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# A new class of operator ideals and approximation numbers

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**Abstract:** In this study, we introduce the class of generalized Stolz mappings by generalized approximation numbers . Also we prove that the class of $\ell_p^\alpha-$type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers and we define a new quasinorm equivalent with $\|T\|_{\phi_{(p)}}^{\alpha}$ . Further we give a new class of operator ideals by using generalized approximation numbers and symmetric norming function and we show that this class is an operator ideal.

**Keywords:** Operator ideal, $s$-numbers, symmetric norming function.

## 1 Introduction

The operator ideal theory has a special importance in functional analysis. One of the most important methods to construct operator ideals is via $s-$ numbers. Pietsch defined the approximation numbers of a bounded linear operator in Banach spaces, in 1963 [13]. Later on, the other examples of $s-$numbers, namely Kolmogorov numbers, Weyl numbers, etc. are introduced to the Banach space setting.

In this paper, by $\mathbb{N}$ and $\mathbb{R}^+$ we denote the set of all natural numbers and non-negative real numbers, respectively.

A bounded linear operator whose dimension of the range space is finite is called a finite rank operator [9].

Let $E$ and $F$ be real or complex Banach spaces and $\mathscr{L}(E,F)$ denotes the space of all bounded linear operators from $E$ to $F$ and $\mathscr{L}$ denotes the space of all bounded linear operators between any two arbitrary Banach spaces.

A map $s=(s_n):\mathscr{L}\to\mathbb{R}^+$ assigning to every operator $T\in\mathscr{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an $s-$number sequence if the following conditions are satisfied for all Banach spaces $E,F,E_0$ and $F_0$:

$(S1)$ $\|T\|=s_1(T)\geq s_2(T)\geq\ldots\geq 0$ for every $T\in\mathscr{L}(E,F)$,

$(S2)$ $s_{m+n-1}(S+T)\leq s_m(S)+s_n(T)$ for every $S,T\in\mathscr{L}(E,F)$ and $m,n\in\mathbb{N}$,

$(S3)$ $s_n(RST)\leq\|R\|s_n(S)\|T\|$ for some $R\in\mathscr{L}(F,F_0)$, $S\in\mathscr{L}(E,F)$ and $T\in\mathscr{L}(E_0,E)$, where $E_0,F_0$ are arbitrary Banach spaces,


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$(S4)$ If $rank(T) \leq n$, then $s_n(T) = 0$,

$(S5)$ $s_n(I : l_2^n \to l_2^n) = 1$, where $I$ denotes the identity operator on the $n-$dimensional Hilbert space $l_2^n$, where $s_n(T)$ denotes the $n-th$ $s-$number of the operator $T$ [12].

An example of $s$-number sequence is the approximation number, which is defined by Pietsch. The $n-$th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf\{\, \|T - A\| : A \in \mathscr{L}(E,F),\, rank(A) < n\,\},$$

where $T \in \mathscr{L}(E,F)$ and $n \in \mathbb{N}$ [13].

Let the space of all real valued sequences be $\omega$. A sequence space is any vector subspace of $\omega$.

The Cesaro sequence space $ces_p$ is defined as ([1], [4], [5])

$$ces_p = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}, \quad 1 < p < \infty.$$

Let $E'$ be the dual of $E$, which is composed of continuous linear functionals on $E$. Let $x' \in E'$ and $y \in F$, then the map $x' \otimes y : E \to F$ is defined by

$$\left(x' \otimes y\right)(x) = x'(x)y,\; x \in E.$$

Pietsch [13] defined an operator $T \in \mathscr{L}(E,F)$ to be $l_p$ type operator if $\sum_{n=1}^{\infty} (a_n(T))^p < \infty$ for $0 < p < \infty$. Then, Constantin [6], generalized the class of $l_p$ type operators to the class of $ces - p$ type operators by using the Cesaro sequence spaces, where an operator $T \in \mathscr{L}(E,F)$ is called $ces - p$ type if $\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k(T) \right)^p < \infty,\; 1 < p < \infty$. As a generalization of $l_p$ type operators, $A - p$ type operators and Stolz mappings were examined in [7], [8]. Also in [9], [10], [11] Maji and Srivastava studied the class $A^{(s)} - p$ of $s-$type $ces_p$ operators using $s-$number sequence and Cesaro sequence spaces and they introduced a new class $A^{(s)}_{p,q}$ of $s-$type $ces(p,q)$ operators by using weighted Cesaro sequence space for $1 < p < \infty$. In [23], the class of $s-$type $Z(u,v;l_p)$ operators is defined and worked on some properties of this class.

