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Some properties of semi-tensor bundle

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Abstract: Using the fiber bundle M over a manifold B, we define a semi-tensor (pull-back) bundle tB of type (p,q). The present paper is devoted to some results concerning with the vertical and complete lifts of some tensor fields from manifold B to its semi-tensor bundle tB of type (p,q).

Keywords: Vector field, complete lift, pull-back bundle, semi-tensor bundle.

1 Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $\pi_1 : M_n \to B_m$ the differentiable bundle determined by a submersion π_1. Suppose that $(x^i) = (x^a, x^\alpha)$, $a, b, ... = 1, ..., n-m; \alpha, \beta, ... = n-m+1, ..., n; i, j, ... = 1, 2, ..., n$ is a system of local coordinates adapted to the bundle $\pi_1 : M_n \to B_m$, where x^α are coordinates in B_m, and x^a are fiber coordinates of the bundle $\pi_1 : M_n \to B_m$. If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another system of local adapted coordinates in the bundle, then we have [14]

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1}$$

The Jacobian of (1) has components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where

$$A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}.$$

Let $\left(T_q^p\right)_x(B_m)(x = \pi_1(\tilde{x}), \tilde{x} = (x^a, x^\alpha) \in M_n)$ be the tensor space at a point $x \in B_m$ with local coordinates $(x^1, ..., x^m)$, we have the holonomous frame field

$$\partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes ... \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes ... \otimes dx^{j_q},$$

for $i \in \{1, ..., m\}^p$, $j \in \{1, ..., m\}^q$, over $U \subset B_m$ of this tensor bundle, and for any (p,q)-tensor field t we have [[4], p.163]:

$$t|U = t_{j_1 \dots j_q}^{i_1 \dots i_p} \partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes ... \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes ... \otimes dx^{j_q},$$

then by definition the set of all points $(x^I) = (x^a, x^\alpha, x^{\overline{\alpha}})$, $x^{\overline{\alpha}} = t_{j_1 \dots j_q}^{i_1 \dots i_p}$, $\overline{\alpha} = \alpha + m^{p+q}$, $I, J, ... = 1, ..., n + m^{p+q}$ is a semi-tensor bundle $t_q^p(B_m)$ over the manifold M_n [14]. The semi-tensor bundle $t_q^p(B_m)$ has the natural bundle structure over B_m, its

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bundle projection $\pi : t_q^p(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$. If we introduce a mapping $\pi_2 : t_q^p(B_m) \to M_n$ by $\pi_2 : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^a, x^\alpha)$, then $t_q^p(B_m)$ has a bundle structure over M_n. It is easily verified that $\pi = \pi_1 \circ \pi_2$ [14].
On the other hand, let $\varepsilon = \pi : E \to B$ denote a fiber bundle with fiber F. Given a manifold B' and a map $f : B' \to B$, one can construct in a natural way a bundle over B' with the same fiber: Consider the subset

$$f^*E = \left\{ (b', e) \in B' \times E \middle| f(b') = \pi(e) \right\}$$

together with the subspace topology from $B' \times E$, and denote by $\pi_1 : f^*E \to B'$, $\pi_2 : f^*E \to E$ the projections. $f^*\varepsilon = \pi_1 : f^*E \to B'$ is a fiber bundle with fiber F, called the pull-back bundle of ε via f [[3], [5], [8], [10], [14]].
From the above definition it follows that the semi-tensor bundle $(t_q^p(B_m), \pi_2)$ is a pull-back bundle of the tensor bundle over B_m by π_1 (see, for example [12], [14]).
In other words, the semi-tensor bundle (induced or pull-back bundle) of the tensor bundle $\left(T_q^p(B_m), \widetilde{\pi}, B_m\right)$ is the bundle $\left(t_q^p(B_m), \pi_2, M_n\right)$ over M_n with a total space $t_q^p(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times \left(T_q^p\right)_x (B_m) : \pi_1(x^a, x^\alpha) = \widetilde{\pi}(x^\alpha, x^{\overline{\alpha}}) = (x^\alpha) \right\} \subset M_n \times \left(T_q^p\right)_x (B_m)$.To a transformation (1) of local coordinates of M_n, there corresponds on $t_q^p(B_m)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}\left(x^b, x^\beta\right), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = t^{\beta_1' \ldots \beta_p'}_{\alpha_1' \ldots \alpha_q'} = A^{\beta_1' \ldots \beta_p'}_{\alpha_1 \ldots \alpha_p} A^{\beta_1 \ldots \beta_q}_{\alpha_1' \ldots \alpha_q'} t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')} x^{\overline{\beta}}. \end{cases} \tag{2}$$

