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Hermite-Hadamard inequality for $M_\varphi A$-strongly convex functions

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Abstract: In this paper we obtain the Hermite-Hadamard Inequality for $M_\varphi A$-strongly convex function. Using this $M_\varphi A-$strongly convex function we get some new theorems and corollaries.

Keywords: $M_\varphi A-$strongly convex function, Hermite-Hadamard type inequality.

1 Introduction

In recent years, several integral inequalities related to various classes of convex functions. Convex functions have played an important role in the development of various fields in pure and applied sciences. A significant class of convex functions is strongly convex functions. The strongly convex functions also play an important role in optimization theory and mathematical economics.
In [1], Noor et. al. gave the following definition.

Definition 1. *Let $I \subseteq \mathbb{R}$ be an interval and c be a positive number. A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$, if*

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y)-ct(1-t)\|y-x\|^2 \in I$$

for $\forall x,y \in I$ and $t \in [0,1]$.

In [7] N. Merentes and K. Nikodem implied Hermite-Hadamard Inequality for strongly convex function as follow:

Theorem 1. *I a function $f : I \to \mathbb{R}$ is strongly convex with modulus c then*

$$f\left(\frac{a+b}{2}\right)+\frac{c}{12}(a-b)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}-\frac{c}{6}(a-b)^2, \tag{1}$$

for all $a,b \in I, a < b$.

Conversely, if f is continuous and satisfies the left of right hand side of (1) for all $a,b \in I, a < b$, then it is strongly convex with modulus c.

In [2], Turhan et. al. revealed the new definition as follow:

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Definition 2. *Let I be a interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function. $f : I \to \mathbb{R}$ is said to be $M_\varphi A-$ convex, if*

$$f\left(\varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right)\right) \leq t f(x)+(1-t) f(y),$$

for every $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be $M_\varphi A-$ concave function.

In [2], Turhan et. al. proved the new theorem for $M_\varphi A-$ convex function as follow:

Theorem 2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a $M_\varphi A-$ convex function and $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $a, b \in I$ with $a < b$. If $f, \varphi \in L[a,b]$ then the following inequality is satisfied almost everywhere:*

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \leq \frac{1}{\varphi(b)-\varphi(a)} \int_a^b f(x)\varphi(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

This inequality known as Hermite-Hadamard inequality for $M_\varphi A-$ convex function.

Many authors have studied the work about $M_\varphi A-$convex and strongly convex function, see [1-10]. In this paper, we firstly list several definitions. Then, we have discussed some properties of $M_\varphi A-$ strongly convex functions and obtained Hermite Hadamard inequality for strongly $M_\varphi A-$ convex.

2 Main results

In this section, we derive Hermite-Hadamard inequalities for strongly $M_\varphi A-$convex function.

Definition 3. *Let I be a interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function. $f : I \to \mathbb{R}$ is said to be $M_\varphi A-$ strongly convex with modulus $c > 0$, if*

$$f\left(\varphi^{-1}\left(t\varphi(y)+(1-t)\varphi(x)\right)\right) \leq (1-t) f(x)+t f(y)-ct(1-t)\|\varphi(y)-\varphi(x)\|^2$$

for all $x, y \in I$ ve $t \in [0,1]$.

Proposition 1.

(1) *If we take $\varphi : I \to \mathbb{R}, \varphi(x) = x$, then we se that $M_\varphi A-$strongly convexity reduces to ordinary strongly convexity on I.*

(2) *If we take $\varphi : I \to \mathbb{R}, \varphi(x) = x^{-1}$, then we see that $M_\varphi A-$strongly convexity reduces to strongly harmonic convexity on I.*

(3) *If we take $\varphi : I \to (0,\infty), \varphi(x) = \ln x$, we see that $M_\varphi A-$strongly convexity reduces to GA strongly convexity on I.*

Lemma 1. *A function $f : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is $M_\varphi A-$ strongly convex with modulus $c > 0$, if and only if, the function $g(x) = f(x) - c\|\varphi(x)\|^2$ is $M_\varphi A-$ convex.*

