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Corrigendum to: on irresolute topological vector spaces

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Abstract: In the paper *On Irresolute Topological Vector Spaces*, Advances in Pure Mathematics, 06(2016), 105-112, Khan and Iqbal state an Example 3, 'Consider the field $F = \mathbb{R}$ with standard topology on F. Let $X = \mathbb{R}$ be endowed with the topology τ generated by the base $\beta = \{\emptyset, X\} \cup \{(a,b), [0,c) : a,b,c \in \mathbb{R}\}$. Then $(\mathbb{R}_{(\mathbb{R})}, \tau)$ is an irresolute topological vector space' but this is false. In this paper, a valid refutation of this example is provided. A general result concerning this error is presented. Some new properties of irresolute topological vector spaces are investigated.

Keywords: Semi-open sets, semi-closed sets, irresolute topological vector spaces.

1 Introduction

The concept of semi-open sets in topological spaces was invented by Norman Levine [3] in 1963. He defines a set S in a topological space X to be semi-open if there exists an open set U in X such that $U \subseteq S \subseteq Cl(U)$; or equivalently, if $S \subseteq Cl(Int(S))$, where $Cl(U)$ and $Int(U)$ denote the closure of U and the interior of U, respectively in X. Every open set is semi-open but the converse is not true, in general. The complement of a semi-open set is called semi-closed; or equivalently, a set S in a topological space X is semi-closed if $Int(Cl(S)) \subseteq S$. The intersection of all semi-closed sets in a topological space X containing a subset $S \subseteq X$ is called the semi-closure of S [1] and is denoted by $sCl(S)$. It is known that a set S in a topological space X is semi-closed if and only if $S = sCl(S)$. A point $x \in sCl(S)$ if and only if $S \cap U \neq \emptyset$ for each semi-open set U in X containing x. The union of all semi-open sets in X that are contained in a subset $S \subseteq X$ is called the semi-interior of S and is denoted by $sInt(S)$. It is known that a set S in X is semi-open if and only if $S = sInt(S)$. A point $x \in X$ is called semi-interior point of a subset S of X if there exists a semi-open set U in X containing x such that $U \subseteq S$. A subset S of a topological space X is called a semi-neighborhood of a point x of X if there exists a semi-open set U in X such that $x \in U \subseteq S$.

Utilizing the concept of semi-open sets in the sense of Levine, in 2016, M.D. Khan and M.A. Iqbal [2] defined the irresolute topological vector spaces as follows.

Let X be a vector space over the field F, where $F = \mathbb{R}$ *or* $\mathbb{C}$ with the usual topology. Let τ be a topology on X such that the following are satisfied:

(i) for each $x, y \in X$, and each semi-open neighborhood W of $x + y$ in X, there exist semi-open neighborhoods U and V of x and y, respectively in X such that $U + V \subseteq W$ and

(ii) for each $\lambda \in F, x \in X$ and each semi-open neighborhood W of $\lambda.x$ in X, there exist semi-open neighborhoods U of λ in F and V of x in X such that $U.V \subseteq W$

Then the pair $(X_{(F)}, \tau)$ is called an irresolute topological vector space.

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Along with the definition of irresolute topological vector spaces, M.D. Khan and M.A. Iqbal studied several of their properties and gave a few examples of them. Among them, example [2, Example 3] is false. In this paper, a valid refutation of this example is provided. A general result concerning this error is proved. Some new depths of irresolute topological vector spaces are investigated.

2 Error

This is Example 3 of the published paper [2]: Consider the field $F = \mathbb{R}$ with standard topology on F. Let $X = \mathbb{R}$ be endowed with the topology τ generated by the base $\beta = \{\emptyset, X\} \cup \{(a,b), [0,c) : a,b,c \in \mathbb{R}\}$. Then $(\mathbb{R}_{(\mathbb{R})}, \tau)$ is an irresolute topological vector space.

3 Refutation of example 3 of [2]

The example [2, Example 3] is not irresolute topological vector space because, for semi-open set $W = [0,1)$ in $X = \mathbb{R}$ containing $0 = -1.0$ ($-1 \in F = \mathbb{R}$, $0 \in X = \mathbb{R}$), there do not exist semi-open sets U in $F = \mathbb{R}$ containing -1 and V in $X = \mathbb{R}$ containing 0 such that $U.V \subseteq [0,1)$, since the sets of the form [a,0] or (a,0] are not semi-open in X where a is negative real number.

Alternatively, Consider the set $A = [-2,-1]$ in $X = \mathbb{R}$. Then A is semi-open set in X. If [2, Example 3] is an irresolute topological vector space, then, by theorem [2, Theorem 2], $B = 1 + A = [-1,0]$ is semi-open set in X which is not true because $Cl(Int(B)) = [-1,0)$ does not contain B.

In fact, the above reasoning paves the way for a more general result.