The Jacobian of (2) is given by [14]:

$$\bar{A} = \left(A^{I'}_J\right) = \begin{pmatrix} A^{a'}_b & 0 & 0 \\ 0 & A^{\alpha'}_\beta & 0 \\ 0 & t^{(\alpha)}_{(\sigma)} \partial_\beta A^{(\beta')}_{(\alpha)} A^{(\sigma)}_{(\alpha')} & A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')} \end{pmatrix}, \tag{3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J ... = 1, \ldots, n + m^{p+q}$, $t^{(\alpha)}_{(\sigma)} = t^{\alpha_1 \ldots \alpha_p}_{\sigma_1 \ldots \sigma_q}$, $A^{\alpha'}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta}$.
It is easily verified that the condition $Det\bar{A} \neq 0$ is equivalent to the condition:

$$Det(A^{a'}_b) \neq 0, Det(A^{\alpha'}_\beta) \neq 0, Det(A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')}) \neq 0.$$

Also, $\dim t_q^p(B_m) = n + m^{p+q}$. In the special case $n = m$, $t_q^p(B_m)$ is a tensor bundle $T_q^p(B_m)$ [[6], p.118]. In the special case, the semi-tensor bundles $t_0^1(B_m)$ $(p = 1,\ q = 0)$ and $t_1^0(B_m)$ $(p = 0,\ q = 1)$ are semi-tangent and semi-cotangent bundles, respectively. We note that semi-tangent and semi-cotangent bundle were examined in [[1], [7], [9]] and [[11], [13], [15], [16]], respectively. Also, Fattaev studied the special class of semi-tensor bundle [2]. We denote by $\Im_q^p(t_q^p(B_m))$ and $\Im_q^p(B_m)$ the modules over $F\left(t_q^p(B_m)\right)$ and $F(B_m)$ of all tensor fields of type (p, q) on $t_q^p(B_m)$ and B_m respectively, where $F\left(t_q^p(B_m)\right)$ and $F(B_m)$ denote the rings of real-valued C^∞ $-$functions on $t_q^p(B_m)$ and B_m, respectively.

2 Some lifts of tensor fields and $\gamma-$operator

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [9] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. On putting

$$^{cc}\widetilde{X} = \begin{pmatrix} ^{cc}\widetilde{X}^b \\ ^{cc}\widetilde{X}^\beta \\ ^{cc}\widetilde{X}^{\overline{\beta}} \end{pmatrix} = \begin{pmatrix} \widetilde{X}^b \\ \widetilde{X}^\beta \\ \sum_{\lambda=1}^p t^{\alpha_1 \ldots \varepsilon \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \partial_\varepsilon X^{\beta_\lambda} - \sum_{\mu=1}^q t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \varepsilon \ldots \beta_q} \partial_{\beta_\mu} X^\varepsilon \end{pmatrix}, \tag{4}$$

we easily see that ${}^{cc}\widetilde{X}' = \bar{A}\left({}^{cc}\widetilde{X}\right)$. The vector field ${}^{cc}\widetilde{X}$ is called the complete lift of $\widetilde{X}$ to the semi-tensor bundle $t_q^p(B_m)$ [14].

