Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Generalization of Chebyshev wavelet collocation method to the rth-order differential equations

Ibrahim celik

Department of Mathematics, Faculty of Arts and Sciences, Pamukkale University, Denizli, Turkey

Received: 20 April 2018, Accepted: 24 September 2018
Published online: 25 September 2018.

Abstract: Chebyshev wavelets operational matrices play an important role for the numeric solution of rth order differential equations. In this study, operational matrices of *rth* integration of Chebyshev wavelets are presented and a general procedure of these matrices is correspondingly given. Disadvantages of Chebyshev wavelets methods is eliminated for rth integration of $\Psi(t)$. The proposed method is based on the approximation by the truncated Chebyshev wavelet series. Algebraic equation system has been obtained by using the Chebyshev collocation points and solved. The proposed method has been applied to the three nonlinear boundary value problems using quasilinearization technique. Numerical examples showed the applicability and accuracy.

Keywords: Chebyshev wavelets, collocation method, nonlinear differential equations, quasilinearization technique, approximate solution.

1 Introduction

Various numerical methods based on orthogonal functions and polynomial series, involving wavelets, have been used to obtain approximate solutions of various problems in recent years, [1,2,3,4,5,6]. Using truncated orthogonal approximate series as:

$$y(t) \cong y_N(t) = \sum_{i=0}^{N-1} c_i\phi_i$$

these problems are reduced to solve a algebraic equation system. Fundamental approach is based on transforming the given differential equations into integral equations by integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix P obtained by eliminating the integral operations as:

$$\int_0^t \Phi(s)ds \cong P\Phi(t).$$

Where $\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{N-1}(t)]^T$ and matrix P is uniquely determined on the basis of particular orthogonal functions defined on the interval [a, b]. The form of P depends on the choice of the orthogonal functions. Typical examples of orthogonal functions or polynomials are Legendre polynomials [2], Chebyshev polynomials [3], Walsh function [7], block-pulse function [8] and Legendre series [9]. All of the orthogonal functions previously mentioned dependent on the interval [a, b]. However, this global dependence on the interval [a, b] is clearly a disadvantage for some analysis works, particularly systems involving abrupt variations or local function vanishing outside a short interval of time or space [5].

Wavelets, known as very well-localized functions, a powerful and recognized tool used in image processing, quantum

* Corresponding author e-mail: i.celik@pau.edu.tr

© 2018 BISKA Bilisim Technology

mechanics, signal processing, computer science, communications and mathematics and many more other areas. Wavelets are greatly useful for solving differential, fractional differential, integral, integro-differential and fractional Volterra integro-differential equations and give accurate solutions [10,11,12,13,?,15,16,17]. The wavelet technique allows the development of extremely fast algorithms when it is compared with the algorithms ordinarily used. Gu and Jiang [18] derived the Haar wavelets operational matrix of integration. Lepik [19] solved Burgers and sine-Gordon equations by using the Haar wavelet method. Geng et al [20] and Hariharan et al. [21,22,23,24,25] used Haar wavelets methods for the solution of some nonlinear PDEs. Kaur et al [26] used Haar wavelet quasilinearization approach for solving nonlinear boundary value problems. Karabacak and Çelik [27] solved fractional differential-algebraic equations with Haar wavelet method. Çelik [28,29] solved generalized Burgers-Huxley equation and magnetohydrodynamic flow equations with Haar wavelet method. In the literature, special attention has been given to the applications of Legendre wavelets [6,30]. The Legendre and Chebyshev wavelets operational matrixes of integration and product operation matrix have been introduced in [31,32,33,34,35]. These matrices may be used to solve problems such as identification, analysis and optimal control. Karabacak and Yiğider [36] analyzed fractional differential-algebraic equations with Legendre wavelets method. Adibi and Assari [37] used Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. Wang and Fan [38] solved fractional differential equations with the second kind Chebyshev wavelet method. Heydari et al [39] used Chebyshev wavelets method for solution of nonlinear fractional integrodifferential equations. Hooshmandasl et al [40] solved one dimensional heat equation by using Chebyshev wavelets method. Yang and Hou [41] used Chebyshev wavelets method for solving Bratu's problem. Hariharan et al. [42] used Chebyshev wavelets based approximation method to water quality assessment model problem. Çelik [43,44,45] solved differential equations, generalized Burgers-Huxley equation and Free vibration problems of non-uniform Euler-Bernoulli beam by Chebyshev wavelet collocation method.

Analyses show that there are some disadvantages in applying Legendre and Chebyshev wavelet methods in terms of taking second integration as P^2. In Razzaghi and Yousefi [15,16] and Babolian and Fattahzadeh [17], the operational matrix of integration P are derived as

$$\left.\begin{array}{l} \int_0^t \Psi(s)ds \cong P\Psi(t) \\ \int_0^t \int_0^s \Psi(w)dwds \cong P^2\Psi(x) \end{array}\right\} \tag{1}$$

Where $\Psi(t)$ is vector and P is a matrix. Some expressions, obtained from integration of $\Psi(t)$, have not been used in the construction of the matrix P. For example, in the Chebyshev wavelet method, T_M and T_{M+1} obtained from the first and second integrations of the $\Psi(t)$ have not been used in the construction of P and P^2. These are the disadvantages of the Legendre wavelet and Chebyshev wavelet methods. Çelik [43,44] showed that P^2 was different from the second integration, directly taken, for Chebyshev wavelet method. That is

$$\int_0^t \int_0^s \Psi(w)dwds \neq P^2\Psi(t) \tag{2}$$

Hence, disadvantages of Chebyshev wavelets methods are eliminated by new construction of P.

This study presents a Chebyshev wavelet collocation method for the solution of rth-order linear ordinary differential equations with variable coefficients given in the following form:

$$y^{(r)}(t) + \sum_{i=1}^{r} A_i(t)\frac{d^{r-i}y(t)}{dt^{r-i}} = G(t) \tag{3}$$

© 2018 BISKA Bilisim Technology

where A_i and G are functions of x defined in the interval of $a \le t \le b$. rth-order nonlinear ordinary differential equations with variable coefficients is given as:

$$y^{(r)}(t) = F\left(t, y(t), y'(t),\ \ldots, y^{(r-1)}(t)\right) \tag{4}$$

The quasilinearization of these equations gives a set of recurrence linear differential equations

$$y_{s+1}^{(r)}(x) = F\left(t, y_s(t), y_s'(t),\ \ldots, y_s^{(r-1)}(t)\right) + \sum_{i=0}^{r-1}\left(y_{s+1}^{(i)}(t) - y_s^{(i)}(t)\right) F_{y_s^{(i)}(t)}\left(t, y_s(t), y_s'(t),\ \ldots, y_s^{(r-1)}(t)\right) \tag{5}$$

where $F_{y_s^{(i)}(t)}\left(t, y_s(t), y_s'(t),\ \ldots, y_s^{(r-1)}(t)\right) = \frac{\partial}{\partial y_s^{(i)}(t)}\left(F\left(x, y_s(t), y_s'(t),\ \ldots, y_s^{(r-1)}(t)\right)\right)$ and $y_0(t)$ is taken as a function satisfying initial/boundary conditions. Any range $a \le x \le b$ can be transformed in to the basic range $0 \le t \le 1$ with the change of variables $x = (b-a)t + a$.

