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On new f-statistical convergence in probabilistic normed spaces

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Abstract: In this paper we introduce and study the concept of density of moduli with respect to the probabilistic norm in a probabilistic normed space, where f is a unbounded modulus function . Also we are trying to investigate some relation between the ordinary convergence and module statistical convergence for every unbounded modulus function.

Keywords: f-statistical convergence, Probabilistic Normed Space, f-density, modulus function.

1 Introduction

Looking through historically to statistical convergence of single sequences, we recall that the concept of statistical convergence of sequences was first introduced by Fast [21] as an extension of the usual concept of sequential limits. Schoenberg [24] gave some basic properties of statistical convergence and also studied the concept as a summability method. Most of the existing works on statistical convergence have been restricted to real or complex sequences except the works of Kolk [9], Maddox [23] and Cakalli [20]. More recently, the notion of statistical convergence has been used as a tool by many mathematicians to solve many open problems in the area of sequence spaces and summability theory and some other applications as well. One may refer to ([25],[23],[19],[31],[32],[36],[15],[10],[11],[12],[13]).

In 2014, A. Aizpuru et al. [27] introduced a new concept of density for sets of natural numbers with respect to the modulus function. They studied and characterized the generalization of this notion of f-density with statistical convergence and proved that ordinary convergence is equivalent to the module statistical convergence for every unbounded modulus function. Savaş and Borgohain [17] introduced some new spaces of lacunary f -statistical A-convergent sequences of order α.

An interesting and very useful generalization of the notion of metric space was introduced by Menger [26] under the name of statistical metric space, which is now called probabilistic metric space. The idea of Menger was to use distribution functions instead of nonnegative real numbers.

The most fascinating application of the probabilistic metric space in quantum physics arises in string and El Naschie's ε^{∞}-theory ([28],[29],[30]). In fact the probabilistic theory has become an area of active research for the last forty years. It has a wide range of applications in functional analysis [33]. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces). The notion of probabilistic normed spaces was introduced by Sherstnev [2] in 1963 and later on studied by various authors, see ([7],[8]).

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In [1], Alotaibi studied the notion of λ-statistical convergence for single sequences in probabilistic normed spaces. Also Savas and Mohiuddine [18] studied $\overline{\lambda}$-statistically convergent double sequences in probabilistic normed spaces.

2 Preliminary Concepts

A sequence (x_i) of real numbers is statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $K(i) = \{i \leq k : |x_i - L| \geq \varepsilon\}$ has natural density zero, .i.e.,

$$\lim_k \frac{1}{k} \sum_{j=1}^{i} \chi_{K(i)}(j) = 0,$$

where $\chi_{K(i)}$ denotes the characteristic function of $K(i)$.

A modulus function ([22]and [10]) is defined as a function $f : R^+ \to R^+$ which satisfies:

(1) $f(x) = 0$ if and only if $x = 0$.
(2) $f(x+y) \leq f(x) + f(y)$ for every $x, y \in R^+$.
(3) f is increasing.
(4) f is continuous from the right at 0.

It is clear that a modulus function must be continuous on $\mathbb{R}_+$. Examples of moduli are $f(x) = \frac{x}{1+x}$ and $f(x) = x^p, 0 < p \leq 1$.

Let $A \subseteq \mathbb{N}$, we mean f-density of A if $\delta_f(A) = \lim_k \frac{f(|A(i)|)}{f(k)}$, (in case this limit exists)where $A(i) = \{k \in A : k \leq i\}$ and f is an unbounded modulus function.

Let (x_i) be a sequence in X (X is a normed space). If for each $L > 0$, $A = \{i \leq k : \|x_i - L\| > \varepsilon\}$ has f-density zero, then it is said that the f-statistical limit of (x_i) is $L \in X$, and we write it as $f - stat \lim_k x_i = L$. Note that $\delta(A) = 1 - \delta(\mathbb{N} \backslash A)$.

A triangular norm (t-norm) is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0,1]$.

Let X be a real linear space and $N : X \to D$, where D is the set of all distribution functions $g : \mathbb{R} \to \mathbb{R}_0^+$ such that it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} g(t) = 0$ and $\sup_{t \in \mathbb{R}} g(t) = 1$.

