,\varphi Y)g(Z,\varphi W)\\&-g(X,W)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)+g(Y,W)\eta(X)\eta(Z)\},\end{aligned} \tag{5}$$

for any $X,Y,Z,W\in\Gamma(T\bar{M})$.

Now, let M be a submanifold of an almost contact metric manifold $\bar{M}$, we denote the induced connections on M and the normal bundle $T^{\perp}M$ by ∇ and $\nabla^{\perp}$, respectively, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_XY=\nabla_XY+h(X,Y) \tag{6}$$

and

$$\bar{\nabla}_XV=-A_VX+\nabla^{\perp}_XV, \tag{7}$$

for any $X,Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_VX,Y)=g(h(X,Y),V). \tag{8}$$

We denote the Riemannian curvature tensor of M by R, then the Gauss equation and Weingarten formulas imply

$$\bar{R}(X,Y)Z = R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{9}$$

for any $X,Y,Z \in \Gamma(TM)$. Taking the normal component of (9), we reach at equation of Codazzi

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \tag{10}$$

If $(\bar{R}(X,Y)Z)^{\perp} = 0$, then submanifold is said to be curvature-invariant.

Next we define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$g(R^{\perp}(X,Y)V,U) = g(\bar{R}(X,Y)U,V) - g([A_V,A_U]X,Y), \tag{11}$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$. If $R^{\perp} = 0$, then normal connection of M is said to be flat.

Definition 1.*If the normal curvature tensor $R^{\perp}$ of M satisfies*

$$R^{\perp}(X,Y)V = 2g(fX,Y)V,$$

for any $X,Y \in (\varphi^2\Gamma(TM))$ and $V \in \Gamma(T^{\perp}M)$, then the normal connection of M is said to be contact flat.

Furthermore, for any $X \in \Gamma(TM)$, we can write

$$\varphi X = fX + \omega X, \tag{12}$$

where fX and ωX denote the tangential and normal components of φX, respectively. Similarly, for $V \in \Gamma(T^{\perp}M)$, φV also can be written

$$\varphi V = BV + CV, \tag{13}$$

where BV and CV denote, respectively, the tangential and normal components of φV. By using (1), (12), (13) and taking into account of ξ being tangent to M, we get

$$f^2 + B\omega = -I + \eta \otimes \xi, \quad \omega f + C\omega = 0, \tag{14}$$

and

$$fB + BC = 0, \quad \omega B + C^2 = -I. \tag{15}$$

Here the covariant derivations of tensor fields f,ω, B and C are defined by

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \tag{16}$$

$$(\bar{\nabla}_X \omega)Y = \nabla_X{}^{\perp}\omega Y - \omega\nabla_X{}^{\perp}Y, \tag{17}$$

$$(\bar{\nabla}_X B)V = \nabla_X BV - B\nabla_X^{\perp}V, \tag{18}$$

$$(\bar{\nabla}_X C)Y = \nabla_X^{\perp}CV - C\nabla_X^{\perp}V, \tag{19}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. By using (4), (6), (7) and (12), we can easily to see that

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X,Y), \tag{20}$$

$$(\nabla_X \omega)Y = -h(X, fY) + Ch(X,Y), \tag{21}$$

$$(\nabla_X C)V = -\omega A_V X - h(X,BV) \tag{22}$$

and

$$(\nabla_X B)V = A_{CV}X - fA_V X, \tag{23}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. By using (21) and (23), we can easily to see that

$$g((\nabla_X \omega)Y,V) = -g((\nabla_X B)V,Y), \tag{24}$$

3 Contact slant submanifolds of a cosymplectic manifold

Let M be a submanifold of an almost contact metric manifold $(\bar{M},\varphi,\xi,\eta,g)$. Then M is said to be a contact slant submanifold if the angle $\theta(X)$ between φX and $T_M(p)$ is constant at any point $p \in M$ for any X linearly independent of ξ. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper contact slant submanifold. The slant submanifolds of an almost contact metric manifold, the following theorem is well known.