In the proposed method, there are no disadvantages of the Legendre wavelet and Chebyshev wavelet methods. The method is based on the approximation by the truncated Chebyshev wavelets series. By using the Chebyshev collocation points, algebraic equation system has been obtained. Solving this algebraic equation system, the coefficients of the Chebyshev wavelet series can be found. Hence, we have the implicit form of the approximate solution of rth-order linear or nonlinear ordinary differential equations. The method is applied to four nonlinear boundary value problems using quasilinearization technique. Calculations demonstrated that the accuracy of the Chebyshev wavelet collocation method is quite high even in the case of a small number of grid points.

2 Chebyshev wavelet method

Wavelets consist of a family of functions defined by dilation and translation of a single function named the mother wavelet. If a as a dilation parameter and b as translation parameter vary continuously, the following family is a continuous wavelets [10].

$$\psi_{a,b}(t) = |a|^{\ 1/2}\,\psi\left(\frac{t-b}{a}\right), \quad a, b \in R,\ \ a \neq 0. \tag{6}$$

Chebyshev wavelets are written as

$$\psi_{nm}(t) = \psi(k, n, m, t), \tag{7}$$

where $k = 0,\ 1,\ 2,\ \ldots,\ \ n = 1,\ 2,\ \ldots,\ 2^k$, m is degree of Chebyshev polynomials of the first kind and x denotes the normalized time. They are defined on the interval $[0,1)$ by:

$$\psi_{nm}(t) = \begin{cases} \frac{\alpha_m 2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \le x \le \frac{n}{2^k}, \\ 0, & otherwise \end{cases} \tag{8}$$

where

$$\alpha_m = \begin{cases} \sqrt{2}, & m = 0 \\ 2, & m = 1,\ 2,\ \ldots \end{cases} \tag{9}$$

and $T_m(2^{k+1}t - 2n + 1)$ are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function $w_n(t) = w(2^{k+1}t - 2n + 1)$on $[-1,\ 1]$ [46].

© 2018 BISKA Bilisim Technology

A function $f(t) \in L_w^2[0,1]$ may be expanded as:

$$f(f) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(t) \tag{10}$$

where

$$f_{nm} = \langle f(x), \psi_{nm}(t)\rangle \tag{11}$$

$\langle .,. \rangle$ denotes the inner product with weight function $w_n(t)$ in Eq. (11).

Truncated form of Eq. (10) can be written as:

$$f(t) \cong \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = C^T\Psi(t) \tag{12}$$

where C and $\Psi(t)$ are $2^kM\times 1$ columns vectors given by:

$$C^T = [f_{1,0},\ f_{1,1},\ \ldots,\ f_{1,M-1},\ f_{2,0},\ \ldots,\ f_{2,M-1},\ \ldots,\ f_{2^k,0},\ \ldots,\ f_{2^k,M-1}\,] \tag{13}$$

$$\Psi(t) = [\psi_{1,0},\ \psi_{1,1},\ \ldots,\ \psi_{1,M-1},\ \psi_{2,0},\ \ldots,\ \psi_{2,M-1},\ \ldots,\ \psi_{2^k,0},\ \ldots,\ \psi_{2^k,M-1}\,]^T \tag{14}$$

The integration of the $\psi_{nm}(t)$ given in Eq. (8) can be shown as

$$p_{nm}(t) = \int_0^t \psi_{nm}(s)ds \tag{15}$$

For $m=0, m=1$ and $m>1$, $p_{nm}(t)$ can be obtained as

$$p_{n0}(t) = \begin{cases} 0, & 0 \le t < \frac{n-1}{2^k} \\ \frac{\alpha_0 2^{-k/2-1}}{\sqrt{\pi}}\left[T_1(2^{k+1}t-2n+1)+T_0(2^{k+1}t-2n+1)\right], & \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ \frac{\alpha_0 2^{-k/2}}{\sqrt{\pi}}T_0(2^{k+1}t-2n+1), & \frac{n}{2^k} \le t < 1 \end{cases} \tag{16}$$

$$p_{n1}(t) = \begin{cases} 0, & 0 \le t < \frac{n-1}{2^k} \\ \frac{\alpha_1 2^{-k/2-3}}{\sqrt{\pi}}\left[T_2(2^{k+1}t-2n+1)-T_0(2^{k+1}t-2n+1)\right], & \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & \frac{n}{2^k} \le t < 1 \end{cases} \tag{17}$$

$$p_{nm}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ \frac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\frac{T_{m+1}(u)-(-1)^{m+1}}{m+1} - \frac{T_{m-1}(u)-(-1)^{m-1}}{m-1}\right], & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ \frac{\alpha_m 2^{-k/2-2}}{\sqrt{\pi}}\left[\frac{1-(-1)^{m+1}}{m+1} - \frac{1-(-1)^{m-1}}{m-1}\right], & \frac{n}{2^k} \le x < 1 \end{cases} \tag{18}$$

where $u = 2^{k+1}x-2n+1$. The integration of the $\Psi(t)$ can be represented as

$$\int_0^t \Psi(s)ds = [p_{1,0},\ p_{1,1},\ \ldots,\ p_{1,M-1},\ p_{2,0},\ \ldots,\ p_{2,M-1},\ \ldots,\ p_{2^k,0},\ \ldots,\ p_{2^k,M-1}\,]^T = P_1\Psi_1(t) \tag{19}$$

where

$$\Psi_1(t) = [\psi_{1,0},\ \psi_{1,1},\ \ldots,\ \psi_{1,M},\ \psi_{2,0},\ \ldots,\ \psi_{2,M},\ \ldots,\ \psi_{2^k,0},\ \ldots,\ \psi_{2^k,M}\,]^T \tag{20}$$

© 2018 BISKA Bilisim Technology

$$L_1 = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3}}{M-3} - \frac{(-1)^{M-1}}{M-1}\right) & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-3)} & 0 & \frac{1}{2(M-1)} & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2} - \frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2(M-2)} & 0 & \frac{1}{2M} \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{2\sqrt{2}}{3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M-1}}{M-1} - \frac{1-(-1)^{M-3}}{M-3}\right) & 0 & \cdots & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix} P_1 = \frac{1}{2^{k+1}} \begin{bmatrix} L_1 & F_1 & F_1 & \cdots & F_1 & F_1 \\ 0 & L_1 & F_1 & \cdots & F_1 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L_1 & F_1 \\ 0 & 0 & 0 & \cdots & 0 & L_1 \end{bmatrix}$$

The second integrations of the $\Psi(t)$can be represented as

$$\int_0^t \int_0^{t_1} \Psi(s)dsdt_1 = \int_0^t P_1\Psi_1(t_1)dt_1 = P_1\int_0^t \Psi_1(t_1)dt_1 = P_1P_2\Psi_2(t) \neq P^2\Psi(t) \tag{21}$$

The rth integrations of the $\Psi(t)$can be represented as

$$\int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{r-1}} \Psi(s)\,dsdt_{r-1}dt_{r-2}\cdots dt_1 = P_1P_2\cdots P_r\Psi_r(t) \neq P^r\Psi(t) \tag{22}$$

where

$$L_r = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-\sqrt{2}}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3}}{M-3} - \frac{(-1)^{M-1}}{M-1}\right) & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-3)} & 0 & \frac{1}{2(M-1)} & 0 & \cdots & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-2}}{M-2} - \frac{(-1)^{M}}{M}\right) & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2(M-2)} & 0 & \frac{1}{2M} & \cdots & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-1}}{M-1} - \frac{(-1)^{M+1}}{M+1}\right) & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{2(M-1)} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-3+r}}{M-3+r} - \frac{(-1)^{M+r-1}}{M+r-1}\right) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2(M-3+r)} & 0 & \frac{1}{2(M-1+r)} \end{bmatrix}$$

