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On the control of a nonlinear beam

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Abstract: In this paper, optimal vibration control of a nonlinear beam is investigated. In order to achieve the control function, existence and uniqueness of solution to the system and controllability of the system are discussed. Deriving maximum principle, optimal control function is obtained analytically and nonlinear optimal control problem is reduced to solve a system of partial differential equation for the state variable and adjoint variable subjected to boundary, initial and terminal conditions.

Keywords: Nonlinear beam, optimal control, vibration, maximum principle.

1 Introduction

Nonlinear control of the mechanical systems is very important and active research area due to its wide applications in science. It is well known that if the governing equation of the motion is linear, then the system to be controlled is called linear control system and linear control systems are excessively studied by several authors. For instance, let us summarize some of them as follows; necessary and sufficient conditions of optimality for one-dimensional hyperbolic systems including several integral constraints are presented in. [1,5,11]. In [4], maximum principle is investigated under minimum hypotheses. In [8], optimality conditions are presented for the systems including several control functions in one space dimension. In [7], an optimal control problem for a beam is studied via maximum principle. In [13], necessary and sufficient conditions to be satisfied by optimal control function for two dimensional systems are derived in the form of maximum principle. In [3,14,15], some linearization procedures are employed for solving nonlinear control problems. In [9], optimal piezoelectric control of a plate subject to time dependent boundary moments and forcing function for vibrating damping is studied. In [10], control of a time-delay system is presented. In [16], active piezoelectric vibration control for a Timoshenko beam is studied by means of maximum principle.
But if the governing equation includes a nonlinear term of state function, it is named nonlinear control system. In this case, because of nonlinearity, it is not easy to obtain the optimal control of these kind of systems. Therefore, researchers generally prefer to using linearized form of the nonlinear term in the equation of motion. But linearized form of the nonlinear term does not entirely reflect the characteristic of the system. Also, linearization leads to losing important mechanical features of the system. Therefore, this paper deals with the control of a nonlinear system for showing that it is possible to obtain the optimal control of a nonlinear system via maximum principle without linearization of nonlinear term.
The original contribution of the present paper to literature is that in this study, optimal control function of a nonlinear mechanical system is obtained analytically without linearization of nonlinear term while other studies existing in the literature include linearization of nonlinear term to get the optimal control of a nonlinear mechanical system. Also, it is easy to extend the introduced control algorithm in this paper to other nonlinear systems to obtain the optimal control function analytically without linearization of the nonlinear term.
With this aim, a nonlinear beam equation is taken into account. In order to achieve the control function, existence and uniqueness of the solution to the system and controllability of the system are given. Performance index of the system consists of a weighted quadratic functional of displacement and velocity of the beam and also includes a quadratic functional of the control function as a penalty term. By means of maximum principle, optimal control function is obtained without linearization and nonlinear optimal control problem is transformed to solving a system of partial differential equation for the state variable and adjoint variable subjected to boundary, initial and terminal conditions.

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This paper is organized as follows; wellposedness and controllability of the nonlinear beam equation are recalled in the next section. Later, optimal control problem is introduced and a maximum principle is derived. In section 4, conclusion is written.

2 Mathematical formulation of the problem

In this study, we consider the nonlinear partial differential equation [2]

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{txxxx} + [g(w_{xx})]_{xx} = f(x,t), \tag{1}$$

where w is the transversal displacement, $x \in (0,\ell)$ is the space variable, ℓ is the length of the beam, $t \in (0,t_f)$ is the time variable, t_f is the terminal time, $\kappa_1 > 0$ and $\kappa_2 > 0$ are constants, $g(w) = O(w^{1+\theta})$ is the nonlinear term and θ is an positive integer, f is the control function to be determined optimally. (1) is subject to the following boundary conditions

$$w(0,t) = 0, \quad w(\ell,t) = 0, \quad w_x(0,t) = 0, \quad w_x(\ell,t) = 0, \tag{2}$$

also following initial conditions

$$w(x,0) = w_0(x) \in H_0^2(0,\ell), \quad w_t(x,0) = w_1(x) \in L^2(0,\ell). \tag{3}$$

