New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.319

Constancy of φ-holomorphic sectional curvature of an indefinite Sasaki-like almost contact manifold with B-metric

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Received: 1 January 2017, Accepted: 22 July 2018
Published online: 10 December 2018.

Abstract: The aim of the present paper is to establish a criterion for an indefinite Sasaki-like almost contact manifold with B-metric to reduce to a space of φ-holomorphic sectional curvature.

Keywords: Sasaki-like almost contact manifolds, B-metric, φ-holomorphic Sectional curvature.

1 Introduction

Ganchev et al. [3] defined the odd-dimensional version of almost complex manifolds with Norden metric [8,4,2] known as the almost contact manifolds with B-metric (Norden metric). Later, Ivanov et al. [5] introduced a new class of almost contact manifolds with B-metric namely Sasaki-like almost contact Complex Riemannian manifolds with B-metric, which is analogue to indefinite Sasakian manifold.

Tanno [9] proved the following result for an almost Hermitian manifold (M^{2n},g,J) to reduce to a space of constant holomorphic sectional curvature.

Theorem 1. [9] *Let dimension* $(2n \geq 4)$, *assume that almost Hermitian manifold* (M^{2n},g,J) *satisfies*

$$R(JX,JY,JZ,JX) = R(X,Y,Z,X), \tag{1}$$

for every tangent vectors X,Y *and* Z. *Then* (M^{2n},g,J) *is of constant holomorphic sectional curvature at* x, *if and only if,*

$$R(X,JX)X \quad \text{is proportional to} \quad JX, \tag{2}$$

for every tangent vector X *at* x *in* M.

Tanno [9] also extended the above Theorem 1 for the Sasakian manifolds as follows.

Theorem 2. [9] *A Sasakian manifold* $(M^{2n+1},\phi,\eta,\xi,g)$ *of dimension* ≥ 5, *is of constant* ϕ*-sectional curvature if and only if*

$$R(X,\phi X)X \quad \text{is proportional to} \quad \phi X \tag{3}$$

for every vector field X *such that* $g(X,\xi)=0$, *where* ξ *is a characteristic vector field of* M.

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Further, Nagaich [7] generalized the Theorem 1 for an *indefinite almost Hermitian manifold* and provided the following characterization.

Theorem 3. *Let (M^{2n}, g, J) of dimension $2n$, where $(n \geq 2)$ be an indefinite almost Hermitian manifold satisfying (1). Then M is of constant holomorphic sectional curvature at p, if and only if,*

$$R(X, JX)X \quad \text{is proportional to} \quad JX, \tag{4}$$

for every tangent vector X at $p \in M$.

And later, Kumar et al.[6] proved the generalized version of the Theorem 2 for an *indefinite Sasakian manifold* as follows.

Theorem 4. *Let $(M^{2n+1}, \phi, \eta, \xi, g)$ $(2n \geq 4)$ be an indefinite Sasakian manifold. Then M is of constant ϕ-sectional curvature if and only if*

$$R(X, \phi X)X \quad \text{is proportional to} \quad \phi X \tag{5}$$

for every tangent vector field X such that $g(X, \xi) = 0$, where ξ is a characteristic vector field of M.

Recently, we have generalized the Theorem 3 to the setting of an *almost complex manifold with Norden metric* as

Theorem 5. [1] *Let (M^{2n}, g, J) $(2n \geq 4)$ be an indefinite almost complex manifold with Norden metric satisfying (1). Then (M^{2n}, g, J) is of constant holomorphic sectional curvature at p if and only if*

$$R(X, JX)X \quad \text{is proportional to} \quad \alpha X + \beta JX, \tag{6}$$

where α and β are the functions of holomorphic sectional curvature $H(X)$, for every tangent vector X at $p \in M$.

In this paper, we have extended the Theorem 5 to the setting of a Sasaki-like almost contact manifold with B-metric to reduce to a space of constant φ-holomorphic sectional curvature.

Theorem 6. *Let $(\overline{M}^{2n+1}, \varphi, \zeta, \eta)$ be an indefinite Sasaki-like almost contact manifold with B-metric. Then $\overline{M}$ is of constant φ-holomorphic sectional curvature if and only if*

$$R(X, \varphi X)X \quad \text{is proportional to} \quad \gamma X + \delta \varphi X, \tag{7}$$

where γ and δ are the functions of φ-holomorphic sectional curvature $H(X)$, for every tangent vector X such that $g(X, \zeta) = 0$, where ζ is a characteristic vector field of $\overline{M}$.