© 2018 BISKA Bilisim Technology

$$F_r = \begin{bmatrix} 2 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \frac{2\sqrt{2}}{3} & 0 \cdots 0 \\ \vdots & 0 \ddots 0 \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M-1}}{M-1} - \frac{1-(-1)^{M-3}}{M-3}\right) & 0 \cdots 0 \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 \cdots 0 \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M+1}}{M+1} - \frac{1-(-1)^{M-1}}{M-1}\right) & 0 \cdots 0 \\ \vdots & 0 \ddots 0 \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M+r-1}}{M+r-1} - \frac{1-(-1)^{M+r-3}}{M+r-3}\right) & 0 \cdots 0 \end{bmatrix} P_r = \frac{1}{2^{k+1}} \begin{bmatrix} L_r & F_r & F_r & \cdots & F_r & F_r \\ 0 & L_r & F_r & \cdots & F_r & F_r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L_r & F_r \\ 0 & 0 & 0 & \cdots & 0 & L_r \end{bmatrix}$$

and

$$\Psi_r(t) = [\psi_{1,0},\ \psi_{1,1},\ \dots,\ \psi_{1,M+r-1},\ \psi_{2,0},\ \dots,\ \psi_{2,M+r-1},\ \dots,\ \psi_{2^k,0},\ \dots,\ \psi_{2^k,M+r-1}]^T \tag{23}$$

The matrices L_r and F_r have the dimension $(M+r-1)\times(M+r)$. Hence P_r has the dimension $2^k(M+r-1)\times 2^k(M+r)$.

3 Chebyshev wavelet collocation method for *R*th-order differential equations

Consider Eq. (3) or Eq. (5) with initial conditions

$$\sum_{i=0}^{r-1} \lambda_{ij} y^{(i)}(0) = \delta_j,\ j = 0, 1, 2, \dots, r-1 \tag{24}$$

or boundary conditions

$$\sum_{i=0}^{r-1} \alpha_{ij} y^{(i)}(0) + \beta_{ij} y^{(i)}(1) = \delta_j,\ j = 0, 1, 2, \dots, r-1. \tag{25}$$

It is assumed that $y^{(r)}(t)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y^{(r)}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = C^T\Psi(t). \tag{26}$$

By successively integration of Eq. (26) with respect to t from 0 to t, following equations are obtained

$$y^{(r-1)}(t) = \int_0^t C^T\Psi(s)ds + y^{(r-1)}(0) = C^T P_1 \Psi_1(t) + y^{(r-1)}(0). \tag{27}$$

$$y^{(r-2)}(t) = C^T P_1 \int_0^t \Psi_1(s)ds + ty^{(r-1)}(0) + y^{(r-2)}(0) = C^T P_1 P_2 \Psi_2(t) + ty^{(r-1)}(0) + y^{(r-2)}(0). \tag{28}$$

$$\begin{aligned} y^{(r-3)}(t) &= C^T P_1 P_2 \textstyle\int_0^t \Psi_2(s)ds + \frac{t^2}{2} y^{(r-1)}(0) + ty^{(r-2)}(0) + y^{(r-3)}(0) \\ &= C^T P_1 P_2 P_3 \Psi_3(t) + \frac{t^2}{2} y^{(r-1)}(0) + ty^{(r-2)}(0) + y^{(r-3)}(0). \end{aligned} \tag{29}$$

$$\vdots$$

© 2018 BISKA Bilisim Technology

$$\begin{aligned} y(x) &= C^T P_1 P_2 \ldots P_{r-1} \int_0^t \Psi_{r-1}(s)ds + \tfrac{t^{r-1}}{(r-1)!} y^{(r-1)}(0) + \tfrac{t^{r-2}}{(r-2)!} y^{(r-2)}(0) + \ \ldots \ + t y'(0) + y(0) \\ &= C^T P_1 P_2 P_3 \ldots P_r \Psi_r(t) + \tfrac{t^{r-1}}{(r-1)!} y^{(r-1)}(0) + \tfrac{t^{r-2}}{(r-2)!} y^{(r-2)}(0) + \ \ldots \ + t y'(0) + y(0). \end{aligned} \tag{30}$$

In these equations, the values of $y(0), y'(0), \ \ldots, y^{(t-1)}(0)$ can be obtained with initial conditions or boundary conditions.

Replacing Eqs. (27)-(30) into the Eq. (3) or Eq. (5), we have the following equations.

$$\begin{aligned} & C^T \left(\Psi(t) + A_1(t) P_1 \Psi_1(t) + \ \ldots \ + A_r(t) P_1 P_2 P_3 \ldots P_r \Psi_r(t)\right) \\ & \quad + y^{(r-1)}(0) \textstyle\sum_{v=0}^{r-1} A_{v+1}(t) \frac{t^v}{v!} + y^{(r-2)}(0) \sum_{v=0}^{r-2} A_{v+2}(t) \frac{t^v}{v!} + y^{(r-3)}(0) \sum_{v=0}^{r-3} A_{v+3}(t) \frac{t^v}{v!} \\ & \quad + \ldots + (A_{r-1}(t) + t A_r(t)) y'(0) + A_r(t) y(0) = G(t) \end{aligned} \tag{31}$$

$$\begin{aligned} & C^T \left(\Psi(t) - F_{y_s^{(r-1)}} P_1 \Psi_1(t) - \ \ldots \ - F_{y_s} P_1 P_2 P_3 \ldots P_r \Psi_r(t)\right) = F\left(t, y_s(t), y_s'(t), \ \ldots, y_s^{(r-1)}(t)\right) \\ & \quad + y^{(r-1)}(0) \textstyle\sum_{v=0}^{r-1} F_{y_s^{(r-1-v)}} \frac{t^v}{v!} + y^{(r-2)}(0) \sum_{v=0}^{r-2} F_{y_s^{(r-2-v)}} \frac{t^v}{v!} + F_{y_s^{(r-3)}} y^{(r-3)}(0) \sum_{v=0}^{r-3} F_{y_s^{(r-3-v)}} \frac{t^v}{v!} \\ & \quad + \ldots + (F_{y_s'} + t F_{y_s}) y'(0) + F_{y_s} y(0) - \textstyle\sum_{i=0}^{r-1} y_n^{(i)} F_{y_n^{(i)}} \left(t, y_s(x), y_s'(x), \ \ldots, y_s^{(r-1)}(t)\right). \end{aligned} \tag{32}$$

The collocation points can be taken as $2^{k+1} t_{n,i} - 2n + 1 = \cos \frac{((M+1)-i)\pi}{(M+1)}$ or

$$t_{n,i} = \frac{1}{2^{k+1}} \left(2n - 1 + \cos \frac{((M+1)-i)\pi}{(M+1)}\right), i = 1, \ 2, \ \ldots, \ M, n = 1, \ 2, \ \ldots, \ 2^k. \tag{33}$$

which are also called the turning points of $T_{M+1}(2^{k+1}t - 2n + 1)$. Substituting the Chebyshev collocation points into (14), (19) and (23), a discretized form of the vectors $\Psi(t_{n,i}), \Psi_1(t_{n,i})$ and $\Psi_r(t_{n,i})$ can be obtained. Hence form Eq. (31) or Eq. (32), the algebraic equation system whose matrix notation is obtained as:

$$C^T U = B \tag{34}$$

where U is a$2^k M \times 2^k M$ matrix. C and B are $2^k M \times 1$ vectors. Hence, by solving algebraic equation system (34), we can find the coefficients of the Chebyshev wavelet series that satisfied differential equation with initial or boundary conditions.