In [2], a weak solution for eq:1 is presented under some assumptions on the nonlinear term. For achieving the optimal control function, let us recall the definition of weak solution (for more details, see [2]. Let $H = L^2(0,\ell)$ and $V = H_0^2(0,\ell)$, so the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with $V^* = H^{-2}(0,\ell)$. Also, $\langle .,. \rangle$ in H and $\langle .,. \rangle_{V^*,V}$ are the inner product and usual duality product, respectively. Denote $\|.\|_S$ the norm of the solution space S and the space of solutions to be

$$S(0,t_f) = \{p | p \in L^2(0,t_f;V), p_t \in L^2(0,t_f;V), p_{tt} \in L^2(0,t_f;V^*)\},$$

with norm

$$\|p\|_{S(0,t_f)} = (\|p\|^2_{L^2(0,t_f;V)} + \|p_t\|^2_{L^2(0,t_f;V)} + \|p_{tt}\|^2_{L^2(0,t_f;V^*)})^{1/2}.$$

A function $w \in S(0,t_f)$ is weak solution of (1) if the following equation holds

$$\langle w_{tt}(t), \phi \rangle_{V^*,V} + \kappa_1 \langle w_{xx}(t), \phi_{xx} \rangle + \kappa_2 \langle w_{txx}(t), \phi_{xx} \rangle = \langle g(w_{xx}(t)), \phi_{xx} \rangle + \langle f(t), \phi \rangle_{V^*,V}, \quad \forall \phi \in V, \tag{4}$$

in which, $w(0) = w_0 \in V$ and $w_t(0) = w_1 \in H$. Assume that the nonlinear function g satisfies the following local Lipschitz condition. Let $B_r(0)$ denote the ball radius r centred at 0 in H, and for some constant L_{B_r}, we have

$$\|g(\rho) - g(\sigma)\|_H \le L_{B_r} \|\rho - \sigma\|_H, \quad \forall \rho, \sigma \in B_r(0). \tag{5}$$

Then, if the nonlinear function g satisfies the local Lipschitz condition given by (5) and the control function $f \in L^2(0,t_f : V^*)$, there exists a t_f such that (1) has a unique weak solution on the interval $[0,t_f]$. Besides, if the non-linear function g satisfies following boundedness criteria

$$\|g(\rho)\|_H \le C_1 \|\rho\|_H + C_2, \quad \rho \in H, \quad \text{for some constants} \quad C_1, C_2 \ge 0, \tag{6}$$

then the weak solution to (1)-(3) is global solution. Namely, (1) has a unique solution. In this case, in order to preserve the uniqueness of the solution, the control function f, corresponding to unique solution, must be unique. Note that the system under consideration has a unique solution and unique control function. Then, it is called observable. Hilbert Uniqueness method proved that observable is equal to the controllable [6, 12] Namely, the system under consideration is controllable.

3 Optimal control problem and maximum principle

The aim of the optimal control problem is to determine an optimum function $f(x,t)$ to minimize the performance index functional of the beam at t_f with the minimum expenditure of the control. Therefore, performance index functional is defined by the weighted dynamic response of the beam and the expenditure of the control over $(0,t_f)$ as follows

$$\mathscr{J}(f(x,t)) = \int_0^{\ell} [\mu_1 w^2(x,t_f) + \mu_2 w_t^2(x,t_f)]dx + \int_0^{t_f}\int_0^{\ell} \mu_3 f^2(x,t)dxdt, \tag{7}$$

where $\mu_1, \mu_2 \geq 0, \quad \mu_1 + \mu_2 \neq 0 \quad$ and $\quad \mu_3 > 0$ are weighting constants. The first integral in (7) is the modified dynamic response of the beam and last integral represents the measure of the total control expense that accumulates over $(0,t_f)$. The optimal control of a nonlinear beam is expressed as

$$\mathscr{J}(f^\circ(x,t)) = \min_{f \in L^2(0,t_f:V^*)} \mathscr{J}(f(x,t)), \tag{8}$$