The probabilistic norm or N-norm [6] is a triangular norm satisfying the following conditions:

(1) $N_p(0) = 0$,
(2) $N_p(t) = 1$ for all $t > 0$ iff $p = 0$,
(3) $N_{\alpha p}(t) = N_p\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R} \backslash \{0\}$ and for all $t > 0$,
(4) $N_{p+q}(s+t) \geq N_p(s) * N_q(t)$ for all $p, q \in X$ and $s, t \in \mathbb{R}_0^+$;

where N_p means $N(p)$ and $N_p(t)$ is the value of N_p at $t \in \mathbb{R}$. $(X, N, *)$ is named as a probabilistic normed space , in short PN-space.

In this paper, we study the density on moduli with respect to the probabilistic norm N in the PN-space $(X, N, *)$. We also investigate some results on the new concept of f_N-statistical convergence with the ordinary convergence. Also we find out some new concepts on f_N^*-statistically convergent and try to find out new results related to this. Moreover, the concepts of f-statistical limits, f-cluster points and f-equivalence are introduced and try to find out the relations among them.

3 Main results

Definition 1. *Let $(X,N,*)$ be a PN-space. Then a sequence $x=(x_i)$ is said to be f-statistically convergent to L with respect to the probabilistic norm N provided that, for every $t>0$ and $\mu>0$,*

$$\lim_k \frac{f(|\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|)}{f(k)}=0.$$

We define it as $f_N-stat-\lim_i x_i=L$ and Ω_N^f is the collection of all f_N-statistically convergent sequences with resepct to the probabilistic norm N.

We begin with the following observation.

Corollary 1. *For any unbounded modulus f, if a sequence (x_i) is convergent to L with respect to the probabilistic norm N , then it is $f_N-stat\lim x_i=L$. But not conversely.*

Proof. Let $\lim_i x_i=L$ with respect to the probabilistic norm N . So for $t>0,\mu>0$, we have $N_{x_i-L}>1-\mu$. Construct $K_{N,\mu}(t)=\{i\le k : N_{x_i-L}(t)\le 1-\mu\}$, which is a finite set of $\mathbb{N}$. Then we have that there exists $k_0,p\in\mathbb{N}$ such that $|K_{N,\mu}(t)|=p$, if $k\ge k_0$, which will show that, $\lim_k \frac{f(|K_{N,\mu}(t)|)}{f(k)}=0\Rightarrow f_N-stat\lim x_i=L$.

For the converse part, let $(\mathbb{R},N,*)$ be a PN-space with $a*b=ab, N_x(t)=\frac{t}{t+|x|}$. Define a sequence ,

$$x_i=\begin{cases}1, \text{ if } k=i^2, i\le k;\\ 0, \text{ otherwise.}\end{cases}$$

(x_i) is f_N-statistical convergent , but not convergent with respect to the probabilistic norm N.

The proofs of the following Therems are easy and thus omitted.

Theorem 1. *Let $(X,N,*)$ be a PN-space. Then the f_N-statistical limit of a sequence (x_i) is unique.*

Corollary 2. *Let $(X,N,*)$ be a PN-space. For f and g two unbounded moduli, if $f_N-st\lim x_n=x$ and $g_N-st\lim x_n=y$ then $x=y$.*

Theorem 2. *Let $(X,N,*)$ be a PN-space.*

(1) *If L_1 and L_2 are two f-statistical limits of (x_i) and (y_i) respectively with respect to the probabilistic norm N, then $f_N-\lim(x_i+y_i)=L_1+L_2$.*
(2) *If (x_i) is f-statistically convergent to L with respect to the probabilistic norm N, then for any $\alpha>0$, $f_N-\lim\alpha x_i=\alpha L$.*

In the following, we investigate the relationships between f_N-statistical convergence and f_N^*-statistical convergence with respect to the probabilistic norm N. However , to define f_N^*-statistically convergence, we first prove Theorem 3.4. which describes the characteristic of f_N^*-statistical convergence in a more clear way.