Proof. Assume that f is $M_\varphi A$ strongly convex with modulus $c > 0$. Using properties of the inner product, we have

$$\begin{aligned}
g\left(\varphi^{-1}\left(t\varphi(y)+(1-t)\varphi(x)\right)\right) &= f\left(\varphi^{-1}\left(t\varphi(y)+(1-t)\varphi(x)\right)\right) - c\left\|\varphi\left(\varphi^{-1}\left(t\varphi(y)+(1-t)\varphi(x)\right)\right)\right\|^2 \\
&\leq (1-t) f(x)+t f(y)-ct(1-t)\|\varphi(y)-\varphi(x)\|^2 - c\|t\varphi(y)+(1-t)\varphi(x)\|^2 \\
&\leq (1-t) f(x)+t f(y)-c\Big(t(1-t)\|\varphi(y)\|^2 - 2t(1-t)\varphi(y)\varphi(x)+t(1-t)\|\varphi(x)\|^2 \\
&\quad + t^2\|\varphi(y)\|^2 + 2t(1-t)\varphi(y)\varphi(x)+(1-t)^2\|\varphi(x)\|^2\Big) \\
&\leq (1-t) f(x)+t f(y)-c\left(t\|\varphi(y)\|^2+(1-t)\|\varphi(x)\|^2\right) \\
&\leq (1-t) f(x)-c(1-t)\|\varphi(x)\|^2+t f(y)-ct\|\varphi(y)\|^2 \\
&= (1-t) g(x)+t g(y)
\end{aligned}$$

which gives that g is $M_{\varphi}A-$ convex function. Conversely, if g is $M_{\varphi}A-$ convex function, then we have

$$\begin{aligned} f\left(\varphi^{-1}\left(t\varphi\left(y\right)+\left(1-t\right)\varphi\left(x\right)\right)\right) &= g\left(\varphi^{-1}\left(t\varphi\left(y\right)+\left(1-t\right)\varphi\left(x\right)\right)\right)+c\left\|\varphi\left(\varphi^{-1}\left(t\varphi\left(y\right)+\left(1-t\right)\varphi\left(x\right)\right)\right)\right\|^{2} \\ &\leq tg\left(y\right)+\left(1-t\right)g\left(x\right)+c\left\|\left(t\varphi\left(y\right)+\left(1-t\right)\varphi\left(x\right)\right)\right\|^{2} \\ &\leq tg\left(y\right)+ct\left\|\varphi\left(y\right)\right\|^{2}+\left(1-t\right)g\left(x\right)+c\left(1-t\right)\left\|\varphi\left(x\right)\right\|^{2} \\ &+ct\left(1-t\right)\left\|\varphi\left(y\right)\right\|^{2}-ct\left(1-t\right)\left\|\varphi\left(x\right)\right\|^{2}+2ct\left(1-t\right)\varphi\left(y\right)\varphi\left(x\right) \\ &= tf\left(y\right)+\left(1-t\right)f\left(x\right)-ct\left(1-t\right)\left\|\varphi\left(y\right)-\varphi\left(x\right)\right\|^{2} \end{aligned}$$

which shows that f is $M_{\varphi}A$ strongly convex with modulus $c>0$.

Theorem 3. *Let $f:I=[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a $M_{\varphi}A-$ strongly convex function with modulus $c>0$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi\left(I^{0}\right)\to I^{0}$ continuous differentiable function and $\forall x,y\in I,\ \ t\in[0,1]$. If $f\in L[a,b]$, then the following inequality is satisfied almost everywhere:*

$$\begin{aligned} &f\left(\varphi^{-1}\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right)+\frac{c}{12}\left\|\varphi\left(b\right)-\varphi\left(a\right)\right\|^{2} \\ &\leq\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{a}^{b}f\left(x\right)\varphi\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{c}{6}\left\|\varphi\left(b\right)-\varphi\left(a\right)\right\|^{2} \end{aligned} \tag{2}$$

Proof. Since $f:I=[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a $M_{\varphi}A$ strongly convex function, we have,

$$f\left(\varphi^{-1}\left(\frac{\varphi\left(x\right)+\varphi\left(y\right)}{2}\right)\right)\leq\frac{f\left(x\right)+f\left(y\right)}{2}-\frac{c}{4}\left\|\varphi\left(y\right)-\varphi\left(x\right)\right\|^{2}$$