2 Preliminaries

2.1 Almost contact manifold with B-metric

Let $(\overline{M}^{2n+1}, \varphi, \zeta, \eta)$ be an almost contact manifold with B-metric $\overline{g}$, i.e., $\overline{M}$ is a $(2n+1)$-dimensional smooth manifold endowed with an almost contact structure (φ, ζ, η) and equipped with a pseudo-Riemannian metric $\overline{g}$, such that the following relations are satisfied [3],

$$\varphi\zeta = 0, \quad \eta(\zeta) = 1, \tag{8}$$

$$\eta(X) = \overline{g}(X, \zeta), \quad \eta(\varphi X) = 0, \tag{9}$$

$$\bar{g}(\varphi X,\varphi Y) = -\bar{g}(X,Y) + \eta(X)\eta(Y), \tag{10}$$

$$\bar{g}(\varphi X,Y) = \bar{g}(X,\varphi Y), \tag{11}$$

$$\bar{\tilde{g}}(X,Y) = \bar{g}(\varphi X,Y) + \eta(X)\eta(Y), \tag{12}$$

for arbitrary tangent vector fields $X,Y \in T\bar{M}$, where $\bar{\tilde{g}}$ is called the associated metric of $\bar{g}$ on $\bar{M}$ and is also a B-metric on $\bar{M}$. Moreover, the manifold $(\bar{M},\varphi,\zeta,\eta,\bar{\tilde{g}})$ is also called an almost contact manifold with B-metric. Infact, both $\bar{g}$ and $\bar{\tilde{g}}$ are indefinite metrics having signature $(n+1,n)$.

Let $\bar{\nabla}$ and $\bar{\tilde{\nabla}}$ be the Levi-Civita connections of $\bar{g}$ and $\bar{\tilde{g}}$, respectively on $\bar{M}$. In [3], the tensor field F of type $(0,3)$ is defined on $\bar{M}$ as follows

$$F(X,Y,Z) = \bar{g}((\bar{\nabla}_X\varphi)Y,Z)$$

and the following properties hold in general [3] :

$$F(X,Y,Z) = F(X,Z,Y) = F(X,\varphi Y,\varphi Z) + \eta(Y)F(X,\zeta,Z) + \eta(Z)F(X,Y,\zeta), \tag{13}$$

for any $X,Y,Z \in T\bar{M}$. The relations of F with $\bar{\nabla}\zeta$ and $\bar{\nabla}\eta$ are given by :

$$(\bar{\nabla}_X\eta)Y = g(\bar{\nabla}_X\zeta,Y) = F(X,\varphi Y,\zeta), \quad \eta(\bar{\nabla}_X\zeta) = 0, \quad \varphi(\bar{\nabla}_X\varphi)\zeta = \bar{\nabla}_X\zeta \tag{14}$$

In [3], Ganchev et al. defined eleven basic classes $F_i(i = 1,2,....11)$ of almost contact manifolds with B-metric and classified the almost contact manifolds with B-metric in terms of the tensor F. The intersection of these basic classes is the class F_0, which is analogue to Kaehler manifold with Norden metric and is determined by the condition

$$F(X,Y,Z) = 0(\bar{\nabla}\varphi = \bar{\nabla}\eta = \bar{\nabla}\zeta = 0).$$

Definition 1. [5] *An almost contact manifold$(\bar{M},\varphi,\zeta,\eta,\bar{g})$ with B- metric is called Sasaki-like if the structure tensors $(\varphi,\zeta,\eta,\bar{g})$ satisfy the following equalities*

$$F(X,Y,Z) = F(\zeta,Y,Z) = F(\zeta,\zeta,Z) = 0, \tag{15}$$

$$F(X,Y,\zeta) = -\bar{g}(X,Y). \tag{16}$$

Also, the covariant derivative $\bar{\nabla}\varphi$ satisfies the following equality

$$(\bar{\nabla}_X\varphi)Y = -\bar{g}(X,Y)\zeta - \eta(Y)X + 2\eta(X)\eta(Y)\zeta. \tag{17}$$

A non-zero tangent vector field U is classified in the following types

(i) spacelike if $\bar{g}(U,U) > 0$,
(ii) timelike if $\bar{g}(U,U) < 0$,
(iii) null (lightlike) if $\bar{g}(U,U) = 0, U \neq 0$.