Now let give the definitions of operator ideal and quasi-norm:

A subcollection $\Im$ of $\mathscr{L}$ is called an operator ideal if each component $\Im(E,F) = \Im \cap \mathscr{L}(E,F)$ satisfies the following conditions:

$(OI-1)$ if $x' \in E'$, $y \in F$, then $x' \otimes y \in \Im(E,F)$,

$(OI-2)$ if $S,T \in \Im(E,F)$, then $S + T \in \Im(E,F)$,

$(OI-3)$ if $S \in \Im(E,F)$, $T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$, then $RST \in \Im(E_0,F_0)$[14].

A function $\alpha : \Im \to \mathbb{R}^+$ is said to be a quasi-norm on the operator ideal $\Im$ if the following conditions hold:

$(QN-1)$ If $x' \in E'$, $y \in F$, then $\alpha(x' \otimes y) = \|x'\|\,\|y\|$;

$(QN-2)$ there exists a constant $C \geq 1$ such that $\alpha(S+T) \leq C[\alpha(S)+\alpha(T)]$;

$(QN-3)$ if $S \in \mathfrak{I}(E,F), T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$, then $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$[14].

In particular if $C=1$ then $\alpha$ becomes a norm on the operator ideal $\mathfrak{I}$.

An ideal $\mathfrak{I}$ with a quasi-norm $\alpha$, denoted by $[\mathfrak{I},\alpha]$ is said to be a quasi-Banach operator ideal if each component $\mathfrak{I}(E,F)$ is complete under the quasi-norm $\alpha$.

A map $s^{\alpha} = (s_n^{\alpha}) : \mathfrak{I}(E,F) \to \mathbb{R}^{+}$ assigning to every operator $T \in \mathfrak{I}(E,F)$ a non-negative scalar sequence $\{s_n^{\alpha}(T)\}_{n \in \mathbb{N}}$ is called a generalized $s-$number sequence if the following conditions are satisfied for all Banach spaces $E,F$ [2],[3]:

$(S^{\alpha}1)$ $\alpha(T) = s_1^{\alpha}(T) \geq s_2^{\alpha}(T) \geq \ldots \geq 0$ for all $T \in \mathfrak{I}(E,F)$,

$(S^{\alpha}2)$ $s_{m+n-1}^{\alpha}(S+T) \leq s_m^{\alpha}(S) + s_n^{\alpha}(T)$ for every $S,T \in \mathfrak{I}(E,F)$ and $m,n \in \mathbb{N}$,

$(S^{\alpha}3)$ $s_n^{\alpha}(RST) \leq \|R\| s_n^{\alpha}(S) \|T\|$ ,for some $R \in \mathscr{L}(F,F)$, $S \in \mathfrak{I}(E,F)$ and $T \in \mathscr{L}(E,E)$,

$(S^{\alpha}4)$ If $\dim(T) \leq n$, then $s_n^{\alpha}(T) = 0$.

Consequently, generalized approximation numbers $\{a_n^{\alpha}(T)\}$ are the examples of generalized $s$-numbers, where

$$a_n^{\alpha}(T) = \inf\{\alpha(T-K) : K \in \mathfrak{I}, \dim K < n\} \text{ [3]}.$$

By using the generalized approximation numbers we define the class of $l_p^{\alpha}-type$ operators as

$$L_p^{\alpha}(E,F) = \left\{ T \in \mathfrak{I}(E,F) : \sum_{n=1}^{\infty} (a_n^{\alpha}(T))^p < \infty \right\}$$

for $0 < p < \infty$.

Let $\ell_{\infty}$ be the space of all bounded real sequences and $K \subset \ell_{\infty}$ be the set of all sequences $x$ such that $card\{i \in \mathbb{N},\ x_i \neq 0\} < n$ and $x_1 \geq x_2 \geq \ldots \geq 0$.