Definition 2.*A subset K of $\mathbb{N}$ is called f-statistically dense if $\delta_f(K)=1$.*

Theorem 3. *Let $(X,N,*)$ be a PN-space. Then $f_N-stat\lim_k x_i=L$ if and only if there exists a subset $I=\{i_n : i_1<i_2<i_3,\ldots\}$ of $\mathbb{N}$ such that I is f-statistically dense with resepct to N and $\lim_n x_{i_n}=L$ with respect to the probabilistic norm N.*

Proof. Assume, $f_N - stat\lim_i x_i = L$. Then for any $t > 0$ and $\mu \in \mathbb{N}$, we have $K_N(t) = \left\{ i \le k \le n : N_{x_i - L}(t) \le 1 - \frac{1}{\mu} \right\}$ so that $\lim_k \frac{f(|K_N(t)|)}{f(k)} = 0$.

Construct

$$M_N(\mu,t) = \left\{ i \le k \le n : N_{x_i - L}(t) > 1 - \frac{1}{\mu} \right\}$$

such that

$$M_N(1,t) \supset M_N(2,t) \supset M_N(3,t) \supset ...M_N(i,t) \supset M_N(i+1,t) \supset ....$$

and

$$\lim_k \frac{|M_N(\mu,t)|}{k} = 1.$$

We have to show that $i \in M_N(\mu,t)$ and (x_i) is convergent to L with respect to the probabilistic norm N. Suppose the sequence (x_i) is not convergent to L with respect to the probabilistic norm N , for $i \in M_N(\mu,t)$. Therefore there is $r > 0$ and one positive integer i_0 such that $N_{x_i - L}(t) \le 1 - r, \forall i \ge i_0$.

Take $N_{x_i - L}(t) > 1 - r$, for all $i \le i_0$ such that $\lim_k \frac{f(|\{i \le k \le n : N_{x_i - L}(t) > 1 - r\}|)}{f(k)} = 0$. Since $r > \frac{1}{\mu}$, we have, $\lim_k \frac{f(|M_N(\mu,t)|)}{f(k)} = 0$, which is a contradiction . Hence (x_i) is convergent to L with respect to the probabilistic norm N.

Conversely, suppose there exists a subset $I = \{i_n : i_1 < i_2 < i_3....\} \subseteq \mathbb{N}$ such that I is f-statistically dense with respect to N , i.e. $\delta_{f_N}(I) = 1$ and $\lim_n x_{i_n} = L$ with respect to the probabilistic norm N, then there exists $i_0 \in \mathbb{N}$ such that for every $t > 0$ and $\mu > 0$, we have

$$N_{x_i - L} > 1 - \mu, \text{ for all } i \ge i_0.$$

Thus,

$$M_N(\mu,t) = \{i \le k : N_{x_i - L}(t) \le 1 - \mu\} \subseteq \mathbb{N} \backslash \{I_{i_0+1}, I_{i_0+2}, I_{i_0+3} \cdots .\}$$

Therefore, $\lim_k \frac{f(|M(\mu,t)|)}{f(k)} = 0$, hence $f_N - stat\lim_i x_i = L$.

Definition 3. *Let $(X,N,*)$ be a PN-space. Then $x = (x_i)$, defined in Theorem 2.4. is said to be f^*-statistical convergent to L with respect to the probabilistic norm N. We define it as f_N^*-statistical convergent to L.*

Definition 4. *Let $(X,N,*)$ be a PN-space. For an unbounded modulus f, a sequence $x = (x_i)$ is said to be f-statistically null with respect to the probabilistic norm N, if for every $\mu > 0$, $\delta_{f_N}(\{i \le k : N_{x_i} \le 1 - \mu\}) = 0$.*

Theorem 4. *Let $(X,N,*)$ be a PN-space. Then if $f_N^* - \lim_k x_i = L$ if and only if there exists two sequences $y = (y_i)$ and $z = (z_i)$ in X such that $x = y + z$, where y is statistically convergent to L with respect to the probabilistic norm N and z is f-statistically null in X.*