Theorem 1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. M is a contact slant submanifold if and only if there exists a constant $\lambda \in (0,1)$ such that*

$$f^2 = \lambda(-I + \eta \otimes \xi). \tag{25}$$

Furthermore, if θ is slant angle of M, then it satisfies $\lambda = \cos^2\theta$.

As a consequence of the above Theorem and (14), we have the following relations;

$$g(fX,fY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\} \tag{26}$$

$$g(\omega X,\omega Y) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{27}$$

$$B\omega = \sin^2\theta(-I + \eta \otimes \xi). \tag{28}$$

For a slant submanifold M of an almost contact metric manifold $\bar{M}$, the normal bundle $T^{\perp}M$ of M is decomposable as

$$T^{\perp}M = \omega(TM) \oplus \mu, \tag{29}$$

where μ is the invariant normal subbundle with respect to φ.

4 Contact Pseudo-slant submanifolds in cosymplectic manifold

Definition 2. *Let M be a submanifold of a cosymplectic manifold $\bar{M}(\varphi,\xi,\eta,g)$. We say that M is a contact pseudo-slant submanifold if there exists a pair of orthogonal distributions $D^\perp$ and D^θ on M such that*

(i) *The distribution $D^\perp$ is totally real, i.e., $\varphi(D^\perp)\subseteq T^\perp M$,*
(ii) *The distribution D^θ is slant with slant angle θ,*
(iii) *The tangent space TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D^\theta$.*

If we denote the dimensions of $D^\perp$ and D^θ by p and q, respectively, the we have the following possible cases;

(i) if $p=0$, then M is a slant submanifold,
(ii) if $q=0$, then M is an anti-invariant submanifold,
(iii) if $pq\neq 0$, $\theta=0$, then M is a contact CR-submanifold.

For a pseudo-slant submanifold M of a cosymplectic manifold $\bar{M}$, the normal bundle $T^\perp M$ of a pseudo-slant submanifold M is decomposable as

$$T^\perp M = \varphi(D^\perp)\oplus\omega(D^\theta)\oplus\mu,\ \ \varphi(D^\perp)\perp\omega(D^\theta). \tag{30}$$

Example 1. Let us consider the Euclidean space $\mathbb{R}^{11}$ with the cartesian coordinates $(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4,x_5,y_5,t)$ and almost contact metric structure

$$\varphi\left(\sum_{i=1}^{5}\left\{X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right\}+Z\frac{\partial}{\partial t}\right)=\sum_{i=1}^{5}\left(-Y_i\frac{\partial}{\partial x_i}+X_i\frac{\partial}{\partial x_i}\right),$$
$$\xi=\frac{\partial}{\partial t},\ \ \eta=dt,\ \ g=\eta\otimes\eta+\sum_{i=1}^{5}(dx_i^2+dy_i^2).$$

It is clear that $\mathbb{R}^{11}$ is a cosymplectic manifold with usual Euclidean metric tensor. Let M be a submanifold of $\mathbb{R}^{11}$ defined by

$$\chi(u,v,s,t)=(v\cos u,-v\sin u,s\cos u,-s\sin u,v+2s,2v-s,-s\cos u,-s\sin u,-v\cos u,-v\sin u,t)$$

with non-zero u,v,s,t. Then the tangent space of M is spanned by the vector fields