A function $\phi : K \to \mathbb{R}$ is called symmetric norming function, if the following conditions satisfied

$(\phi 1)$ $\phi(x) > 0$, $\forall x \neq 0$,

$(\phi 2)$ $\phi(\alpha x) = \alpha \phi(x)$, $x \in K, \alpha \geq 0$

$(\phi 3)$ $\phi(x+y) \leq \phi(x) + \phi(y)$

$(\phi 4)$ $\phi(1,0,0,\ldots) = 1$

$(\phi 5)$ if $\sum_{1}^{k} x_i \leq \sum_{1}^{k} y_i$, $k = 1,2,\ldots$ then $\phi(x) \leq \phi(y)$.

It is given that ([15], [16]) for all symmetric norming functions $\phi$, the function $\phi_{(p)}$ defined as

$$\phi_{(p)} : (x_i) \in K \to \left(\phi\left(\left\{x_i^p\right\}\right)\right)^{\frac{1}{p}}, 1 \leq p \leq \infty$$

is also a symmetric norming function. For more details on symmetric norming functions we refer to [2], [15], [17], [18], [19], [20].

By using the properties of symmetric norming function and the sequence $(a_n(T))$, the class $\mathscr{L}_\phi(E,F)$ is defined in [18] and [21] as follows

$$\mathscr{L}_\phi(E,F) = \{T \in \mathscr{L}(E,F) : \phi(\{a_n(T)\}) < \infty\}.$$

By using the relations $a_{2n-1}^\alpha(T_1+T_2) \leq a_n^\alpha(T_1) + a_n^\alpha(T_2)$, $n = 1,2,\ldots$ and $a_n^\alpha(\beta T) = |\beta| a_n^\alpha(T)$, ($\beta$ is a scalar) and the properties of the function $\phi$, $\|T\|_\phi^\alpha = \phi(\{a_n^\alpha(T)\})$ and $\|T\|_{\phi_{(p)}}^\alpha = \phi_{(p)}(\{a_n^\alpha(T)\})$ are quasinorms.

In this study, we introduce the class of generalized Stolz mappings by generalized approximation numbers. Also we prove that the class of $\ell_p^\alpha-$type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers and we define a new quasinorm equivalent with $\|T\|_{\phi_{(p)}}^\alpha$. Further we introduce a new class of operator ideals by using approximation numbers and symmetric norming function. We prove that this class is an operator ideal.

## 2 Main results

Throughout this paper $(u_n)$ and $(w_n)$ sequences satisfies the following conditions:

Let $(u_n)$ and $(w_n)$ be sequences of non-negative real numbers such that $u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots$ and $w_1 \leq w_2 \leq \ldots \leq w_n \leq \ldots$ and $w_n \leq n \leq \frac{w_n}{u_n}$. Let $T \in \mathscr{L}(E,F)$ then the class of generalized Stolz mappings $L_{GSTOL,p}(E,F)$, in [22] defined as

$$L_{GSTOL,p}(E,F) = \left\{T : \sum_{n=1}^{\infty}\left[\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i(T)\right]^p < \infty\right\},\ \ 0 < p < \infty.$$

Now we define the class of generalized Stolz mappings by generalized approximation numbers $L_{GSTOL,p}^\alpha(X)$ as:

Let $T \in \mathfrak{I}(E,F)$ and $\alpha$ be an ideal norm:

$$L_{GSTOL,p}^\alpha(X) = \left\{T : \sum_{n=1}^{\infty}\left[\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i^\alpha(T)\right]^p < \infty\right\},\ \ 0 < p < \infty.$$

Then we introduce a new class of operator ideals with the help of symmetric norming function as follows:

$$\mathfrak{I}_{\phi_{(p)}}^\alpha(E,F) = \left\{T \in \mathfrak{I}(E,F) : \phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i^\alpha(T)\right\}\right) < \infty\right\}.$$

In the following theorem, we prove that $\ell_p^\alpha-$ type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers.

**Theorem 1.** *If $\lim_{n\to\infty} u_n = u \neq 0$, the class of $\ell_p^\alpha-$ type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers* $(1 < p < \infty)$.