Now, consider $A \in \Im_q^p(B_m)$ and $\varphi \in \Im_1^1(B_m)$, then ${}^{vv}A \in \Im_0^1(t_q^p(B_m))$ (vertical lift), $\gamma\varphi \in \Im_0^1(t_q^p(B_m))$ and $\widetilde{\gamma}\varphi \in \Im_0^1(t_q^p(B_m))$ have respectively, components on the semi-tensor bundle $t_q^p(B_m)$ [14]

$$ {}^{vv}A = \begin{pmatrix} 0 \\ 0 \\ A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix}, \gamma\varphi = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\varphi_\varepsilon^{\alpha_\lambda} \end{pmatrix}, \widetilde{\gamma}\varphi = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\varphi_{\beta_\mu}^{\varepsilon} \end{pmatrix} \tag{5} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t_q^p(B_m)$, where $A_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}$, $\varphi_\varepsilon^{\alpha_\lambda}$ and $\varphi_{\beta_\mu}^{\varepsilon}$ are local components of A and φ.

On the other hand, ${}^{vv}f$ the vertical lift of function $f \in \Im_0^0(B_m)$ on $t_q^p(B_m)$ is defined by [14]:

$$ {}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{6} $$

Theorem 1. *For any vector fields $\widetilde{X}, \widetilde{Y}$ on M_n and $f \in \Im_0^0(B_m)$, we have*

(i) ${}^{cc}(\widetilde{X} + \widetilde{Y}) = {}^{cc}\widetilde{X} + {}^{cc}\widetilde{Y}$,
(ii) ${}^{cc}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf)$.

Proof. (i) This immediately follows from (4).
(ii) Let $\widetilde{X} \in \Im_0^1(M_n)$. Then we get by (4) and (6):

$$ {}^{cc}\widetilde{X}\,{}^{vv}f = {}^{cc}\widetilde{X}^I\partial_I({}^{vv}f) = {}^{cc}\widetilde{X}^a\underbrace{\partial_a({}^{vv}f)}_{0} + {}^{cc}\widetilde{X}^\alpha\partial_\alpha({}^{vv}f) + {}^{cc}\widetilde{X}^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} = X^\alpha\partial_\alpha({}^{vv}f) = {}^{vv}(Xf), $$

which gives *(ii)* of Theorem 1.

Theorem 2. *If $\varphi \in \Im_1^1(B_m)$, $f \in \Im_0^0(B_m)$ and $A \in \Im_q^p(B_m)$, then*

(i) $({}^{vv}A)\,{}^{vv}f = 0$,
(ii) $(\gamma\varphi)\,({}^{vv}f) = 0$,
(iii) $(\widetilde{\gamma}\varphi)\,({}^{vv}f) = 0$.

Proof. (i) If $A \in \Im_q^p(B_m)$, then, by (5) and (6), we find

$$ ({}^{vv}A)\,{}^{vv}f = ({}^{vv}A)^I\partial_I({}^{vv}f) = ({}^{vv}A)^a\underbrace{\partial_a({}^{vv}f)}_{0} + \underbrace{({}^{vv}A)^\alpha}_{0}\partial_\alpha({}^{vv}f) + ({}^{vv}A)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} = 0. $$

Thus, we have *(i)* of Theorem 2.
(ii) If $\varphi \in \Im_1^1(B_m)$, then we have by (5) and (6):

$$ (\gamma\varphi)\,({}^{vv}f) = (\gamma\varphi)^I\partial_I({}^{vv}f) = (\gamma\varphi)^a\underbrace{\partial_a({}^{vv}f)}_{0} + \underbrace{(\gamma\varphi)^\alpha}_{0}\partial_\alpha({}^{vv}f) + (\gamma\varphi)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} = 0. $$

Thus, we have *(ii)* of Theorem 2.
(iii) If $\varphi \in \Im_1^1(B_m)$, then we have by (5) and (6):

$$ (\widetilde{\gamma}\varphi)\,({}^{vv}f) = (\widetilde{\gamma}\varphi)^I\partial_I({}^{vv}f) = (\widetilde{\gamma}\varphi)^a\underbrace{\partial_a({}^{vv}f)}_{0} + \underbrace{(\widetilde{\gamma}\varphi)^\alpha}_{0}\partial_\alpha({}^{vv}f) + (\widetilde{\gamma}\varphi)^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{vv}f)}_{0} = 0. $$

Thus, we have *(iii)* of Theorem 2.