4 Error analysis

Lemma 1. *If the Chebyshev wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev wavelet expansion converges to the function $f(t)$ [37].*

Theorem 1. *A function $f(t) \in L^2_\omega([0,1])$ with bounded second derivative, $|f''(t)| \leq N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(t)$ [37]. That is*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \psi_{nm}(t). \tag{35}$$

© 2018 BISKA Bilisim Technology

Since the truncated Chebyshev wavelets series is an approximate solution of problem, so one has an error function $f(t)$ for $y(t)$ as follows:

$$E(t) = \left|y(t) - C^T\Psi(t)\right|. \tag{36}$$

The error bound of the approximate solution using Chebyshev wavelets series is given by the following theorem.

Theorem 2. *Suppose that $y(t) \in C^m[0,1]$ and $C^T\Psi(t)$ is the approximate solution of problem using the Chebyshev wavelets method. Then the error bound would be obtained as follows [41]:*

$$E(t) \leq \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^{(m)}(t)\right| \right\|^2 \tag{37}$$

5 Numerical results

Example 1. Consider the nonlinear boundary value problem [26]

$$\begin{aligned} &y''(t) = y^2(t) + 2\pi^2\cos(2\pi t) - \sin^4(\pi t), \quad 0 < t < 1, \\ &y(0) = y(1) = 0, \end{aligned} \tag{38}$$

with analytic solution$y(t) = \sin^2(\pi t)$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique. First approximate solution is taken as $y_0(t) = \sin(\pi t)$ and $y_1^2(t) \cong 2y_0(t)y_1(t) - y_0^2(t)$ can be obtained by quasilinearization technique. Hence converted problem is obtained as

$$y''_{s+1}(t) - 2y_s(t)y_{s+1}(t) = -y_s^2(t) + 2\pi^2\cos(2\pi t) - \sin^4(\pi t), \tag{39}$$

where $y_s(t)$ is known. It is assumed that $y''(t)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y''_{s+1}(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = C^T\Psi(t). \tag{40}$$

By integrating this equation twice with respect to t from 0 to $t$and using boundary condition, following equations are obtained.

$$y'_{s+1}(t) = \int_0^t C^T\Psi(s)ds + y'_{s+1}(0) = C^T P_1\Psi_1(t) + y'_{s+1}(0). \tag{41}$$

$$y_{s+1}(x) = C^T P_1 \int_0^t \Psi_1(s)ds + ty'_{s+1}(0) + y_{s+1}(0) = C^T P_1P_2\Psi_2(t) + ty'_{s+1}(0). \tag{42}$$

$$y_{s+1}(1) = C^T P_1P_2\Psi_2(1) + y'_{s+1}(0) = 0 \ \Rightarrow y'_{s+1}(0) = -\,C^T P_1P_2\Psi_2(1). \tag{43}$$

$$y_{s+1}(t) = C^T P_1P_2\Psi_2(t) - tC^T P_1P_2\Psi_2(1) = C^T\left(P_1P_2\Psi_2(t) - tP_1P_2\Psi_2(1)\right). \tag{44}$$

Replacing Eqs. (40) and (44) into the Eq. (39), we have

$$C^T\left(\Psi(t) - 2y_s(t)P_1P_2\Psi_2(t) + 2ty_s(t)P_1P_2\Psi_2(1)\right) = -y_s^2(t) + 2\pi^2\cos(2\pi t) - \sin^4(\pi t). \tag{45}$$

Substituting the Chebyshev collocation points (33) into the (45), we can obtain algebraic equation system as given (34). Hence, by solving algebraic equation system, coefficients C^T of the Chebyshev wavelet series can be obtained. By substituting the Chebyshev wavelet coefficients into Eq. (44), we have the implicit form of the approximate solution of

© 2018 BISKA Bilisim Technology

(39) satisfied differential equation and whose boundary conditions. Table 1 shows the absolute errors in collocation points for $M = 4,\ k = 1$, $M = 8,\ k = 1$ and $M = 16,\ k = 0$. Graphical presentations of the approximate solution and absolute error are depicted in Figures 1 and 2 for $M = 8,\ k = 1$. As can be seen in Table 1, Figures 1-2 and Figure 2 given [26], it is clear that the results obtained by the presented method are superior to [26].

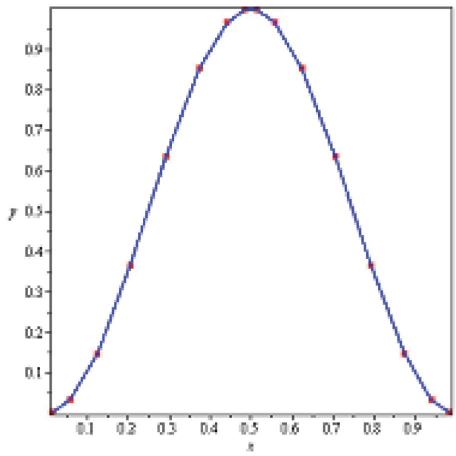

Fig. 1: Approximate solution of Example 1 for $M = 8,\ k = 1$.

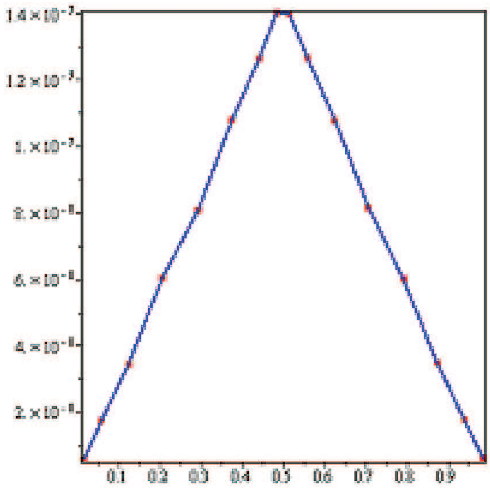

Fig. 2: The absolute error of Example 1 for $M = 8,\ k = 1$.