*Proof.* Let $T \in \Im(E,F)$ and $(u_n)$ and $(w_n)$ be sequences of non-negative real numbers such that $u_1 \geq u_2 \geq ... \geq u_n \geq \dots$ and $w_1 \leq w_2 \leq ... \leq w_n \leq ...$ and $w_n \leq n \leq \frac{w_n}{u_n}$ and $\lim_{n\to\infty} u_n \neq 0$. Then we can write

$$\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right)^p \leq \sum_{n=1}^{\infty}\left(\frac{u_1}{nu_n}\sum_{i=1}^{n}a_i^{\alpha}(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\alpha}(T)\right)^p.$$

Since $\sum_{n=1}^{\infty}(a_n^{\alpha}(T))^p < \infty$, we obtain from Hardy's inequality that

$$\left(\frac{u_1}{u}\right)^p\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\alpha}(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}(a_n^{\alpha}(T))^p < \infty.$$

It follows that

$$\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right)^p < \infty.$$

Hence the class of $\ell_p^\alpha-$type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers $(1 < p < \infty)$.

**Theorem 2.** *Let $\lim_{n\to\infty} u_n \neq 0$ then the quasinorm $\|T\|_{\phi_{(p)}}^{\alpha}$ is equivalent with*

$$\|\hat{T}\|_{\phi_{(p)}}^{\alpha,\gamma} = \phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right\}\right) \quad (1 < p < \infty).$$

*Proof.* Since the sequences $(u_n)$ and $(a_n^{\alpha}(T))$ are decreasing we can write

$$\frac{1}{n}nu_n a_n^{\alpha}(T) \leq \frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T) \leq \frac{1}{nu_n}u_1\sum_{i=1}^{n}a_i^{\alpha}(T).$$

Summing from $n = 1$ to $k$, we get

$$\sum_{n=1}^{k}(u_n a_n^{\alpha}(T))^p \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right)^p \leq \sum_{n=1}^{k}\left(\frac{u_1}{nu_n}\sum_{i=1}^{n}a_i^{\alpha}(T)\right)^p.$$

If $\lim_{n\to\infty} u_n = u \neq 0$ then we obtain

$$u^p\sum_{n=1}^{k}(a_n^{\alpha}(T))^p \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\sum_{n=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\alpha}(T)\right)^p$$

for every $k \in \mathbb{N}$. By using Hardy's inequality

$$u^p\sum_{n=1}^{k}(a_n^{\alpha}(T))^p \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^{\alpha}(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{k}(a_n^{\alpha}(T))^p$$

for every $k \in \mathbb{N}$. From the properties of the function $\phi$ we have that

$$u\|T\|^{\alpha}_{\phi_{(p)}} \leq \left\|\hat{T}\right\|^{\alpha,\gamma}_{\phi_{(p)}} \leq \left(\frac{u_1}{u}\right)\left(\frac{p}{p-1}\right)\|T\|^{\alpha}_{\phi_{(p)}}.$$

To prove the next theorem we need the following lemma:

**Lemma 1.** [2] *Generalized approximation numbers verify the inequality:*

$$\sum_{n=1}^{k} a_n^{\alpha}(S+T) \leq 2\sum_{n=1}^{k}\left(a_n^{\alpha}(S)+a_n^{\alpha}(T)\right), k=1,2,\dots \quad . \tag{1}$$

*Proof.*

$$\begin{aligned}\sum_{n=1}^{k} a_n^{\alpha}(S+T) &\leq \sum_{n=1}^{2k} a_n^{\alpha}(S+T) = \sum_{n=1}^{k} a_{2n-1}^{\alpha}(S+T) + \sum_{n=1}^{k} a_{2n}^{\alpha}(S+T) \\ &\leq 2\sum_{n=1}^{k} a_{2n-1}^{\alpha}(S+T) \leq 2\sum_{n=1}^{k}\left(a_n^{\alpha}(S)+a_n^{\alpha}(T)\right).\end{aligned}$$

**Theorem 3.** *If $\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right) < \infty$, then the class $\Im^{\alpha}_{\phi_{(p)}}(E,F)$ is a quasi-normed operator ideal by*

$$\|T\|^{a,\gamma}_{\phi_{(p)}} = \frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i^{\alpha}(T)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}, \quad (1 < p < \infty).$$