subject to the (1)-(3). In order to achieve the maximum principle, let us introduce an adjoint variable $\nu(x,t)$ satisfying the following equation

$$\nu_{tt} + \kappa_1 \nu_{xxxx} - \kappa_2 \nu_{txxxx} = 0, \tag{9}$$

and subjects to the following boundary conditions

$$\nu(x,t) = \nu_x(x,t) = 0 \quad \text{at} \quad x = 0, \ell, \tag{10}$$

and terminal conditions

$$\nu_t(x,t) - \kappa_2 \nu_{xxxx}(x,t) = -2\mu_1 w(x,t), \quad \nu(x,t) = 2\mu_2 w_t(x,t) \quad \text{at} \quad t = t_f. \tag{11}$$

A maximum principle in terms of Hamiltonian functional is derived as a necessary condition for the optimal control function. It is proved in [1] that under some convexity assumption, which are satisfied by (7) on performance index function, maximum principle is also sufficient condition for the optimal control function. Deriving the maximum principle, the nonlinear control problem is reduced to solving a system of partial differential equations for the state variable and the adjoint variable subjected to boundary, initial and terminal conditions. Also, the maximum principle gives an explicit expression for the optimal control function and relates the optimal control to the state variable implicitly. Then, the maximum principle can be given as follows

Theorem 1. *The maximization problem states that if*

$$\mathscr{H}[t; \nu^\circ, f^\circ(x,t)] = \max_{f \in L^2(0,t_f:V^*)} \mathscr{H}[t; \nu, f(x,t)], \tag{12}$$

in which $\nu = \nu(x,t)$ satisfies the adjoint system given by (9), (11) and the Hamiltonian function is defined by

$$\mathscr{H}[t; \nu, f(x,t)] = -\nu f(x,t) - \mu_3 f^2(x,t) + \nu[g(w_{xx})]_{xx}, \tag{13}$$

then

$$\mathscr{J}[f^\circ(x,t)] \leq \mathscr{J}[f(x,t)], \tag{14}$$

where $f^\circ(x,t)$ is the optimal control function.

Proof. Before starting the proof, let us introduce an operator and its adjoint operator as follows

$$\Upsilon(w) = w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{txxxx}, \quad \Upsilon^*(\nu) = \nu_{tt} + \kappa_1 \nu_{xxxx} - \kappa_2 \nu_{txxxx}. \tag{15}$$

The deviations are given by $\Delta w = w - w^\circ, \quad \Delta w_t = w_t - w_t^\circ$ in which w° is the optimal displacement. The operator $\Upsilon(\Delta w)(x,t) = \Delta f(x,t) - \Delta G(x,t)$, in which $G(\Delta w) = [g(w_{xx})]_{xx} - [g(w_{xx}^\circ)]_{xx}$, is subject to the following boundary conditions

$$\Delta w(x,t) = \Delta w_x(x,t) = 0 \quad \text{at} \quad x = 0, \ell, \tag{16}$$

and initial conditions

$$\Delta w(x,t) = \Delta w_t(x,t) = 0 \quad \text{at} \quad t = 0. \tag{17}$$

Consider the following functional

$$\int_0^\ell \int_0^{t_f} \Big\{ \nu \Upsilon(\Delta w) - \Delta w \Upsilon^*(\nu) \Big\} dt dx = \int_0^\ell \int_0^{t_f} \Big\{ \nu [\Delta f(x,t) - G(\Delta w)] \Big\} dt dx. \tag{18}$$

Integrating the left side of (18) twice integration by parts with respect to t and four times integration by parts with respect to x, using (16), (17), one observes the following relation

$$\int_0^\ell \int_0^{t_f} \Big\{ \nu \Upsilon(\Delta w) - \Delta w \Upsilon^*(\nu) \Big\} dt dx = \int_0^\ell \Big(\nu(x,t_f) \Delta w_t(x,t_f) - \Delta w(x,t_f) \big[\nu_t(x,t_f) - \kappa_2 \nu_{xxxx}(x,t_f) \big] \Big) dx. \tag{19}$$