$$e_1=-v\sin u\frac{\partial}{\partial x_1}-v\cos u\frac{\partial}{\partial y_1}-s\sin u\frac{\partial}{\partial x_2}-s\cos u\frac{\partial}{\partial y_2}+s\sin u\frac{\partial}{\partial x_4}-s\cos u\frac{\partial}{\partial y_4}+v\sin u\frac{\partial}{\partial x_5}-v\cos u\frac{\partial}{\partial y_5}$$
$$e_2=\cos u\frac{\partial}{\partial x_1}-\sin u\frac{\partial}{\partial y_1}+\frac{\partial}{\partial x_3}+2\frac{\partial}{\partial y_3}-\cos u\frac{\partial}{\partial x_5}-\sin u\frac{\partial}{\partial y_5}$$
$$e_3=\cos u\frac{\partial}{\partial x_2}-\sin u\frac{\partial}{\partial y_2}+2\frac{\partial}{\partial x_3}-\frac{\partial}{\partial y_3}-\cos u\frac{\partial}{\partial x_4}-\sin u\frac{\partial}{\partial y_4},$$
$$e_4=\xi=\frac{d}{dt}.$$

Furthermore, with respect to complex structure of $\mathbb{R}^{11}$, we have

$$\begin{aligned}
\varphi e_1 &= v\cos u\frac{\partial}{\partial x_1} - v\sin u\frac{\partial}{\partial y_1} + s\cos u\frac{\partial}{\partial x_2} - s\sin u\frac{\partial}{\partial y_2} + s\cos u\frac{\partial}{\partial x_4} + s\sin u\frac{\partial}{\partial y_4} + v\cos u\frac{\partial}{\partial x_5} + v\sin u\frac{\partial}{\partial y_5}\\
\varphi e_2 &= \cos u\frac{\partial}{\partial x_1} + \cos u\frac{\partial}{\partial y_1} - 2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3} + \sin u\frac{\partial}{\partial x_5} - \cos u\frac{\partial}{\partial y_5}\\
\varphi e_3 &= \sin u\frac{\partial}{\partial x_2} + \cos u\frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} - 2\frac{\partial}{\partial y_3} + \sin u\frac{\partial}{\partial x_4} - \cos u\frac{\partial}{\partial y_4},\\
\varphi e_4 &= 0.
\end{aligned}$$

Since $g(\varphi e_1, e_2) = g(\varphi e_1, e_3) = 0$, φe_1 is orthogonal to M and

$$\cos\theta = \frac{g(\varphi e_2, e_3)}{\|e_2\|\|e_3\|} = \frac{3}{7},$$

it is easy to see that $D^\theta = Sp\{e_2, e_3\}$ is a slant distribution and $D^\perp = Sp\{e_1\}$ is an anti-invariant distribution. Thus M is a 4-dimensional proper contact pseudo-slant submanifold of $\mathbb{R}^{11}$. It is easy to check that the distributions $D^\perp$ and D^θ are integrable. We denote the integral manifolds of $D^\perp$ and D^θ by $M_\perp$ and M_θ, we can conclude that $M = M_\perp x M_\theta$.

Theorem 2. *Let M be a contact pseudo-slant submanifold of a cosymplectic manifold $\bar{M}$. Then anti-invariant distribution $D^\perp$ is always integrable.*

Proof. For any $Y, Z \in \Gamma(D^\perp)$, we have

$$\bar{\nabla}_Z \varphi Y = \varphi\bar{\nabla}_Z Y - A_{\omega Y}Z + \nabla^\perp_Z \omega Y = \omega\nabla_Z Y + f\nabla_Z Y + Bh(Z,Y) + Ch(Z,Y),$$

which implies that

$$-A_{\omega Y}Z = f\nabla_Z Y + Bh(Y,Z).$$

Thus we have

$$f[Y,Z] = A_{\omega Z}Y - A_{\omega Y}Z. \tag{31}$$

Since the ambient manifold $\bar{M}$ is cosymplectic, we have

$$\begin{aligned}
g(A_{\omega Z}Y - A_{\omega Y}Z, U) &= g(h(Y,U), \omega Z) - g(h(Z,U), \omega Y)\\
&= g(h(Y,U), \omega Z) - g(\bar{\nabla}_U Z, \omega Y)\\
&= g(h(Y,U), \omega Z) + g(\bar{\nabla}_U \varphi Y, Z)\\
&= g(h(Y,U), \omega Z) - g(\bar{\nabla}_U Y, \varphi Z)\\
&= g(h(Y,U), \omega Z) - g(h(U,Y), \omega Z) = 0,
\end{aligned}$$

for any $U \in \Gamma(TM)$, that is,

$$A_{\omega Z}Y = A_{\omega Y}Z. \tag{32}$$

From (31) and (32) we conclude that $f[Y,Z] = 0$, i.e., $[Y,Z] \in \Gamma(D^\perp)$. The proof is completes.