For every $x,y\in I$, (with $t=\frac{1}{2}$ in the inequality (2)). By choosing

$$x=\varphi^{-1}\left(t\varphi\left(b\right)+\left(1-t\right)\varphi\left(a\right)\right)\ \texttt{and}\ y=\varphi^{-1}\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right).$$

We get

$$\begin{aligned} f\left(\varphi^{-1}\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right) &\leq\frac{f\left(\varphi^{-1}\left(t\varphi\left(b\right)+\left(1-t\right)\varphi\left(a\right)\right)\right)+f\left(\varphi^{-1}\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right)}{2} \\ &-\frac{c}{4}\left\|\varphi\left(\varphi^{-1}\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right)-\varphi\left(\varphi^{-1}\left(t\varphi\left(b\right)+\left(1-t\right)\varphi\left(a\right)\right)\right)\right\|^{2}. \end{aligned}$$

By integrating for $t\in[0,1]$, we have

$$\begin{aligned} f\left(\varphi^{-1}\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right) &\leq\frac{1}{2}\left[\int_{0}^{1}f\left(\varphi^{-1}\left(t\varphi\left(b\right)+\left(1-t\right)\varphi\left(a\right)\right)\right)dt+\int_{0}^{1}f\left(\varphi^{-1}\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right)dt\right] \\ &-\frac{c}{4}\left\|\varphi\left(b\right)-\varphi\left(a\right)\right\|^{2}\int_{0}^{1}\left(1-2t\right)^{2}dt \end{aligned}$$

and so

$$f\left(\varphi^{-1}\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right)+\frac{c}{12}\left\|\varphi\left(b\right)-\varphi\left(a\right)\right\|^{2}\leq\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{a}^{b}f\left(x\right)\varphi\left(x\right)dx.$$

Thus, we get the left hand side of the inequality (2). Furthermore, we observe that for all $t\in[0,1]$

$$f\left(\varphi^{-1}\left(t\varphi\left(b\right)+\left(1-t\right)\varphi\left(a\right)\right)\right)\leq\left(1-t\right)f\left(a\right)+tf\left(b\right)-ct\left(1-t\right)\left\|\varphi\left(b\right)-\varphi\left(a\right)\right\|^{2}$$

By integrating this inequality with respect to t over $[0,1]$, we have the right-hand side of the inequality (2).

$$\leq \int_0^1 ((1-t)f(a)+tf(b))\,dt - c\|\varphi(b)-\varphi(a)\|^2 \int_0^1 t(1-t)\,dt = \frac{f(a)+f(b)}{2} - \frac{c}{6}\|\varphi(b)-\varphi(a)\|^2.$$

Theorem 4. *Let $f: I=[a,b]\subset \mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a $M_{\varphi}A-$ strongly convex function with modulus $c>0$, $\varphi: I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi\left(I^0\right)\to I^0$ continuous differentiable function and $x,y\in I,\ t\in [0,1]$. Then the following inequality is satisfied almost everywhere:*

$$\begin{aligned} f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)+\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2 \leq \phi(\mathrm{x}) \leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi(x)\,dx \\ \leq \psi(x) \leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\|\varphi(b)-\varphi(a)\|^2, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \phi(\mathrm{x}) &= \frac{1}{2}\left[f\left(\varphi^{-1}\left(\frac{3\varphi(a)+\varphi(b)}{4}\right)\right)+f\left(\varphi^{-1}\left(\frac{\varphi(a)+3\varphi(b)}{4}\right)\right)\right]+\frac{c}{48}\|\varphi(b)-\varphi(a)\|^2 \\ \psi(x) &= \frac{1}{2}\left[f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)+\frac{f(a)+f(b)}{2}\right]-\frac{c}{24}\|\varphi(b)-\varphi(a)\|^2. \end{aligned}$$