Example 2. Consider the nonlinear boundary value problem [47,48]

$$y^{(4)}(t) = y^2(t) - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48, \quad 0 < t < 1,$$
$$y(0) = y'(0) = 0,\ y(1) = y'(1) = 1, \tag{46}$$

© 2018 BISKA Bilisim Technology

Table 1: Absolute error of Example 1 with Chebyshev wavelet collocation method for various collocation points.

t	$M=4,k=1$	t	$M=8,k=1$	t	$M=16,k=0$
0.0477457514	2.4974593112 e-4	0.0150768448	5.9826586536 e-9	0.0085134500	1.12782311 e-13
0.1727457517	8.1982196612 e-4	0.0584888893	1.7469014807 e-8	0.0337638850	3.10584000 e-15
0.3272542486	1.2376539594 e-3	0.1250000000	3.4378819800 e-8	0.0748914320	1.08110000 e-13
0.4522542486	1.8483044111 e-3	0.2065879555	6.0398887234 e-8	0.1304955415	1.32348500 e-14
0.5477457514	1.8483044111 e-3	0.2934120442	8.0761774170 e-8	0.1986826815	9.71398900 e-14
0.6727457514	1.2376539594 e-3	0.3750000000	1.0781454947 e-7	0.2771308225	3.12829100 e-14
0.8272542486	8.1982196613 e-4	0.4415111108	1.2650986336 e-7	0.3631685050	7.87285800 e-14
0.9522542486	2.4974593112 e-4	0.4849231552	1.3973346048 e-7	0.4538658200	5.49689900 e-14
		0.5150768448	1.3973346048 e-7	0.5461341795	5.49689800 e-14
		0.5584888892	1.2650986336 e-7	0.6368314950	7.87286400 e-14
		0.6250000000	1.0781454947 e-7	0.7228691780	3.12830900 e-14
		0.7065879555	8.0761774170 e-8	0.8013173180	9.71398600 e-14
		0.7934120442	6.0398887236 e-8	0.8695044585	1.32348200 e-14
		0.8750000000	3.4378819800 e-8	0.9251085680	1.08109800 e-13
		0.9415111108	1.7469014807 e-8	0.9662361145	3.10576000 e-15
		0.9849231552	5.9826586535 e-9	0.9914865500	1.12782471 e-13

with analytic solution $y(x)=t^5-2t^4+2t^2$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique. First approximate solution is taken as $y_0(t)=t^3(2-t)^2$ and $y_{s+1}^2(t)\cong 2y_s(t)y_{s+1}(t)-y_s^2(t)$ can be obtained. Hence converted problem is obtained as

$$y_{s+1}^{(4)}(t)-2y_s(t)y_{s+1}(t)=y_s^2(t)-t^{10}+4t^9-4t^8-4t^7+8t^6-4t^4+120t-48, \quad (47)$$

where $y_s(t)$ is known. It is assumed that $y_{s+1}^{(4)}(t)$ can be expanded in terms of truncated Chebyshev wavelet series as

$$y_{s+1}^{(4)}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t)=C^T\Psi(t). \quad (48)$$

By integrating this equation four times with respect to t from 0 to $t$and using condition in Eq. (46), following equations are obtained.

$$y'''_{s+1}(t)=\int_0^t C^T\Psi(s)ds+y'''_{s+1}(0)=C^TP_1\Psi_1(t)+y'''_{s+1}(0). \quad (49)$$

$$y''_{s+1}(t)=C^TP_1P_2\Psi_2(t)+ty'''_{s+1}(0)+y''_{s+1}(0). \quad (50)$$

$$y'_{s+1}(t)=C^TP_1P_2P_3\Psi_3(t)+\frac{t^2}{2}y'''_{s+1}(0)+ty''_{s+1}(0). \quad (51)$$

$$y_{s+1}(t)=C^TP_1P_2P_3P_4\Psi_4(t)+\frac{t^3}{6}y'''_{s+1}(0)+\frac{t^2}{2}y''_{s+1}(0). \quad (52)$$

By using boundary conditions $y'_{s+1}(1)=1$ and $y_{s+1}(1)=1$, the following two equations are obtained.

$$C^TP_1P_2P_3\Psi_3(1)+\frac{1}{2}y'''_{s+1}(0)+y''_{s+1}(0)=1. \quad (53)$$

$$C^TP_1P_2P_3P_4\Psi_4(1)+\frac{1}{6}y'''_{s+1}(0)+\frac{1}{2}y''_{s+1}(0)=1. \quad (54)$$

© 2018 BISKA Bilisim Technology

Table 2: Absolute error of Example 2 with Chebyshev wavelet collocation method for various collocation points.

t	$M=4, k=0$	t	$M=4, k=1$	t	$M=4, k=2$
0.0954915028	1.0500 e-20	0.0477457514	1.4000 e-22	0.0238728757	4.0000 e-23
0.3454915028	1.1000 e-21	0.1727457514	2.0000 e-21	0.0863728757	4.0000 e-23
0.6545084972	2.3000 e-20	0.3272542486	1.0000 e-21	0.1636271243	9.0000 e-22
0.9045084972	5.2700 e-20	0.4522542486	2.1000 e-20	0.2261271243	3.9000 e-21
		0.5477457514	1.9000 e-20	0.2738728757	9.0000 e-21
		0.6727457514	2.4000 e-20	0.3363728757	1.0000 e-20
		0.8272542486	4.6000 e-20	0.4136271243	4.0000 e-21
		0.9522542486	2.5000 e-20	0.4761271243	6.0000 e-21
				0.5238728757	8.0000 e-21
				0.5863728757	5.5000 e-20
				0.6636271243	5.0000 e-21
				0.7261271243	1.6000 e-20
				0.7738728757	1.7000 e-20
				0.8363728757	1.1000 e-21
				0.9136271243	2.4100 e-20
				0.9761271243	2.7260 e-20

Table 3: Comparison between the maximum errors for Example 2.

Method in [47]	Method in [48]	Present Method
7.092 e-6	6.6613 e-15	2.7260 e-20

By solving this system of equations,

$$\frac{1}{2}y''_{s+1}(0) = C^T(-3P_1P_2P_3P_4\Psi_4(1)+P_1P_2P_3\Psi_3(1))+2, \tag{55}$$

$$\frac{1}{6}y'''_{s+1}(0) = C^T(2P_1P_2P_3P_4\Psi_4(1)-P_1P_2P_3\Psi_3(1))-1 \tag{56}$$

are obtained. Hence replacing Eqs. (55) and (56) into the Eqs. (51) and (52), we have

$$y'_{s+1}(t) = C^T\left(P_1P_2P_3\Psi_3(t)+(6t^2-6t)P_1P_2P_3P_4\Psi_4(1)+(2t-3t^2)P_1P_2P_3\Psi_3(1)\right)+4t-3t^2 \tag{57}$$

$$y_{s+1}(t) = C^T\left(P_1P_2P_3P_4\Psi_4(t)+(2t^2-3t^2)P_1P_2P_3P_4\Psi_4(1)+(t^2-t^3)P_1P_2P_3\Psi_3(1)\right)+2t^2-t^3. \tag{58}$$

Replacing Eqs. (48) and (58) into Eq. (47), we have

$$\left.\begin{aligned} C^T\left(\Psi(t)-2y_s(t)P_1P_2P_3P_4\Psi_4(t)-2ty_s(t)P_1P_2P_3P_4\Psi_4(1)-2y_s(t)P_1P_2P_3\Psi_3(1)\right) = \\ -y_s^2(t)+2(2t^2-t^3)y_s(t)-t^{10}+4t^9-4t^8-4t^7+8t^6-4t^4+120t-48. \end{aligned}\right\} \tag{59}$$

Algebraic equation system achieved from Eq. (59) by using collocation points can be solved and the coefficients C^T of Eq. (58) satisfied differential equation and whose boundary conditions are obtained. Table 2 shows the absolute errors in collocation points for $M=4,\ k=0$, $M=4,\ k=1$ and $M=4,\ k=2$. Numerical results for this problem presented in [47] with the maximum absolute error $7.092e-6$ and in [48] with the maximum absolute error $6.6613e-15$. As can be seen in Table 3, it is clear that the results obtained by the presented method are superior with respect to [47] and [48].