2.2 Curvature properties

Let the curvature tensor R of $\overline{\nabla}$ on $\overline{M}$ is given by

$$R(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z.$$

The corresponding curvature $(0,4)$-tensor with respect to $\overline{g}$ is given by

$$R(X,Y,Z,W) = \overline{g}(R(X,Y)Z,W)$$

and satisfies the following properties

$$RX,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z),$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$

$$R(X,Y,Z,W) = -R(X,Y,\varphi Z,\varphi W),$$

for all tangent vector fields X,Y,Z and W on $\overline{M}$.

The associated curvature tensor $\tilde{R}$ of $\tilde{\overline{\nabla}}$ on $\overline{M}$ is defined as

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,\varphi W).$$

Thus, for the curvature tensor R, we have

$$R(X,Y,Z,\varphi W) = R(X,Y,\varphi Z,W). \tag{18}$$

Let α denote a non-degenerate 2-plane in the tangent space $T_p\overline{M}$. Then the sectional curvature for α with respect to $\overline{g}$ and R is given by

$$K(\alpha,p) = \frac{R(U,V,U,V)}{\overline{g}(U,U)\overline{g}(V,V) - \overline{g}(U,V)^2}. \tag{19}$$

where $\{U,V\}$ is an orthogonal basis of α and $p \in \overline{M}$.

Definition 2. *A 2-plane $\alpha = \{U,\varphi U\}$, where U is orthonormal to ζ is known as φ-holomorphic section (respectively, a ζ-section) if $\alpha = \varphi\alpha$ (respectively, $\zeta \in \alpha$) and the curvature associated with this is said to be φ-holomorphic sectional curvature, denoted by $H(U)$ and given as*

$$H(U) = \frac{R(U,\varphi U,U,\varphi U)}{\overline{g}(U,U)\overline{g}(\varphi U,\varphi U) - \overline{g}(U,\varphi U)^2}. \tag{20}$$

Moreover, if $H(U)$ is always constant with respect to every unit tangent vector $U \in T\overline{M}$, then $\overline{M}$ is said to be of constant φ-holomorphic sectional curvature or a Sasakian space form.

2.3 Sasaki-like almost contact manifold with B-metric

In [5], Ivanov defined the odd dimensional version of an indefinite Kaehler manifold known as Sasaki-like almost contact manifold with B-metric and proved the following result.

Lemma 1. [5] *For a Sasaki-like almost contact manifold* $(\overline{M},\varphi,\zeta,\eta,\overline{g})$ *with B-metric the next formula holds*

$$\begin{aligned}R(X,Y,\varphi Z,\varphi U)-R(X,Y,Z,\varphi U)=&\{\overline{g}(Y,Z)-2\eta(Y)\eta(Z)\}\overline{g}(X,\varphi U)+\{\overline{g}(Y,U)-2\eta(Y)\eta(U)\}\overline{g}(X,\varphi Z)\\&-\{\overline{g}(X,Z)-2\eta(X)\eta(Z)\}\overline{g}(Y,\varphi U)-\{\overline{g}(X,U)-2\eta(X)\eta(U)\}\overline{g}(Y,\varphi Z).\end{aligned}\tag{21}$$

In particular, we have

$$R(X,Y)\zeta=\eta(Y)X-\eta(X)Y\tag{22}$$

and

$$R(\zeta,X)\zeta=-X$$

The equation (21) further implies

$$\begin{aligned}R(X,Y)\varphi Z=&\varphi R(X,Y)Z-2\varphi\eta(Z)R(X,Y)\zeta-\overline{g}(X,Z)\varphi Y+\overline{g}(X,\varphi Z)Y\\&-\overline{g}(Y,\varphi Z)X+2\{\overline{g}(Y,\varphi Z)\eta(X)-\overline{g}(X,\varphi Z)\eta(Y)\}\zeta.\end{aligned}\tag{23}$$

Replacing Y by φX and Z by φX in above equation (23) and use of (22) yields,

$$R(X,\varphi X)X=-\{R(X,\varphi X)\varphi X+(\eta(X))^2\varphi X+2\overline{g}(X,\varphi X)X+2\overline{g}(\varphi X,\varphi X)\varphi X-3\overline{g}(X,\varphi X)\eta(X)\}\zeta\}\tag{24}$$

3 Constancy of φ-holomorphic sectional curvature

Now we will prove the main result.