Proof. By applying the inequality (2) on each of the intervals $\left[a,\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right]$ and $\left[\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right),b\right]$, we have

$$\begin{aligned} f\left(\varphi^{-1}\left(\frac{3\varphi(a)+\varphi(b)}{4}\right)\right)+\frac{c}{48}\|\varphi(b)-\varphi(a)\|^2 \leq \frac{2}{\varphi(b)-\varphi(a)}\int_a^{\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)} f(x)\varphi^{-1}(x)\,dx \\ \leq \frac{1}{2}\left[f(a)+f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\right]-\frac{c}{24}\|\varphi(b)-\varphi(a)\|^2 \end{aligned} \tag{4}$$

and

$$\begin{aligned} f\left(\varphi^{-1}\left(\frac{\varphi(a)+3\varphi(b)}{4}\right)\right)+\frac{c}{48}\|\varphi(b)-\varphi(a)\|^2 \leq \frac{2}{\varphi(b)-\varphi(a)}\int_{\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}^b f(x)\varphi^{-1}(x)\,dx \\ \leq \frac{1}{2}\left[f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)+f(b)\right]-\frac{c}{24}\|\varphi(b)-\varphi(a)\|^2 \end{aligned} \tag{5}$$

respectively. Summing up side by side, we obtain

$$\begin{aligned} \phi(\mathrm{x}) &= \frac{1}{2}\left[f\left(\varphi^{-1}\left(\frac{3\varphi(a)+\varphi(b)}{4}\right)\right)+f\left(\varphi^{-1}\left(\frac{\varphi(a)+3\varphi(b)}{4}\right)\right)\right]+\frac{c}{48}\|\varphi(b)-\varphi(a)\|^2 \\ &\leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi(x)\,dx \\ &\leq \frac{1}{2}\left[f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)+\frac{f(a)+f(b)}{2}\right]-\frac{c}{24}\|\varphi(b)-\varphi(a)\|^2 \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2}-\frac{c}{4}\|\varphi(b)-\varphi(a)\|^2\right]-\frac{c}{24}\|\varphi(b)-\varphi(a)\|^2 \\ &\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\|\varphi(b)-\varphi(a)\|^2. \end{aligned} \tag{6}$$

Theorem 5. *Let $f,g: I=[a,b]\subset \mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a $M_{\varphi}A-$ strongly convex function with modulus $c>0$, $\varphi: I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi\left(I^0\right)\to I^0$ continuous differentiable function and $x,y\in$*

$I,\ t\in[0,1]$. *If* $f,g\in L[a,b]$, *then the following inequality is satisfied almost everywhere:*

$$\begin{aligned}&\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\,g\left(\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(x))\right)\varphi(x)\,dx\\&\leq\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2S(a,b)-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4\end{aligned}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b), \tag{7}$$
$$N(a,b)=f(a)g(b)+f(b)g(a), \tag{8}$$
$$S(a,b)=f(a)+f(b)+g(a)+g(b). \tag{9}$$

*Proof.*Let f,g be $M_\varphi A-$ strongly convex functions with modulus $c>0$. Then

$$\begin{aligned}&\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\left\{f(x)\,g\left(\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(x))\right)\right\}\varphi(x)\,dx\\&=\int_0^1 f\left(\varphi^{-1}(t\varphi(b)+(1-t)\varphi(a))\right)g\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)dt\\&\leq\int_0^1\left[(1-t)f(a)+tf(b)-ct(1-t)\|\varphi(b)-\varphi(a)\|^2\right]\\&\left[tg(a)+(1-t)g(b)-ct(1-t)\|\varphi(b)-\varphi(a)\|^2\right]dt\\&=f(a)g(b)\int_0^1(1-t)^2dt+f(b)g(a)\int_0^1t^2dt\\&+[f(a)g(a)+f(b)g(b)]\int_0^1t(1-t)\,dt\\&-c\|\varphi(b)-\varphi(a)\|^2[f(a)+g(b)]\int_0^1t(1-t)^2dt-c\|\varphi(b)-\varphi(a)\|^2\\&[f(b)+g(a)]\int_0^1t^2(1-t)\,dt-c^2\|\varphi(b)-\varphi(a)\|^4\int_0^1t^2(1-t)^2dt\\&=\frac{f(a)g(b)+f(b)g(a)}{3}+\frac{f(a)g(a)+f(b)g(b)}{6}\\&-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2[f(a)+f(b)+g(a)+g(b)]-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4\\&=\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2S(a,b)-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4.\end{aligned}$$