© 2018 BISKA Bilisim Technology

Example 3. Consider the nonlinear boundary value problem [47,48]

$$
\begin{aligned}
&y^{(4)}(t) = \sin t + \sin^2 t - (y''(t))^2, \quad 0 < x < 1, \\
&y(0) = 0,\ y'(0) = 1,\ y(1) = \sin 1,\ y'(1) = \cos 1,
\end{aligned}
\tag{60}
$$

with analytic solution $y(t) = \sin t$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique. First approximate solution is taken as

$$
y_0(t) = (1 + \cos 1 - 2\sin 1)t^3 + (3\sin 1 - \cos 1 - 2)t^2 + t \tag{61}
$$

and

$$
\left(y''_{s+1}(t)\right)^2 \cong 2y''_s(t)y''_{s+1}(t) - \left(y''_s(t)\right)^2 \tag{62}
$$

can be obtained. Hence converted problem is obtained as

$$
y^{(4)}_{s+1}(t) + 2y''_s(t)y''_{s+1}(t) = \left(y''_s(t)\right)^2 + \sin t + \sin^2 t, \tag{63}
$$

where $y_s(t)$ is known. It is assumed that $y^{(4)}_{s+1}(t)$ can be expanded in terms of truncated Chebyshev wavelet series as in Eq. (48). By the similar processing, we have

$$
y''_{s+1}(t) = C^T \left(P_1P_2\Psi_2(t) + (12t-6)P_1P_2P_3P_4\Psi_4(1) + (2-6t)P_1P_2P_3\Psi_3(1)\right) + (6t-2)\cos 1 + (6-12t)\sin 1 + 6t - 4, \tag{64}
$$

$$
\begin{aligned}
y_{s+1}(t) = C^T &\left(P_1P_2P_3P_4\Psi_4(t) + (2t^3-3t^2)P_1P_2P_3P_4\Psi_4(1) + (t^2-t^3)P_1P_2P_3\Psi_3(1)\right) \\
&+ (t^3-t^2)\cos 1 + (3t^2-2t^3)\sin 1 + t^3 - 2t^2 + t
\end{aligned}
\tag{65}
$$

equations and the following algebraic equations system

$$
\begin{aligned}
&C^T \left(\Psi(t) + 2y''_s(t)P_1P_2\Psi_2(t) + (24t-12)y''_s(t)P_1P_2P_3P_4\Psi_4(1) + (4-12t)y_s(t)P_1P_2P_3\Psi_3(1)\right) = \\
&(y''_s(t))^2 - (12t-4)y''_s(t)\cos 1 - (12-24t)y''_s(t)\sin 1 - (12t-8)y''_s(t) + \sin t + \sin^2 t.
\end{aligned}
\tag{66}
$$

Algebraic equation system achieved from the above equation by using collocation points can be solved and the coefficients C^T of Eq. (65) satisfied differential equation and whose boundary conditions are obtained. Table 4 shows the absolute errors in collocation points for $M = 4,\ k = 1$, $M = 8,\ k = 1$ and $M = 16,\ k = 0$. Numerical results for this problem were presented in [47] with the maximum absolute error $1.358\,e-5$ and in [48] with the maximum absolute error $1.0502\,e-6$. As can be seen in Table 5, it is clear that the results obtained by the presented method are superior than [47] and [48].

Example 4. Consider the nonlinear boundary value problem [49,50,51,52,53,54]

$$
\begin{aligned}
&y^{(4)}(t) = 6e^{-4y} - \tfrac{12}{(1+t)^2} \quad 0 < t < 1, \\
&y(0) = 0,\ y(1) = \ln(2),\ y''(0) = -1,\ y''(1) = -\tfrac{1}{4},
\end{aligned}
\tag{67}
$$

with analytic solution $y(t) = \ln(1+t)$. This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique. First approximate solution satisfying boundary

© 2018 BISKA Bilisim Technology

Table 4: Absolute error of Example 3 with Chebyshev wavelet collocation method for various collocation points.

t	$M=4,k=1$	t	$M=8,k=1$	t	$M=16,k=0$
0.0477457514	6.1312946582 e-11	0.0150768448	5.18630 e-19	0.0085134500	1.1000 e-22
0.1727457517	6.9209690356 e-10	0.0584888893	7.46650 e-18	0.0337638850	1.1000 e-22
0.3272542486	1.7781891083 e-9	0.1250000000	3.05100 e-17	0.0748914320	0
0.4522542486	2.3627312327 e-9	0.2065879555	7.25950 e-17	0.1304955415	3.1000 e-21
0.5477457514	2.3537433020 e-9	0.2934120442	1.22920 e-16	0.1986826815	8.1000 e-21
0.6727457514	1.8017223110 e-9	0.3750000000	1.64720 e-16	0.2771308225	1.1000 e-20
0.8272542486	7.1472435460 e-10	0.4415111108	1.87870 e-16	0.3631685050	1.1000 e-20
0.9522542486	5.5932879400 e-11	0.4849231552	1.94570 e-16	0.4538658200	0
		0.5150768448	1.94480 e-16	0.5461341795	1.1000 e-20
		0.5584888892	1.87700 e-16	0.6368314950	1.1000 e-20
		0.6250000000	1.64260 e-16	0.7228691780	4.1000 e-20
		0.7065879555	1.22660 e-16	0.8013173180	2.1000 e-20
		0.7934120442	7.24900 e-17	0.8695044585	3.1000 e-20
		0.8750000000	3.04000 e-17	0.9251085680	4.1000 e-20
		0.9415111108	7.62000 e-18	0.9662361145	3.1000 e-20
		0.9849231552	4.90000 e-19	0.9914865500	2.1000 e-20

Table 5: Comparison between the maximum errors for Example 3.

Method in [47]	Method in [48]	Present Method
7.092 e-6	6.6613 e-15	4.1000 e-20

condition is taken as:

$$y_0(t)=\frac{t^3}{8}-\frac{t^2}{2}+\left(\ln(2)+\frac{3}{8}\right)t \tag{68}$$

and

$$e^{-4y_{s+1}(t)}\cong -4e^{-4y_s(t)}y_{s+1}(t)+4e^{-4y_s(t)}y_s(t)+e^{-4y_s(t)} \tag{69}$$

can be obtained. Hence converted problem is obtained as

$$y_{s+1}^{(4)}(t)+24e^{-4y_s(t)}y_{s+1}(t)=6e^{-4y_s(t)}+24e^{-4y_s(t)}y_s(t)-\frac{12}{(1+t)^2}, \tag{70}$$

where $y_s(t)$ is known. It is assumed that $y_{s+1}^{(4)}(t)$ can be expanded in terms of truncated Chebyshev wavelet series as in Eq. (48). By the similar processing given above, we have a equation as:

$$y_{s+1}(t)=C^T\left(P_1P_2P_3P_4\Psi_4(t)-tP_1P_2P_3P_4\Psi_4(1)+\left(\tfrac{t}{6}-\tfrac{t^3}{6}\right)P_1P_2\Psi_2(1)\right)+\tfrac{t^3}{8}-\tfrac{t^2}{2}+\left(\ln(2)+\tfrac{3}{8}\right)t \tag{71}$$

and the following algebraic equations system

$$C^T\left(\Psi(t)+24e^{-4y_s(t)}P_1P_2P_3P_4\Psi_4(t)-24te^{-4y_s(t)}P_1P_2P_3P_4\Psi_4(1)+4(t-t^3)e^{-4y_s(t)}P_1P_2\Psi_2(1)\right)=$$

$$6e^{-4y_s(t)}+24e^{-4y_s(t)}y_s(t)-\tfrac{12}{(1+x)^2}-3t^3e^{-4y_s(t)}+12t^2e^{-4y_s(t)}-24\left(\ln(2)+\tfrac{3}{8}\right)te^{-4y_s(t)}$$