Proof. Let us assume that $f_N^* - \lim x = L$ which implies that there exists a subset $I = \{i_n : i_1 < i_2 < ...\}$ of $\mathbb{N}$ such that $\delta_{f_N}(I) = 1$ and $\lim_m x_{i_m} = L$ with respect to the probabilistic norm N. We define the sequences $y = (y_i)$ and $z = (z_i)$ as

$$y_i = \begin{cases} \overline{\theta}, \text{ if } i \in K \text{ and } \overline{\theta} \text{ is the zero element of } X; \\ L, \text{ if } i \in K^c. \end{cases}$$

and

$$z_i = \begin{cases} x_i, & \text{if } i \in K \\ x_i - L, & \text{if } i \in K^c. \end{cases}$$

For given $t>0,\mu>0$, we have, $N_{y_i-L}(t)=1>1-\mu, i\in K^c$. Which implies that $x=(x_i)$ is convergent to L with respect to the probabilistic norm N. Since $Z_N=\{i\leq k : z_i$ is f-statistically null in $X\}\subset K^c$, we have $\lim_k \frac{f(|Z_N|)}{f(k)}=0$. To prove the converse part, let us take $Z_N=\{i\leq k : z_i$ is f-statistically null in $X\}$ is an infinite set such that $\delta_{f_N}(Z_N)=1$. Let $I=\{i_m : i_1<i_2<i_3......\}$. Since $x_{i_m}=y_{i_m}$ and both y and (x_{i_m}) convergent to the same limit L with respect to the probabilistic norm N, it implies that $f_N^*-\lim x_i=L$. This completes the proof.

Corollary 3. *Let $(X,N,*)$ be a PN-space. Then if (x_i) is f_N-statistical convergent sequence, then it has a convergent subsequence with respect to N.*

Definition 5. *Let $(X,N,*)$ be a PN-space. Then (x_i) is said to be f_N-statistically Cauchy if $\lim_k \frac{f(|B_N(t)|)}{f(k)}=0$ where $B_N(t)=\{i,l\leq k : N_{x_i-x_l}(t)\leq 1-\mu\}$.*

Theorem 5. *Let $(X,N,*)$ be PN-space. Then f_N-statistically Cauchy implies f_N-statistical convergent.*

Proof. Let $B_N(t)=\{i,l\leq k : N_{x_i-x_l}(t)\leq 1-\mu\}$ such that $\lim_k \frac{f(|B_N(t)|)}{f(k)}=0$, which follows that the set $B_N^c(t)$ is a non-empty set. Construct $B_N^c(t)=\left\{i,l\leq k : N_{x_i-x_l}\left(\frac{t}{2}\right)>1-r\right\}$.

For given $t>0,\mu>0$, let us take $r>0$ such that $(1-r)*(1-r)\geq 1-\mu$. Assume that $x=(x_i)$ is not f_N-statistically convergent but convergent to L with respect to the probabilistic norm N , so $N_{x_i-l}\left(\frac{t}{2}\right)>1-r$. Now, for $i\in B_N^c(t)$, we have,

$$N_{x_i-L}(t)=N_{x_i-x_l+x_l-L}\left(\frac{t}{2}+\frac{t}{2}\right)\geq N_{x_i-x_l}\left(\frac{t}{2}\right)*N_{x_l-L}\left(\frac{t}{2}\right)>(1-r)*(1-r)\geq 1-\mu.$$

Hence,

$$\begin{aligned}
&\{i\leq k : N_{x_i-x_l}(t)>1-\mu\}\subset\{i,l\leq k : N_{x_i-L}(t)>1-\mu\}\\
&\Rightarrow \lim_k \frac{f(\{i,l\leq k : N_{x_i-x_l}(t)>1-\mu\})}{f(k)}\leq \lim_k \frac{f(\{i\leq k : N_{x_i-L}(t)>1-\mu\})}{f(k)}\\
&\Rightarrow \lim_k \frac{f(\{i,l\leq k : N_{x_i-L}(t)\leq 1-\mu\})}{f(k)}\leq \lim_k \frac{f(\{i\leq k : N_{x_i-x_l}(t)\leq 1-\mu\})}{f(k)}.
\end{aligned}$$

Since (x_i) is f_N-statistically Cauchy, so this leads to the conclusion that $x=(x_i)$ is f_N-statistically convergent. This completes the proof.