Proposition 1. *If τ is any topology on the real vector space $X = \mathbb{R}$ such that $\tau_u \subsetneq \tau$, where τ_u is the usual topology, then $(X_{(\mathbb{R})}, \tau)$ is not an irresolute topological vector space.*

Proof. Suppose that $(X_{(\mathbb{R})}, \tau)$ is an irresolute topological vector space. Choicely, consider $A \in \tau$ such that $A \notin \tau_u$. Then the second condition of the definition of irresolute topological vector spaces forces that A must be an interval whereby the following cases arise:

Case (I) If $A = [a,b)$.
In this case, consider $B = [x,y]$ and $z = a - y$ in X, for some appropriate $x,y \in X$. Then we see that B is semi-open but $z + B$ is not semi-open (otherwise, τ must be lower limit topology on $X = \mathbb{R}$ which do not satisfy the conditions of the definition of irresolute topological vector spaces). Anyway, violation of translationally invariance of semi-open sets is found which irresolute topological vector spaces cannot tolerate.

Case(II) If $A = (a,b]$.
The same line of reasoning as in case (I) rule out this case as well.

Case (III) If $A = [a,b]$.
Consider a semi-open set $B = [x,y]$, for some suitable $x,y \in X$ such that $B \notin \tau$ (this is possible, otherwise τ must be the discrete topology). Let $z = a - y$ be an element of X. We will see that B is semi-open but $z + B = [a + x - y, a]$ is not semi-open.

Hence, we observe that $(X_{(\mathbb{R})}, \tau)$ does not preserve the depths of an irresolute topological vector space. Therefore, our assumption is wrong and thereby the assertion follows.

From here on, we simply write X for an irresolute topological vector space $(X_{(F)},\tau)$ and by a scalar, we mean an element of the topological field F.

Theorem 1. *Let A be any subset of an irresolute topological vector space X. Then the following statements hold:*

(1) $x+sCl(A)=sCl(x+A)$ *for any* $x\in X$.
(2) $sCl(\lambda A)=\lambda sCl(A)$ *for any non-zero scalar* λ.

Proof. (1) Let $y\in sCl(x+A)$. Consider $z=-x+y$ and let W be any semi-open set in X containing z. Then by the definition of irresolute topological vector spaces, there exist semi-open sets U and V in X such that $-x\in U$, $y\in V$ and $U+V\subseteq W$. This results in $(x+A)\cap V\neq\emptyset$ and hence there is $a\in(x+A)\cap V$. Now $-x+a\in A\cap(U+V)\subseteq A\cap W \Rightarrow A\cap W\neq\emptyset$. Consequently, $z\in sCl(A)$; that is, $y\in x+sCl(A)$. Therefore, $sCl(x+A)\subseteq x+sCl(A)$. For the reverse inclusion, let $z\in x+sCl(A)$. Then $z=x+y$, for some $y\in sCl(A)$. Let W be any semi-open neighborhood of z in X. Then, there exist semi-open neighborhoods U and V of x and y respectively in X such that $U+V\subseteq W$. Since $y\in sCl(A)$, $A\cap V\neq\emptyset$. Consider $a\in A\cap V$. Then $x+a\in(x+A)\cap(U+V)\subseteq(x+A)\cap W$ implies $(x+A)\cap W\neq\emptyset$. Consequently, $z\in sCl(x+A)$. Therefore, $x+sCl(A)\subseteq sCl(x+A)$. Hence $sCl(x+A)=x+sCl(A)$.

(2) Let $y\in\lambda sCl(A)$. Then $y=\lambda x$, for some $x\in sCl(A)$. Let W be a semi-open neighborhood of y in X. By definition of irresolute topological vector spaces, there exist semi-open neighborhoods U of λ in F and V of x in X such that $U.V\subseteq W$. Since $x\in sCl(A)$, there is $a\in A\cap V$ and thereby, $\lambda a\in\lambda A\cap(U.V)\subseteq\lambda A\cap W \Rightarrow (\lambda A)\cap W\neq\emptyset$. Consequently, $y\in sCl(\lambda A)$. That is, $\lambda sCl(A)\subseteq sCl(\lambda A)$.

Next, let $x\in sCl(\lambda A)$ and let W be any semi-open neighborhood of $\frac{1}{\lambda}x$ in X. Then we get semi-open sets U in F containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V\subseteq W$. Since $x\in sCl(\lambda A)$, there is $a\in(\lambda A)\cap V$ and thus, $\frac{1}{\lambda}a\in A\cap W \Rightarrow A\cap W\neq\emptyset$. This implies that $\frac{1}{\lambda}x\in sCl(A)$; that is, $x\in\lambda sCl(A) \Rightarrow sCl(\lambda A)\subseteq\lambda sCl(A)$. Hence the assertion follows.

Theorem 2. *For any subset A of an irresolute topological vector space X, the following statements hold:*

(1) $sInt(x+A)=x+sInt(A)$ *for any* $x\in X$.
(2) $sInt(\lambda A)=\lambda sInt(A)$ *for any non-zero scalar* λ.