Algebraic equation system achieved from the above equation by using collocation points can be solved and the coefficients C^T of Eq. (71) satisfied differential equation and whose boundary conditions are obtained. Table 6 shows the absolute errors in collocation points for $M=4,\ k=1,\ M=8,\ k=1$ and $M=16,\ k=0$. Numerical results for this

© 2018 BISKA Bilisim Technology

Table 6: Absolute error of Example 4 with Chebyshev wavelet collocation method for various collocation points.

t	$M=4,k=1$	t	$M=8,k=1$	t	$M=16,k=0$
0.0477457514	9.25288664 e-7	0.0150768448	8.82647223 e-11	0.0085134500	2.09962400 e-16
0.1727457517	3.40945573 e-6	0.0584888893	3.43160523 e-10	0.0337638850	7.44817000 e-16
0.3272542486	5.91635023 e-6	0.1250000000	7.21292980 e-10	0.0748914320	1.70714300 e-15
0.4522542486	6.96205374 e-6	0.2065879555	1.15346796 e-9	0.1304955415	2.57566500 e-15
0.5477457514	7.02766347 e-6	0.2934120442	1.54466057 e-9	0.1986826815	3.71569000 e-15
0.6727457514	6.03425421 e-6	0.3750000000	1.81704188 e-9	0.2771308225	4.45748000 e-15
0.8272542486	3.58175110 e-6	0.4415111108	1.95548622 e-9	0.3631685050	5.06856000 e-15
0.9522542486	1.02677122 e-6	0.4849231552	1.99469362 e-9	0.4538658200	5.33904000 e-15
		0.5150768448	1.99680551 e-9	0.5461341795	5.10591000 e-15
		0.5584888892	1.96233301 e-9	0.6368314950	4.79461000 e-15
		0.6250000000	1.83191940 e-9	0.7228691780	3.90409000 e-15
		0.7065879555	1.56296740 e-9	0.8013173180	3.18099000 e-15
		0.7934120442	1.17301237 e-9	0.8695044585	2.11543000 e-15
		0.8750000000	7.37052694 e-10	0.9251085680	1.34364000 e-15
		0.9415111108	3.50650517 e-10	0.9662361145	5.90880000 e-16
		0.9849231552	9.07713233 e-11	0.9914865500	1.58960000 e-16

Table 7: Comparison between the maximum errors for Example 4.

Method in [49]	Method in [50]	Method in [51]	Method in [52]	Method in [53]	Method in [54]	Present Method
5.40 e-8	6.30 e-11	1.25 e-12	2.70 e-12	6.70 e-12	3.44 e-15	1.5896 e-16

problem were presented in [49,50,51,52,53,54] with the best absolute error given in the Table 7. As can be seen in Table 5, it is clear that the results obtained by the presented method are superior than [49,50,51,52,53,54].

6 Conclusion

Chebyshev wavelet collocation method is proposed to eliminate disadvantages of Chebyshev wavelet and Legendre wavelet methods and to obtain approximate solution of rth order differential equations. The method has been applied to the three nonlinear boundary value problems by using quasilinearization technique. Approximate and exact solutions of examples are correspondingly compared. For Example 1, comparison of present results in Table 1, Figure 2 and Figure 2 given in [26], it is clear that the results obtained by the proposed method are better than the provided examples. Numerical results for Example 2 were presented in [47,48] with the maximum absolute error $7.092e-6$ and $6.6613e-15$ respectively Also maximum absolute errors of Example 3 were given as $1.358e-5$ and $1.0502e-6$ in [47,48] respectively. The best absolute errors of Example 4 presented in [49,50,51,52,53,54] are given in the Table 5. As can be seen from Tables 1-7, the present method is highly efficient and accurate. All of the calculations have been made by Maple program with 20 digits. These calculations demonstrated that the accuracy of the Chebyshev wavelet collocation method is quite high even in the case of a small number of grid points. In the proposed method, there are no complex integrals or methodology. Applications of this method are very simple. It is also very convenient for solving the initial and boundary value problems since the initial and boundary conditions in the solution are automatically taken. Moreover, the proposed method, which gives accurate solution even in the case of a small number of grid points M and k, is reliable, simple, fast, minimal computation costs, flexible, and convenient alternative method.

Competing interests

The authors declare that they have no competing interests.

© 2018 BISKA Bilisim Technology

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] C. Hwang, Y.P. Shih, Laguerre series direct method for variational problems, J. Optim. Theory Appl. 39 (1983) 143-149.

[2] R.Y. Chang, M.L. Wang, Shifted Legendre direct method for variational problems series, J. Optim. Theory Appl. 39 (1983) 299-307.

[3] I.R. Horng, J.H. Chou, Shifted Chebyshev direct method for variational problems, Int. J. Syst. Sci. 16 (1985) 855-861.

[4] M. Razzaghi, M. Razzaghi, Fourier series direct method for variational problems, Int. J. Control 48 (1988) 887-895.

[5] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed parameter systems, IEEE Proc. Control Theory Appl. 144 (1997) 87-93.

[6] K. Maleknejad, M. T. Kajani, Y. Mahmoudi, Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets, Kybernetes, Int. J. Syst. Math. 32 (2003) 1530-1539.

[7] F.C. Chen and C.H. Hsiao, A Walsh series direct method for solving variational problems, J. Franklin Inst. 300 (1975), pp. 265-280.

[8] F.C. Chen, Y.T. Tsay, and T.T. Wu, Walsh operational matrices for fractional calculus and their application to distributed parameter system, J. Franklin Inst. 503 (1977), pp. 267-284.

[9] I.R. Howang and Y.P. Sheh, Solution of integral equations via Legendre polynomials, J. Comput. Electrical Engineering 9 (1982), pp. 123-129.

[10] I. Daubechies, Ten lectures on wavelets, SIAM, Philadelphia, PA, 1992.

[11] C. Cattani, Fractional calculus and shannon wavelets, Mathematical Problems in Engineering, vol. 2012(2012), Article ID 502812, 25.

[12] C. Cattani, Shannon wavelets for the solution of integro-differential equations, Mathematical Problems in Engineering, vol. 2010(2010), Article ID 408418, 22.

[13] M. Karabacak, M. Yiğider. Shannon wavelet transform for solving fractional differential algebraic equations numerically.. International Journal of Engineering and Applied Sciences 3(12), (2016), 56-61.

[14] C. Cattani and A. Kudreyko, Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind, Applied Mathematics and Computation, 215/12 (2010), 4164-4171.