Proof. Initially assume that $\overline{M}$ be an indefinite Sasaki-like almost contact manifold with B-metric, then using formula (20), we obtain

$$R(X,\varphi X)X=-H(X)\rho X+H(X)\varphi X.\tag{25}$$

where X denotes a unit tangent vector such that $\overline{g}(X,\varphi X)=\rho(\neq 0)$. By using the fact that $\overline{M}$ is having constant φ-holomorphic sectional curvature and the equation (25), the necessity of the assertion follows. To prove the converse part, the following two cases have been considered.

Case I. For the space-like,or in other words, $\overline{g}(X,X)=\overline{g}(Y,Y)$. Let $\{X,Y\}$ denote an orthonormal pair of vectors in $\overline{M}$ such that

$$\overline{g}(X,X)=-\overline{g}(\varphi X,\varphi X)=1,$$

$$\overline{g}(Y,Y)=-\overline{g}(\varphi Y,\varphi Y)=1,$$

$$\overline{g}(X,\varphi X)=\overline{g}(Y,\varphi Y)=\rho(\neq 0)$$

and

$$\overline{g}(X,\varphi Y)=\overline{g}(\varphi X,Y)=0.$$

In this case, X^{**} and Z^{**} be defined by

$$X^{**}=cos\theta X+sin\theta Y$$

and

$$Z^{**}=-sin\theta X+cos\theta Y.$$

Clearly, $\{X^{**}, Z^{**}\}$ also form an orthonormal pair of vectors in $\bar{M}$ and using the above relation (7), we have

$$R(X^{**}, \varphi X^{**})X^{**} \quad \sim \quad \gamma X^{**} + \delta\varphi X^{**}.$$

Taking inner product of above equation with φZ^{**}, we have

$$R(X^{**}, \varphi X^{**}, X^{**}, \varphi Z^{**}) = 0.$$

Also, by using the linear properties of Riemannian curvature tensor R, we obtain

$$cos\theta sin\theta\{-cos^2\theta R(X,\varphi X,X,\varphi X) + sin^2\theta R(Y,\varphi Y,Y,\varphi Y) + (cos^2\theta - sin^2\theta)R(Y,\varphi Y,X,\varphi X)\} = 0. \tag{26}$$

Considering $\theta = \frac{\pi}{4}$ yields,

$$H(X) = H(Y).$$

If $\{Z,W\}$ is a φ-holomorphic section then $\varphi Z = pZ + qZ$, for any scalars p and q. Thus, $\{Z,\varphi Z\} = \{Z, pZ+qZ\} = \{Z,W\}$ and similarly $\{W,\varphi W\} = \{Z,W\}$. therefore $\{Z,\varphi Z\} = \{W,\varphi W\}$ and hence $H(Z) = H(W)$.
On the contrary if $\{Z,W\}$ is not a φ-holomorphic section then there must exist unit vectors $X \in \{Z,\varphi Z\}^{\perp}$ and $Y \in \{W,\varphi W\}^{\perp}$ that determine a φ-holomorphic section $\{X,Y\}$ and thus, we have

$$H(Z) = H(X) = H(Y) = H(W),$$

which proves that any φ-holomorphic section has the same φ-holomorphic sectional curvature.