Theorem 3. *Let $A,B\in\Im_q^p(B_m)$. For the Lie product, we have*

$$[^{vv}A,^{vv}B]=0.$$

Proof. If $A,B\in\Im_q^p(B_m)$ and $\begin{pmatrix}[^{vv}A,^{vv}B]^b\\ [^{vv}A,^{vv}B]^\beta\\ [^{vv}A,^{vv}B]^{\overline{\beta}}\end{pmatrix}$ are components of $[^{vv}A,^{vv}B]^J$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t_q^p(B_m)$, then we have

$$\begin{aligned}[^{vv}A,^{vv}B]^J &= (^{vv}A)^I\partial_I(^{vv}B)^J-(^{vv}B)^I\partial_I(^{vv}A)^J\\ &=\underbrace{(^{vv}A)^a}_{0}\partial_a(^{vv}B)^J+\underbrace{(^{vv}A)^\alpha}_{0}\partial_\alpha(^{vv}B)^J+(^{vv}A)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{vv}B)^J-\underbrace{(^{vv}B)^a}_{0}\partial_a(^{vv}A)^J-\underbrace{(^{vv}B)^\alpha}_{0}\partial_\alpha(^{vv}A)^J-(^{vv}B)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{vv}A)^J\\ &=A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}(^{vv}B)^J-B^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}(^{vv}A)^J.\end{aligned}$$

Firstly, if $J=b$, we have

$$[^{vv}A,^{vv}B]^b=A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}\underbrace{(^{vv}B)^b}_{0}-B^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}\underbrace{(^{vv}A)^b}_{0}=0,$$

by virtue of (5). Secondly, if $J=\beta$, we have

$$[^{vv}A,^{vv}B]^\beta=A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}\underbrace{(^{vv}B)^\beta}_{0}-B^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}\underbrace{(^{vv}A)^\beta}_{0}=0,$$

by virtue of (5). Thirdly, if $J=\overline{\beta}$, then we have

$$[^{vv}A,^{vv}B]^{\overline{\beta}}=A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}(^{vv}B)^{\overline{\beta}}-B^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\overline{\alpha}}(^{vv}A)^{\overline{\beta}}=A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\underbrace{\partial_{\overline{\alpha}}B^{\beta_1\ldots\beta_p}_{\theta_1\ldots\theta_q}}_{0}-B^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\underbrace{\partial_{\overline{\alpha}}A^{\beta_1\ldots\beta_p}_{\theta_1\ldots\theta_q}}_{0}=0$$

by virtue of (5). Thus, we have Theorem 3.

Theorem 4. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projections X and Y on B_m, respectively. For the Lie product, we have*

$$[^{cc}\widetilde{X},^{cc}\widetilde{Y}]=^{cc}[\widetilde{X},\widetilde{Y}](i.e.L^{cc}_{^{cc}\widetilde{X}}\widetilde{Y}=^{cc}\left(L_{\widetilde{X}}\widetilde{Y}\right)).$$

Proof. If $\widetilde{X}$ and $\widetilde{Y}$ are projectable vector fields on M_n with projection $X,Y\in\Im_0^1(B_m)$ and $\begin{pmatrix}[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^b\\ [^{cc}\widetilde{X},^{cc}\widetilde{Y}]^\beta\\ [^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{\overline{\beta}}\end{pmatrix}$ are components of $[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^J$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^J=(^{cc}\widetilde{X})^I\partial_I(^{cc}\widetilde{Y})^J-(^{cc}\widetilde{Y})^I\partial_I(^{cc}\widetilde{X})^J.$$