Proof. (1) Let $z\in sInt(x+A)$. Then $z=x+y$ for some $y\in A$. By the definition of irresolute topological vector spaces, there exist semi-open sets U and V in X containing x and y respectively, such that $U+V\subseteq x+A$. This gives that $z=x+y\in x+V\subseteq x+sInt(A)$. Therefore, $sInt(x+A)\subseteq x+sInt(A)$. Next, let $y\in x+sInt(A)$. Then $-x+y\in sInt(A)$. Since X is irresolute topological vector space, there exist semi-open sets U and V in X such that $-x\in U$, $y\in V$ and $U+V\subseteq A \Rightarrow V\subseteq x+A$. Since V is semi-open, $y\in sInt(x+A)$. This proves that $x+sInt(A)\subseteq sInt(x+A)$. Hence the assertion follows.

(2) Suppose that $x\in\lambda sInt(A)$. Then there exist semi-open sets U in F containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V\subseteq sInt(A)$. This implies that $V\subseteq\lambda A$. Since V is semi-open, $x\in sInt(\lambda A)$. Thus it follows that $\lambda sInt(A)\subseteq sInt(\lambda A)$. Next, if $y\in sInt(\lambda A)$, then $y=\lambda x$ for some $x\in A$. By the definition of irresolute topological vector spaces, there exist semi-open neighborhoods U of λ in F and V of x in X such that $U.V\subseteq sInt(\lambda A)$. Consequently, $y=\lambda x\in\lambda V\subseteq\lambda sInt(A)$. Hence $sInt(\lambda A)=\lambda sInt(A)$.

Theorem 3. *For any semi-closed set A in an irresolute topological vector space X, the following are true:*

(1) $Int(Cl(x+A))\subseteq x+A$ *for each* $x\in X$.
(2) $Int(Cl(\lambda A))\subseteq\lambda A$ *for each non-zero scalar* λ.

Proof. (1) Suppose that $y \in sCl(x+A)$ and let W be any semi-open set in X containing $z = -x+y$. Then there exist $U, V \in SO(X)$ such that $-x \in U$, $y \in V$ and $U+V \subseteq W$. By assumption, there is $a \in (x+A) \cap V$. This gives $-x+a \in A \cap (U+V) \subseteq A \cap W \Rightarrow A \cap W \neq \emptyset$. This shows that $z \in sCl(A) = A$, i.e., $y \in x+A$. This proves that $sCl(x+A) = x+A$ and hence $Int(Cl(x+A)) \subseteq x+A$.

(2) Suppose that $x \in sCl(\lambda A)$. Consider $y = \frac{1}{\lambda}x$ and let W be a semi-open set in X such that $y \in W$. Then we get an inclusion $U.V \subseteq W$ for some semi-open sets U in F containing $\frac{1}{\lambda}$ and V in X containing x. By assumption, we have $(\lambda A) \cap V \neq \emptyset$. So there is $a \in (\lambda A) \cap V$. This gives $\frac{1}{\lambda}a \in A \cap W$. This reflects that $y \in sCl(A) = A \Rightarrow x \in \lambda A$. This proves that λA is semi-closed and therefore the assertion follows.

The following is an improvement of Theorem 3.

Theorem 4. *For any subset A of an irresolute topological vector space X, the following are true:*

(1) $Int(Cl(x+A)) \subseteq x+sCl(A)$ *for each* $x \in X$.
(2) $x+Int(Cl(A)) \subseteq sCl(x+A)$ *for each* $x \in X$.

Proof.

(1) Let A be any subset of X. Then $sCl(A)$ is semi-closed set in X. In view of Theorem 3, $x+sCl(A)$ is semi-closed in X. Thus it follows that $Int(Cl(x+A)) \subseteq x+sCl(A)$.
(2) Since $sCl(x+A)$ is semi-closed set in X, by Theorem 3, $-x+sCl(x+A)$ is semi-closed set in X. Consequently, $Int(Cl(A)) \subseteq -x+sCl(x+A) \Rightarrow x+Int(Cl(A)) \subseteq sCl(x+A)$.

The analog of Theorem 4 is the following:

Theorem 5. *For any subset A of an irresolute topological vector space X, the following are true:*

(1) $Int(Cl(\lambda A)) \subseteq \lambda sCl(A)$ *for each non-zero scalar* λ.
(2) $\lambda Int(Cl(A)) \subseteq sCl(\lambda A)$ *for each non-zero scalar* λ.

Theorem 6. *Let A be any semi-open subset of an irresolute topological vector space X. Then*

(1) $x+A \subseteq Cl(Int(x+A))$ *for each* $x \in X$.
(2) $\lambda A \subseteq Cl(Int(\lambda A))$ *for each non-zero scalar* λ.

Proof. A direct consequence of [2, Theorem 2].

A generalization of Theorem 6 is the following.

Theorem 7. *For any subset A of an irresolute topological vector space X, the following are valid.*

(1) $x+sInt(A) \subseteq Cl(Int(x+A))$ *for each* $x \in X$.
(2) $\lambda sInt(A) \subseteq Cl(Int(\lambda A))$ *for each non-zero scalar* λ.

Proof. A simple consequence of [2, Theorem 2].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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