In this condition, we take $f=g$ in Theorem 2.3, then it reduces to the following result:

Corollary 1. *Let* $f:I=[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *be a* $M_\varphi A-$ *strongly convex function with modulus* $c>0$, $\varphi:I\to\mathbb{R}$ *be a continuous and strictly monotonic function and* $x,y\in I$, $t\in[0,1]$. *If* $f\in L[a,b]$, *then the following inequality is satisfied almost everywhere:*

$$\begin{aligned}&\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\,f\left(\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(x))\right)\varphi(x)\,dx\\&\leq\frac{2[f(a)f(b)]}{3}+\frac{f^2(a)+f^2(b)}{6}-\frac{c}{6}\|\varphi(b)-\varphi(a)\|^2[f(a)+f(b)]-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4.\end{aligned}$$

Theorem 6. *Let* $f,g:I=[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *be a* $M_\varphi A-$ *strongly convex function with modulus* $c>0$, $\varphi:I\to\mathbb{R}$ *be a continuous and strictly monotonic function and* $x,y\in I,\ t\in[0,1]$. *If* $fg\in L[a,b]$, *then the following inequality is satisfied* *almost everywhere:*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\{f(x)g(x)\}\varphi(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2S(a,b)-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4,$$

where $M(a,b)$, $N(a,b)$ and $S(a,b)$ are given by (7),(8) and (9) respectively.

Proof. Let f,g be $M_\varphi A$ strongly convex functions with modulus $c>0$. In this case, the following inequality is satisfied almost everywhere:

$$\begin{aligned}
\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\{f(x)g(x)\}\varphi(x)dx &= \int_0^1 f\left(\varphi^{-1}(t\varphi(b)+(1-t)\varphi(a))\right)g\left(\varphi^{-1}(t\varphi(b)+(1-t)\varphi(a))\right)dt\\
&\le \int_0^1\left[(1-t)f(a)+tf(b)-ct(1-t)\|\varphi(b)-\varphi(a)\|^2\right]\\
&\left[(1-t)g(a)+tg(b)-ct(1-t)\|\varphi(b)-\varphi(a)\|^2\right]dt\\
&= f(a)g(a)\int_0^1(1-t)^2dt+f(b)g(b)\int_0^1 t^2dt\\
&+[f(a)g(b)+f(b)g(a)]\int_0^1 t(1-t)dt\\
&-c\|\varphi(b)-\varphi(a)\|^2[f(a)+g(a)]\int_0^1 t(1-t)^2dt\\
&-c\|\varphi(b)-\varphi(a)\|^2[f(b)+g(b)]\int_0^1 t^2(1-t)dt\\
&-c^2\|\varphi(b)-\varphi(a)\|^4\int_0^1 t^2(1-t)^2dt\\
&=\frac{f(a)g(a)+f(b)g(b)}{3}+\frac{f(a)g(b)+f(b)g(a)}{6}\\
&-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2[f(a)+f(b)+g(a)+g(b)]-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4\\
&=\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)-\frac{c}{12}\|\varphi(b)-\varphi(a)\|^2S(a,b)-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4.
\end{aligned}$$

We take $f=g$ in Theorem 2.4, then it reduces to the following result.

Corollary 2. *Let $f:I=[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a $M_\varphi A-$ strongly convex function with modulus $c>0$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function and $x,y\in I$, $t\in[0,1]$. If $f\in L[a,b]$, then the following inequality is satisfied almost everywhere:*

$$\begin{aligned}
\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\{f^2(x)\}\varphi(x)dx \le \frac{[f(a)f(b)]}{3}+\frac{f^2(a)+f^2(b)}{3}-\frac{c}{6}\|\varphi(b)-\varphi(a)\|^2[f(a)+f(b)]\\
-\frac{c^2}{30}\|\varphi(b)-\varphi(a)\|^4.
\end{aligned}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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