Theorem 3. *Let M be contact pseudo-slant submanifold of a cosymplectic manifold $\bar{M}$. Then the slant distribution D^θ is integrable if and only if*

$$g(A_{C\omega Y}Z + fA_{\omega Z}Y, X) = g(A_{C\omega X}Z + fA_{\omega Z}X, Y), \tag{33}$$

for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$.

*Proof.*By using (6), (7) and (27), we have

$$\begin{aligned}
g([X,Y],Z) &= g(\nabla_X Y, Z) - g(\nabla_Y X, Z) \\
&= g(\bar{\nabla}_Y Z, X) - g(\bar{\nabla}_X Z, Y) \\
&= g(\bar{\nabla}_Y \varphi Z, \varphi X) - g(\bar{\nabla}_X \varphi Z, \varphi Y) \\
&= g(\bar{\nabla}_Y \varphi Z, fX) + g(\bar{\nabla}_Y \varphi Z, \omega X) - g(\bar{\nabla}_X \varphi Z, fY) - g(\bar{\nabla}_X \varphi Z, \omega Y) \\
&= -g(A_{\varphi Z} fX, Y) + g(A_{\varphi Z} fY, X) + g(\bar{\nabla}_Z X, \varphi\omega Y) - g(\bar{\nabla}_Y Z, \varphi\omega X) \\
&= g(A_{\varphi Z} fY, X) - g(A_{\varphi Z} fX, Y) + g(\nabla_X Z, B\omega Y) - g(\nabla_Y Z, B\omega X) + g(\bar{\nabla}_X Z, C\omega Y) - g(\bar{\nabla}_Y Z, C\omega X) \\
&= g(A_{\varphi Z} fY + fA_{\varphi Z} Y, X) - \sin^2\theta g(\nabla_X Z, Y) + \sin^2\theta g(\nabla_Y Z, X) + g(h(X,Z), C\omega Y) - g(h(Y,Z), C\omega X) \\
&= g(fA_{\varphi Z} Y + A_{\varphi Z} fY, X) + \sin^2\theta g([X,Y],Z) + g(A_{C\omega Y}X - A_{C\omega X}Y, Z),
\end{aligned}$$

for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. Consequently, we reach at

$$\cos^2\theta g([X,Y],Z) = g(fA_{\varphi Z}Y + A_{\varphi Z} fY, X) + g(A_{C\omega Y}X - A_{C\omega X}Y, Z),$$

which proves our assertion.

Theorem 4. *Let M be a contact pseudo-slant submanifold of a cosymplectic manifold $\bar{M}$. Then the anti-invariant distribution $D^\perp$ defines totally geodesic foliation in M if and only if*

$$A_{\omega Z} fX - A_{\omega fX} Z \in \Gamma(D^\theta), \tag{34}$$

for any $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. For any $X \in \Gamma(D^\theta)$ and $Y, Z \in \Gamma(D^\perp)$, we have

$$\begin{aligned}
g(\nabla_Y Z, X) = g(\bar{\nabla}_Y \varphi Z, \varphi X) &= g(\bar{\nabla}_Y \varphi Z, \omega X) + g(\bar{\nabla}_Y \varphi Z, fX) \\
&= -g(A_{\varphi Z} fX, Y) - g(\bar{\nabla}_Y Z, B\omega X) - g(\bar{\nabla}_Y Z, C\omega X) \\
&= -g(A_{\varphi Z} fX, Y) + \sin^2\theta g(\nabla_Y Z, X - \eta(X)\xi) + g(A_{\omega fX} Y, Z),
\end{aligned}$$

that is,

$$\cos^2\theta g(\nabla_Y Z, X) = g(A_{\omega fX} Z - A_{\varphi Z} fX, Y). \tag{35}$$

This proves our assertion.