Now, let the $dim(\bar{M}) = 5$ and using the properties of curvature tensor R, the following relations hold.

$$\begin{aligned}
R(X,\varphi X)X &= H(X)\{-\rho X + \varphi X\} && (27)\\
R(X,\varphi X)Y &= \frac{1}{1+\rho^2}\{R(X,\varphi X,Y,\varphi Y)(\rho Y - \varphi Y)\}\\
R(X,\varphi Y)X &= \frac{1}{1+\rho^2}\{R(X,\varphi Y,X,Y)(Y+\rho\varphi Y) + R(X,\varphi Y,X,\varphi Y)(\rho Y - \varphi Y)\}\\
R(Y,\varphi X)X &= \frac{1}{1+\rho^2}\{R(Y,\varphi X,X,Y)(Y+\rho\varphi Y) + R(Y,\varphi X,X,\varphi Y)(\rho Y - \varphi Y)\}\\
R(X,\varphi Y)Y &= \frac{1}{1+\rho^2}\{R(X,\varphi Y,Y,X)(X+\rho\varphi X) + R(X,\varphi Y,Y,\varphi X)(\rho X - \varphi X)\}\\
R(Y,\varphi X)Y &= \frac{1}{1+\rho^2}\{R(Y,\varphi X,Y,X)(X+\rho\varphi X) + R(Y,\varphi X,Y,\varphi X)(\rho X - \varphi X)\}\\
R(Y,\varphi Y)X &= \frac{1}{1+\rho^2}\{R(Y,\varphi Y,X,\varphi X)(\rho X - \varphi X)\}\\
R(Y,\varphi Y)Y &= H(Y)\{-\rho Y + \varphi Y\}. && (28)
\end{aligned}$$

Now, define $X^{**} = dX + eY$ where $d^2 + e^2 = 1$, then making use of the above algebraic relations (28), we have

$$R(X^{**}, \varphi X^{**})X^{**} = E_1 X + E_2 Y + E_3\varphi X + E_4\varphi Y, \tag{29}$$

where

$$E_3 = d^3 H(X) - \frac{de^2}{(1+\rho^2)}E_5, \quad E_4 = e^3 H(X) - \frac{d^2 e}{(1+\rho^2)}E_5,$$

and

$$E_5 = R(X,\varphi Y,Y,\varphi X) + R(Y,\varphi X,Y,\varphi X) + R(Y,\varphi Y,X,\varphi X).$$

On the other hand, equation (27) yields,

$$R(X^{**},\varphi X^{**})X^{**} = H(X^{**})\{-\rho X^{**} + \varphi X^{**}\} = H(X^{**})\{\rho dX + \rho eY - d\varphi X - e\varphi Y\}. \tag{30}$$

Comparing the equations (29) and (30), we obtain

$$d^2 H(X) - \frac{e^2}{(1+\rho^2)} E_5 = H(X^{**}), \quad e^2 H(X) - \frac{d^2}{(1+\rho^2)} E_5 = H(X^{**}),$$

upon solving the above equations, we have

$$E_5 = -(1+\rho^2)H(X)$$

and hence consequently

$$H(X^{**}) = (d^2 + e^2)H(X) = H(X).$$

Similarly, on the parallel lines, we prove that

$$H(Y^{**}) = H(Y).$$

Thus, we have proved that the manifold $\bar{M}$ is of constant φ-holomorphic sectional curvature.

Case II: When the metric is timelike, or in other words, $\bar{g}(X,X) = -\bar{g}(Y,Y)$, where either X and Y are spacelike and timelike vectors, respectively or vice versa. Let $\{X,Y\}$ denote a pair of orthonormal vectors in $\bar{M}$ such that

$$\bar{g}(X,X) = -\bar{g}(\varphi X,\varphi X) = 1,$$

$$\bar{g}(Y,Y) = -\bar{g}(\varphi Y,\varphi Y) = -1,$$

$$\bar{g}(X,\varphi X) = -\bar{g}(Y,\varphi Y) = \rho(\neq 0)$$

and

$$\bar{g}(X,\varphi Y) = \bar{g}(\varphi X,Y) = 0.$$

Further, we define X'' and Z'' by

$$X'' = cosh\theta X + sinh\theta Y$$

and

$$Z'' = -sinh\theta\varphi X + cosh\theta\varphi Y$$

then X'',Z'' form an orthonormal pair of vectors in $\bar{M}$ and therefore making use of the relation (7), we have

$$R(X'',\varphi X'')X'' \quad \sim \quad \gamma X'' + \delta\varphi X''.$$

Taking inner product of above equation with Z'', we obtain,

$$R(X'',\varphi X'',X'',Z'') = 0,$$

further, using the linearity properties of curvature tensor, we have

$$cosh\theta sinh\theta\{cos^2h\theta H(X) - sin^2h\theta H(Y) - (cos^2h\theta - sin^2h\theta)R(X,\varphi X,Y,\varphi Y)\} = 0. \tag{31}$$