Corollary 4. *Let $(X,N,*)$ be a PN-space. Then if (x_i) is f_N-statistically Cauchy sequence then it has a Cauchy subsequence with respect to the probabilistic norm N.*

In this section,we define the concept of f_N-statistical summability with respect to the probabilistic norm N and prove the following result which investigates the relation between f_N-statistical convergence and f_N-statistical summability for the bounded sequence $x=(x_i)$ also.

Definition 6. *Let $(X,N,*)$ be a PN-space. Then a sequence $x=(x_i)$ is said to be f_N-statistically summable to L with respect to the probabilistic norm N provided that , for every $t>0$ and $\mu>0$,*

$$\lim_k \frac{f(|A_N(t)|)}{f(k)}=0, \text{ where } A_N(t)=\{i\leq k : \sum_i N_{x_i-L}(t)\leq 1-\mu\}.$$

We define Π_M^f be the collection of all f_N-statistically summable sequences with respect to the probabilistic norm N.

Finally we conclude this paper by stating the following important theorem.

Theorem 6. *Let $(X,N,*)$ be a PN-space. Then*

(1) *If $x=(x_i)$ is f_N-statistically summable to L then it is f_N-statistically convergent to L.*
(2) *For the bounded sequence $x=(x_i)$, f_N-statistically convergence implies f_N-statistically summable.*
(3) $\Omega_N^f \cap \ell_N^\infty = \Pi_N^f \cap \ell_N^\infty$.

Proof. (1) Let $x=(x_i)$ is f_N-statistically summable to L with resepct to the probabilistic norm N. Then for every $t>0$ and $\mu>0$, we have,

$$\lim_k \frac{f(|A_N(t)|)}{f(k)} = 0, \text{ where } A_N(t) = \{i \le k : \sum_i N_{x_i-L}(t) \le 1-\mu\}. \tag{1}$$

Also, we can write

$$\sum_{i\in\mathbb{N}} N_{x_i-L}(t) \ge \sum_{i\in\mathbb{N}, N_{x_i-L}\le 1-\mu} N_{x_i-L} \ge |\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|\mu$$

So,

$$f(\{i\le k : \sum_{i=1}^{k} N_{x_i-L}(t)\le 1-\mu\}) \ge f(|\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|\mu) \ge cf(|\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|)f(\mu)$$

Then for any $t>0$ and $\mu>0$, we have

$$\frac{1}{f(k)} f(\{i\le k : \sum_{i=1}^{k} N_{x_i-L}(t)\le 1-\mu\}) \ge \frac{cf(|\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|)f(\mu)}{f(k)}$$

which follows that (x_i) is f_N-statistically convergent to L. (from (1))

Proof. (2) Let $x=(x_i)\in \ell_N^\infty$ and (x_i) is f_N-statistically convergent to L. Then there exists an $C>0$ such that for $t>0, N_{x_i-L}(t)\ge 1-C$. Now for any $\mu>0$,we have,

$$\sum_{i\in\mathbb{N}} N_{x_i-L}(t) = \sum_{i\in\mathbb{N}, N_{x_i-L}(t)\le 1-\frac{1}{\mu}} N_{x_i-L}(t) + \sum_{i\in\mathbb{N}, N_{x_i-L}(t)>1-\frac{1}{\mu}} N_{x_i-L}(t) \le C|\{i\le k : N_{x_i-L}(t)\le 1-\frac{1}{\mu}\}| + \frac{1}{\mu}$$

Consequently, we get,

$$\frac{1}{f(k)} f(\{i\le k : \sum_{i=1}^{k} N_{x_i-L}(t)\le 1-\mu\}) \le \frac{f(|\{i\le k : N_{x_i-L}(t)\le 1-\mu\}|)}{f(k)}$$

which shows that (x_i) is f_N-statistically summable to L. This completes the proof.

Proof. (3) Proof follows from (1) and (2). So omitted.

4 Conclusion

In this paper, we studied the concept of f-statistical convergence in probabilistic normed spaces, which can be extended in terms of λ-statistical convergence as well. Results can be generalized in other sequence spaces also along with probabilistic normed spaces. Moreover f-statistical convergence can be studied using the fuzzy real numbers too.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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