Theorem 5. *Let M be a contact pseudo-slant submanifold of a csoymplectic manifold $\bar{M}$. The slant distribution D^θ defines totally geodesic foliation in M is if and only if*

$$A_{\varphi Z} fY - A_{\omega fY} Z \in \Gamma(D^\perp), \tag{36}$$

for any $Y \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$.

Proof. By using, (2), (6) and (7), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= -g(\overline{\nabla}_X Z, Y) = -g(\nabla_X \varphi Z, \varphi Y) \\ &= -g(\overline{\nabla}_X \varphi Z, fY) - g(\overline{\nabla}_X \varphi Z, \omega Y) \\ &= g(A_{\varphi Z} fY, X) + g(\overline{\nabla}_X Z, B\omega Y) + g(\overline{\nabla}_X Z, C\omega Y) \\ &= g(A_{\varphi Z} fY, X) - \sin^2\theta g(\overline{\nabla}_X Z, Y - \eta(Y)\xi) + g(A_{C\omega Y} X, Z), \end{aligned}$$

for any $X, Y \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$. This implies that

$$\cos^2\theta g(\nabla_X Y, Z) = g(A_{\varphi Z} fY - g(A_{C\omega Y} Z, X).$$

Let M be a contact pseudo-slant submanifold of a cosymplectic manifold $\overline{M}$. M is said to be contact pseudo-slant product if the distributions $D^{\perp}$ and D^{θ} are totally geodesic in M.

From Theorems 4 and 5, we have the following statement.

Proposition 1. *Let M be a contact proper pseudo-slant submanifold of a cosymplectic manifold $\overline{M}$. Then M is a contact pseudo-slant product if and only if and only if the shape operator of M satisfies*

$$A_{\varphi D^{\perp}} f(D^{\theta}) = A_{\omega f D^{\theta}} D^{\perp}. \tag{37}$$

Definition 3. *If the second fundamental form h of M satisfies*

$$(\overline{\nabla}_X h)(Y, Z) = g(fX, Y)\omega Z + g(fX, Z)\omega Y, \tag{38}$$

for any $X, Y, Z \in \varphi^2(\Gamma(TM)) = \Gamma(TM - \xi)$, then h is said to be contact parallel.

Theorem 6. *Let M be a contact pseudo-slant submanifold of a cosymplectic space form $\overline{M}(c)$. If the second fundamental form h of M is contact parallel, then M is either invariant or anti-invariant.*

Proof. From (38), we have

$$\begin{aligned} \overline{\nabla}_Y h)(X, Z) - \overline{\nabla}_X h)(Y, Z) &= g(fY, X)\omega Z + g(fY, Z)\omega X - g(fX, Y)\omega Z - g(fX, Z)\omega Y \\ &= 2g(fY, X)\omega Z + g(fY, Z)\omega X - g(fX, Z)\omega Y, \end{aligned} \tag{39}$$

for any $X, Y, Z \in \Gamma\varphi^2(\Gamma M) = \Gamma(TM - \xi)$. Corresponding (5), (9) and (39), we reach at

$$(\frac{c}{4} - 1)\{2g(fY, X)\omega Z + g(fY, Z)\omega X - g(fX, Z)\omega Y\} = 0. \tag{40}$$

Setting $Y = Z$ in (40), we conclude that

$$3\left(\frac{c}{4} - 1\right) g(fY, X)\omega Y = 0,$$

which proves our assertion.