*Proof.* We prove the properties of an operator ideal and the ideal quasi-norm. Let $E$ and $F$ be any two Banach spaces. Let $x' \in E'$, $y \in F$ then $x' \otimes y$ is a rank one operator. So

$$a_n^{\alpha}\left(x' \otimes y\right) = 0 \ \text{ for all } n \geq 2.$$

By using the properties of symmetric norming function and the generalized approximation number we can get;

$$\begin{aligned}\left\|x' \otimes y\right\|^{\alpha,\gamma}_{\phi_{(p)}} &= \frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i^{\alpha}(x' \otimes y)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)} \\ &= \frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}u_1 a_1^{\alpha}(x' \otimes y)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)} \\ &= \frac{u_1\alpha(x' \otimes y)\,\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)} \\ &= \left\|x'\right\|\|y\|.\end{aligned}$$

Hence, $x'\otimes y\in\Im^{\alpha}_{\phi_{(p)}}(E,F)$ and $\|x'\otimes y\|^{\alpha,\gamma}_{\phi_{(p)}}=\|x'\|\,\|y\|$. Let $S,T\in\Im^{\alpha}_{\phi_{(p)}}(E,F)$. By using (1) and the properties of symmetric norming function we can calculate;

$$\begin{aligned}\|S+T\|^{\alpha,\gamma}_{\phi_{(p)}}&=\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_ia_i^{\alpha}(S+T)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\le\frac{2\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_i\left(a_i^{\alpha}(S)+a_i^{\alpha}(T)\right)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\\&=2\left[\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_ia_i^{\alpha}(S)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}+\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_ia_i^{\alpha}(T)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\right]\\&=2\left[\|S\|^{\alpha,\gamma}_{\phi_{(p)}}+\|T\|^{\alpha,\gamma}_{\phi_{(p)}}\right]<\infty.\end{aligned}$$

Hence $S+T\in\Im^{\alpha}_{\phi_{(p)}}(E,F)$ and $\|S+T\|^{\alpha,\gamma}_{\phi_{(p)}}\le 2\left[\|S\|^{\alpha,\gamma}_{\phi_{(p)}}+\|T\|^{\alpha,\gamma}_{\phi_{(p)}}\right]$. Now let $S\in\Im^{\alpha}_{\phi_{(p)}}(E,F)$, $T\in\Im(E_0,E)$ and $R\in\Im(F,F_0)$. Then

$$\begin{aligned}\|RST\|^{\alpha,\gamma}_{\phi_{(p)}}&=\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_ia_i^{\alpha}(RST)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\\&\le\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_i\|R\|\,a_i^{\alpha}(S)\,\|T\|\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\\&=\|R\|\,\|T\|\frac{\phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}u_ia_i^{\alpha}(S)\right\}\right)}{u_1\phi_{(p)}\left(\left\{\frac{1}{w_n}\right\}\right)}\\&=\|R\|\,\|T\|\,\|S\|^{\alpha,\gamma}_{\phi_{(p)}}<\infty.\end{aligned}$$

Hence $RST\in\Im^{\alpha}_{\phi_{(p)}}(E_0,F_0)$ and $\|RST\|^{\alpha,\gamma}_{\phi_{(p)}}\le\|R\|\,\|T\|\,\|S\|^{\alpha,\gamma}_{\phi_{(p)}}$. $\Im^{\alpha}_{\phi_{(p)}}(E,F)$ is an operator ideal and $\|\cdot\|^{\alpha,\gamma}_{\phi_{(p)}}$ is an ideal quasi-norm.

For the particular case, if we choose $(u_n)=\left(\frac{2n+5}{10n}\right)$, $(w_n)=(n)$ and $\phi(x)=\sum_{i=1}^{n}x_i$, we can get a quasi-normed operator ideal by

$$\|T\|^{\alpha,\gamma}_{\phi_{(p)}}=\frac{\phi_{(p)}\left(\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2n+5}{10n}\right)a_i^{\alpha}(T)\right\}\right)}{\left(\frac{7}{10}\right)\phi_{(p)}\left(\left\{\frac{1}{n}\right\}\right)}.$$

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## Competing interests

The authors declare that they have no competing interests.

## Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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