Considering $\theta = \frac{i\pi}{4}$, we get

$$H(X) = H(Y).$$

Further, using the same argument given in **Case I**, we obtain that any holomorphic section has same sectional curvature. Now, assuming the $dim(\bar{M}) = 5$ and using the curvature properties of curvature tensor R, we have the following relations

$$R(X,\varphi X)X = -H(X)\{\rho X - \varphi X\} \tag{32}$$

$$\begin{aligned}
R(X,\varphi X)Y &= \frac{1}{1+\rho^2}\{R(X,\varphi X,Y,\varphi Y)(-\rho Y + \varphi Y)\}\\
R(X,\varphi Y)X &= \frac{1}{1+\rho^2}\{R(X,\varphi X,Y,\varphi Y)(-\rho Y + \varphi Y)\}\\
R(X,\varphi Y)X &= \frac{1}{1+\rho^2}\{R(X,\varphi Y,X,Y)(-Y - \rho\varphi Y) + R(X,\varphi Y,X,\varphi Y)(-\rho Y + \varphi Y)\}\\
R(Y,\varphi X)X &= \frac{1}{1+\rho^2}\{R(Y,\varphi X,X,Y)(-Y - \rho\varphi Y) + R(Y,\varphi X,X,\varphi Y)(-\rho Y + \varphi Y)\}\\
R(X,\varphi Y)Y &= \frac{1}{1+\rho^2}\{R(X,\varphi Y,Y,X)(X + \rho\varphi X) + R(X,\varphi Y,Y,\varphi X)(\rho X - \varphi X)\}\\
R(Y,\varphi X)Y &= \frac{1}{1+\rho^2}\{R(Y,\varphi X,Y,X)(X + \rho\varphi X) + R(Y,\varphi X,Y,\varphi X)(\rho X - \varphi X)\}\\
R(Y,\varphi Y)X &= \frac{1}{1+\rho^2}\{R(Y,\varphi Y,X,\varphi X)(\rho X - \varphi X)\}
\end{aligned}$$

$$R(Y,\varphi Y)Y = -H(Y)\{-\rho Y + \varphi Y\}. \tag{33}$$

Now, define $X'' = dX + eY$ with $d^2 - e^2 = 1$, then using the above relations, we have

$$R(X'',\varphi X'')X'' = E_1 X + E_2 Y + E_3 \varphi X + E_4 \varphi Y, \tag{34}$$

where

$$E_3 = d^3 H(X)(1+\rho^2) - \frac{de^2}{(1+\rho^2)}E_5, \quad E_4 = -e^3 H(X)(1+\rho^2) + \frac{d^2 e}{(1+\rho^2)}E_5,$$

and $E_5 = R(X,\varphi Y,Y,\varphi X) + R(Y,\varphi X,Y,\varphi X) + R(Y,\varphi Y,X,\varphi X)$. On the other hand, using (32), we have

$$R(X'',\varphi X'')X'' = -H(X'')\{\rho dX + \rho eY - d\varphi X - e\varphi Y\}. \tag{35}$$

Comparing (34) and (35), we obtain

$$d^2 H(X) - \frac{e^2}{(1+\rho^2)}E_5 = H(X''), \quad -e^2 H(X) + \frac{d^2}{(1+\rho^2)}E_5 = H(X''),$$

on solving these equations, we obtain

$$E_5 = (1+\rho^2)H(X)$$

and consequently

$$H(X'') = (d^2 - e^2)H(X) = H(X).$$

Similarly, we can prove

$$H(Y'') = H(Y).$$

Thus, the manifold $\bar{M}$ is of constant φ-holomorphic sectional curvature.

Hence, we conclude that Theorem 2 can be derived by considering $g(X,\varphi X) = \rho = 0$, in Theorem 6.

Similarly, by taking $g(X,\varphi X) = \rho = 0$, in Theorem 6, the constancy of φ-holomorphic sectional curvature can be derived for an indefinite almost Sasakian manifold with some minor changes and thus, Theorem 6 provides a generalization of Theorem 4.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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