Theorem 7. *Let M be a proper contact pseudo-slant submanifold of a cosymplectic manifold $\overline{M}$. If the tensor field B is parallel, then M is a contact pseudo-slant product.*

Proof. Since B is parallel, from (23), we have

$$fA_{\omega Z}U = 0, \;\; U \in \Gamma(TM), \;\; Z \in \Gamma(D^{\perp}).$$

This implies that $A_{\varphi Z}U \in \Gamma(D^{\perp})$ and $Bh(U,Z) = 0$. The proof is completes.

Theorem 8. *Let M be a contact pseudo-slant submanifold of a cosymplectic space form $\overline{M}(c)$. M is either anti-invariant submanifold or $\overline{M}$ is flat if ω is parallel.*

Proof. Since ω is parallel, we can easily to see that

$$h(fX,Y) = Ch(X,Y) = h(X,fY),$$

which is equivalent to

$$fA_V X + A_V fX = 0, \tag{41}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. From (41), we have

$$g(A_V fX, BU) = g(A_V X, fBU) = -g(A_V X, BCU) = -g(\omega A_V X, CU) = 0, \tag{42}$$

for any vector fields U,V normal to M. Taking the covariant derivative of (42), for $Y \in \Gamma(TM)$, we obtain

$$g(\overline{\nabla}_Y A_V fX, BU) + g(A_V fX, \overline{\nabla}_Y BU) = 0.$$

This means that

$$g((\nabla_Y A)_V fX + A_{\nabla^{\perp}_Y V} fX + A_V \nabla_Y fX, BU) + g(A_V fX, (\nabla_Y B)U + B\nabla^{\perp}_Y U) = 0.$$

Taking into account (24) and (42), we reach at

$$g((\nabla_Y A)_V fX + A_V\{(\nabla_Y f)X + f\nabla_Y X\}, BU) = 0,$$

from which

$$g((\nabla_Y A)_V fX, BU) + g(A_V\{A_{\omega X}Y + Bh(X,Y)\}, BU) = 0,$$

or,

$$g((\nabla_Y A)_V fX, BU) + g(A_V BU, A_{\omega X}Y) + g(A_V BU, Bh(X,Y)) = 0.$$

This implies that

$$g((\nabla_{fY} h)(fX, BU), V) = g((\nabla_{fY} A)_V fX, BU) = -g(A_V BU, A_{\omega X} fY) - g(A_V BU, Bh(fY,X)).$$

Thus we conclude that

$$\begin{aligned} g((\nabla_{fX}h)(fY,BU)-(\nabla_{fY}h)(fX,BU),V) &= g(A_V BU, A_{\omega X}fY)-g(A_V BU, A_{\omega Y}fX) \\ &= g(A_V BU, A_{\omega X}fY+fA_{\omega Y}X) \\ &= g(A_V A_{\omega X}fY, BU)-g(A_V fA_{\omega Y}X, BU) \\ &= g(A_V fA_{\omega X}Y, BU)=0. \end{aligned} \quad (43)$$

On the other hand, form the Codazzi equation, we have

$$\begin{aligned} g((\nabla_{fX}h)(fY,BU)-(\nabla_{fY}h)(fX,BU),V) &= \frac{c}{4}\{g(f^2Y,BV)g(\omega X,U)-g(f^2X,BV)g(\omega Y,U)+2g(fX,f^2Y)g(\omega BV,U)\} \\ &= -\cos^2\theta\frac{c}{4}\{g(Y,BV)g(\omega X,U)-g(X,BV)g(\omega Y,U)+2g(fX,Y)g(\omega BV,U)\}. \end{aligned} \quad (44)$$

In (44), taking $X,Y\in\Gamma(D^\theta)$ and $U=V=\omega Z\in\Gamma(T^\perp M)$ for $Z\in\Gamma(D^\perp)$, and corresponding (43) and (44), we get

$$\cos^2\theta\frac{c}{2}g(fX,Y)g(Z,Z)=0.$$

This proves our assertion.