In view of (11), (19) becomes

$$\int_0^\ell \int_0^{t_f} \Big\{ \nu \Upsilon(\Delta w) - \Delta w \Upsilon^*(\nu) \Big\} dt dx = 2 \int_0^\ell \big(\mu_1 w(x,t_f) \Delta w(x,t_f) + \mu_2 w_t(x,t_f) \Delta w_t(x,t_f) \big) dx. \tag{20}$$

Consider the difference of the performance index

$$\begin{aligned} \Delta \mathscr{J}[f(x,t)] = \mathscr{J}[f(x,t)] - \mathscr{J}[f^\circ(x,t)] = \\ \int_0^\ell \Big\{ \mu_1 [w^2(x,t_f) - w^{\circ^2}(x,t_f)] + \mu_2 [w_t^2(x,t_f) - w_t^{\circ^2}(x,t_f)] \Big\} dx + \\ \int_0^\ell \int_0^{t_f} \mu_3 [f^2(x,t) - f^{\circ^2}(x,t)] dt dx. \end{aligned} \tag{21}$$

Let us expand the $w^2(x,t_f)$ and $w_t^2(x,t_f)$ to Taylor series around $w^{\circ^2}(x,t_f)$ and $w_t^{\circ^2}(x,t_f)$, respectively. Then, one observes the following

$$w^2(x,t_f) - w^{\circ^2}(x,t_f) = 2 w^\circ(x,t_f) \Delta w(x,t_f) + r, \tag{22a}$$

$$w_t^2(x,t_f) - w_t^{\circ^2}(x,t_f) = 2 w_t^\circ(x,t_f) \Delta w_t(x,t_f) + r_t, \tag{22b}$$

where $r = 2(\Delta w)^2$ + higher order terms > 0 and $r_t = 2(\Delta w_t)^2$ + higher order terms > 0. Substituting (22) into eq. (21) gives

$$\begin{aligned} \Delta \mathscr{J}[f(x,t)] = \int_0^\ell \Big\{ \mu_1 [2 w^\circ(x,t_f) \Delta w(x,t_f) + r] \\ + \mu_2 [2 w_t^\circ(x,t_f) \Delta w_t(x,t_f) + r_t] \Big\} dx + \int_0^{t_f} \int_0^\ell \mu_3 [f^2(x,t) - f^{\circ^2}(x,t)] dx dt. \end{aligned} \tag{23}$$

From Eq. (20) and because of $\mu_1 r + \mu_2 r_t > 0$, one obtains

$$\Delta \mathscr{J}[f(x,t)] \geq \int_0^{t_f} \int_0^\ell \nu [\Delta f(x,t) - G(\Delta w)] dx dt + \int_0^{t_f} \int_0^\ell \mu_3 [f^2(x,t) - f^{\circ^2}(x,t)] dx dt \geq 0,$$

which leads to

$$\nu f(x,t) + \mu_3 f^2(x,t) - \nu [g(w_{xx})]_{xx} \geq \nu f^\circ(x,t) + \mu_3 f^{\circ^2}(x,t) - \nu [g(w_{xx}^\circ)]_{xx}, \tag{24}$$

that is

$$\mathscr{H}[t; \nu^\circ, f] \geq \mathscr{H}[t; \nu, f].$$

Hence, we obtain

$$\mathscr{J}[f] \geq \mathscr{J}[f^\circ], \quad \forall f \in L^2(0, t_f : V^*).$$

By taking the first variation of the $\mathscr{H}$, control function is obtained optimally as follows

$$f(x,t) = \frac{-v(x,t)}{2\mu_3}. \tag{25}$$

4 Conclusion

In this study, optimal control of a nonlinear beam is studied. Optimal control function is analytically obtained by means of a maximum principle without linearization of the nonlinear term in the equation of motion. Hence, it is shown that maximum principle approach can be used for nonlinear system without linearization process. Obtained results have the potential to be used for the other nonlinear systems.

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