[15] M. H. Heydari, M. R. Hooshmandasl, F. M. M. Ghaini, and F. Fereidouni, Two-dimensionallegendre wavelets forsolving fractional poisson equation with dirichlet boundary conditions, Engineering Analysis with Boundary Elements, 37(2013) 1331-1338.

[16] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and F. Mohammadi, Wavelet collocation method for solving multiorder fractional differential equations, Journal of Applied Mathematics, vol. 2012(2012), Article ID 542401, 19.

[17] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi, and C. Cattani, Wavelets method for solving systems of nonlinear singular fractional volterra integro-differential equations, Communications in Nonlinear Science and Numerical Simulation, 19/1 (2014), 37-48.

[18] J.S. Gu and W.S. Jiang, The Haar wavelets operational matrix of integration, Int. J. Syst. Sci. 27 (1996), pp. 623-628.

[19] U. Lepik, Numerical solution of evolution equations by the Haar wavelet method, Applied Mathematics and Computation, 185 (2007) 695-704.

[20] W. Geng, Y. Chen, Y. Li, D. Wang, Wavelet method for nonlinear partial differential equations of fractional order, Computer and information science, 4/5 (2011) 28-35.

[21] G. Hariharan, K. Kannan, K.R. Sharma, Haar wavelet method for solving Fisher’s equation, Applied Mathematics and Computation, 211 (2009) 284-292.

[22] G. Hariharan and K. Kannan, Haar wavelet method for solving FitzHugh-Nagumo equation, Int. Journal of Mathematical and Statistical Sciences, 2:2 (2010) 59-63.

[23] G.Hariharan, K.Kannan, A Comparative Study of a Haar Wavelet Method and a Restrictive Taylor’s Series Method for Solving Convection-diffusion equations, Int. J. Comput. Methods in Eng. Sci. Mechanics, 11: 4 (2010) 173 -184.

[24] G.Hariharan, An efficient wavelet analysis method to film-pore diffusion model arising in mathematical chemistry, The Journal of Membrane Biology,, 247/4 (2014) 339-343.

© 2018 BISKA Bilisim Technology

[25] G.Hariharan, K.Kannan, Haar wavelet method for solving nonlinear parabolic equations, J.Math.Chem. 48 (2010) 1044-1061.
[26] H. Kaur, R. C. Mittal, V. Mishra, Haar wavelet quasilinearization approach for solving nonlinear boundary value problems, Amer. J. Comput. Math., 1, (2011) 176-182.
[27] M Karabacak, E Çelik. The Numerical Solution of Fractional Differential-Algebraic Equations (FDAEs) by Haar Wavelet Functions, Int. J. Eng. Appl. Sci. 2(10) (2015), 65-71.
[28] İ. Çelik, Haar wavelet method for solving generalized Burgers-Huxley equation, Arab Journal of Mathematical Sciences, 18 (2012)25-37.
[29] İ. Çelik, Haar wavelet approximation for magnetohydrodynamic flow equations Applied Mathematical Modelling, 37 (2013) 3894-3902.
[30] M. T. Kajani and A. H. Vencheh, Solving linear integro-differential equation with Legendre wavelet, Int. J. Comput. Math. 81 (6) (2004), pp. 719-726.
[31] M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, Mathematics and Computers in Simulation 53 (2000) 185-192.
[32] M. Razzaghi, S. Yousefi, Legendre wavelets operational matrix of integration, Int. J. Syst. Sci. 32 (4) (2001) 495-502.
[33] E. Babolian, F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comput. 188 (2007) 417-426.
[34] E.Babolian, and F. Fattahzadeh, Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration, Applied Mathematics and Computation 188.1 (2007): 1016-1022.
[35] M. T. Kajania, A. H. Vencheha and M. Ghasemib, The Chebyshev wavelets operational matrix of integration and product operation matrix, Int. J. of Comput. Math. 86/7 (2009) 1118-1125.
[36] M. Karabacak, M Yiğider, The Legendre wavelets method for approximate solution of fractional differential-algebraic equations, International Journal of Applied Mathematics and Statistics 57(4) (2018), 161-173
[37] H. Adibi, P. Assari, Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind, Mathematical Problems in Engineering V 2010, Article ID 138408, p 17.
[38] Y.X. Wang, Q.B. Fan, The second kind Chebyshev wavelet method for solving fractional differential equations, Appl. Math. Comput., 218 (2012), 8592-8601.
[39] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and M. Li, Chebyshev wavelets method for solution of nonlinear fractional integrodifferential equations in a large interval, Advances in Math. Physics, vol. 2013(2013), Article ID 482083, 12.
[40] M. R. Hooshmandasl, M. H. Heydari, and F. M. M. Ghaini, Numerical solution of the one dimensional heat equation by using chebyshev wavelets method, Applied and Computational Mathematics, 1/6 (2012), 1-7, 2012.
[41] C. Yang, J, Hou, Chebyshev wavelets method for solving Bratu's problem, Boundary Value Problems (2013) 2013:142.
[42] G.Hariharan, An efficient wavelet based approximation method to water quality assessment model in a uniform channel, Ain Shams Engineering Journal, 5/2 (2014), 525-532.
[43] İ. Çelik, Numerical Solution of Differential Equations by Using Chebyshev Wavelet Collocation Method, Cankaya University Journal of Science and Engineering, 10(2) (2013) 169-184.
[44] İ. Çelik, Chebyshev Wavelet collocation method for solving generalized Burgers-Huxley equation, Math. Meth. Appl. Sci., 39 (2016) 366-377.
[45] İ. Çelik, Free vibration of non-uniform Euler-Bernoulli beam under various supporting conditions using Chebyshev wavelet collocation method, Applied Mathematical Modelling 54 (2018): 268-280.
[46] L. Fox, I.B. Parker, Chebyshev polynomials in numerical analysis, Oxford University Press, London, 1968.
[47] K.N.S. K. Viswanadham, P. M. Krishna, R. S. Koneru, Numerical solutions of fourth order boundary value problems by Galerkin method with Quintic B-splines, Int. J. Nonli. Sci. 10 (2010) 222-230.
[48] S. Momani, K. Moadi, A reliable algorithm for solving fourth-order boundary value problems, J. Appl. Math. Comput. 22(3), (2006) 185-197.
[49] R.P. Agarwall, G. Akrivis, Boundary value problems occurring in plate deflection theoryJ. Comput. Appl. Math., 8 (1982), 145-154.
[50] M.M. Chawla, C.P. Katti, Finite difference methods for two-point boundary value problems involving high order differential equations, BIT Numer. Math., 19 (1979), 27-33.

© 2018 BISKA Bilisim Technology

[51] J. Rashidinia, M. Ghasemia, B-spline collocation for solution of two-point boundary value problems, J. Comput. Appl. Math., 235 (2011), 2325-2342.

[52] E.H. Twizell, A sixth-order extrapolation method for special nonlinear fourth-order boundary value problems, Comp. Meth. Appl. Mech. Eng., 62 (1987), 293-303.

[53] E.H. Twizell, S.I.A. Tirmizi, Multiderivative methods for nonlinear beam problems, Commun. Appl. Numer. Meth., 4 (1988), 43-50.

[54] Doha, E.H.; Abd-Elhameed, W.M.; Youssri, Y.H. New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems. J. Egypt. Math. Soc., 24/2, (2015), 319-327.

© 2018 BISKA Bilisim Technology