Theorem 9. *Let M be a contact pseudo-slant curvature-invariant submanifold of a cosymplectic space form $\bar{M}(c)$. Then M is a either anti-invariant submanifold or $\bar{M}$ is flat space form.*

Proof. If M is a contact pseudo-slant curvature-invariant submanifold, then from (5) and (10), we conclude

$$\frac{c}{4}\{g(\varphi Y,Z)\omega X-g(\varphi X,Z)+2g(X,\varphi Y)\omega Z\}=0,$$

for any $X,Y,Z\in\Gamma(TM)$. This implies that

$$3\frac{c}{4}g(fY,X)\omega Y=0,$$

from which, we obtain

$$3\frac{c}{4}\cos^2\theta g\{(Y,Y)-\eta^2(Y)\}\omega Y=0.$$

The proof is completes.

Theorem 10. *Let M be a invariant submanifold of a cosymplectic space form $\bar{M}(c)$ such that the normal connection of M is contact flat. Then M is totally geodesic submanifold if and only if $\bar{M}$ is flat space.*

Proof.

$$g([A_V,A_U]X,Y)+g(R^\perp(X,Y)U,V)=\frac{c}{4}\{g(\varphi Y,U)g(\varphi X,V)-g(\varphi X,U)g(\varphi Y,V)+2g(X,\varphi Y)g(\varphi U,V)\},$$

for any $X,Y\in\Gamma(TM)$ and $U,V\in\Gamma(T\perp M)$. Since the normal connection is contact flat, we get

$$g([A_V,A_U]X,Y)=-\frac{c}{2}g(fX,Y)g(\varphi U,V)-2g(fX,Y)g(U,V). \quad (45)$$

In (45), taking $V=\varphi U$, we reach

$$g([A_{\varphi U},A_U]X,Y)=-\frac{c}{2}g(fX,Y)g(\varphi U,\varphi U). \quad (46)$$

Since M is an invariant submanifold, we can derive

$$A_{\varphi U}Y = \varphi A_U Y = -A_U fY.$$

Thus we have

$$\begin{aligned} g(A_{\varphi U}A_U X - A_U A_{\varphi U}X, Y) =& g(A_{\varphi U}Y, A_U X) - g(A_U Y, A_{\varphi U}X) \\ =& g(\varphi A_U Y, A_U X) - g(A_U Y, \varphi A_U X) \\ =& 2g(A_U Y, A_U fX) \\ =& -\frac{c}{2} g(fX, Y) g(\varphi U, \varphi U), \end{aligned}$$

from which

$$g(A_U Y, A_U Y) = -\frac{c}{4} g(Y,Y) g(U,U). \tag{47}$$

Since g is a positive definite, this tell us that M is totally geodesic submanifold if and only if $c = 0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Atçeken, M. Contact CR-warped product submanifolds in cosymplectic space forms. Collectanea Math. 62(1), 17-26(2011).
[2] Chen, B-Y. Slant immersion. Bull. Austral. Math. Soc. 41, 135-147(1990).
[3] Papaghuic, N. Semi-Slant Submanifolds of a Kaehlerian Manifold. Ann. Şt. Al. I. Cuza Univ. Iaşi. 40, 55-61(1994).
[4] Carriazo, A. Bi-Slant Immersions. In. Proc. ICARAMS 2000, Kharagpur, India, pp: 88-97(2000).
[5] V. A. Khan, M. A. Khan. Pseudo-Slant Submanifolds of Sasakian Manifold. Indian J. Pure and Appl. Math. 38, 31-42(2007).
[6] U. C. De and A. sarkar. On Pseudo-Slant Submanifolds of Trans-Sasakian Manifolds. Proc. East Acad. Sci., 60, 1-11(2011).
[7] S. Dirik and M. Atceken. Pseudo-Slant Submanifolds in Cosymplectic Space Forms. Acta Math. Sapientiae 8.1, 53-74(2016).
[8] D. E. Blair and D. K. Showers. Almost Contact Manifolds with Killing Structures II. J. Diff. Geom. 9, 577-582(1974).