Firstly, if $J=b$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}]^{b} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{b}-({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{b}\\
&= ({}^{cc}\widetilde{X})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{Y})^{b}}_{0}+({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{b}+({}^{cc}\widetilde{X})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{b}}_{0}-({}^{cc}\widetilde{Y})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{X})^{b}}_{0}-({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{b}-({}^{cc}\widetilde{Y})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{b}}_{0}\\
&= ({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{cc}Y)^{b}-({}^{cc}Y)^{\alpha}\partial_{\alpha}({}^{cc}X)^{b}\\
&= X^{\alpha}\partial_{\alpha}\widetilde{Y}^{b}-Y^{\alpha}\partial_{\alpha}\widetilde{X}^{b}\\
&= \widetilde{[X,Y]}^{b}
\end{aligned}$$

by virtue of (4). Secondly, if $J=\beta$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}]^{\beta} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{\beta}-({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{\beta}\\
&= ({}^{cc}\widetilde{X})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{Y})^{\beta}}_{0}+({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\beta}+({}^{cc}\widetilde{X})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{\beta}}_{0}-({}^{cc}\widetilde{Y})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{X})^{\beta}}_{0}-({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\beta}-({}^{cc}\widetilde{Y})^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{\beta}}_{0}\\
&= ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\beta}-({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\beta}\\
&= X^{\alpha}\partial_{\alpha}Y^{\beta}-Y^{\alpha}\partial_{\alpha}X^{\beta}\\
&= [X,Y]^{\beta}
\end{aligned}$$

by virtue of (4). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}
[{}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}]^{\overline{\beta}} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{\overline{\beta}}-({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{\overline{\beta}}\\
&= ({}^{cc}\widetilde{X})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{Y})^{\overline{\beta}}}_{0}+({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\overline{\beta}}+({}^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{\overline{\beta}}-({}^{cc}\widetilde{Y})^{a}\underbrace{\partial_{a}({}^{cc}\widetilde{X})^{\overline{\beta}}}_{0}-({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\overline{\beta}}-({}^{cc}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{\overline{\beta}}\\
&= X^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\varepsilon}\Big)\\
&\quad+\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\alpha_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\gamma}\Big)\partial_{\overline{\alpha}}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}\partial_{\sigma}Y^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\gamma}\Big)\\
&\quad-Y^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}X^{\varepsilon}\Big)\\
&\quad-\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\alpha_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}Y^{\varepsilon}\Big)\partial_{\overline{\alpha}}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}\partial_{\sigma}X^{\beta_\lambda}-\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\beta_\mu}X^{\gamma}\Big)\\
&= X^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\beta_\lambda}\Big)-X^{\alpha}\partial_{\alpha}\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}Y^{\varepsilon})\\
&\quad+\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\alpha_\lambda}\underbrace{\partial_{\overline{\alpha}}\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}}_{\delta_{\alpha_\lambda}^{\sigma}}\partial_{\sigma}Y^{\beta_\lambda}-Y^{\alpha}\partial_{\alpha}\Big(\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}X^{\beta_\lambda}\Big)+Y^{\alpha}\partial_{\alpha}\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}(\partial_{\beta_\mu}X^{\varepsilon})\\
&\quad-\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}\partial_{\varepsilon}Y^{\alpha_\lambda}\underbrace{\partial_{\overline{\alpha}}\sum_{\lambda=1}^{p}t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\sigma\ldots\alpha_p}}_{\delta_{\alpha_\lambda}^{\sigma}}\partial_{\sigma}X^{\beta_\lambda}+\underbrace{\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\varepsilon}\underbrace{\partial_{\overline{\alpha}}\sum_{\mu=1}^{q}t_{\beta_1\ldots\gamma\ldots\beta_q}^{\alpha_1\ldots\alpha_p}}_{\delta_{\gamma}^{\alpha}}\partial_{\beta_\mu}Y^{\gamma}}_{\sum_{\mu=1}^{q}t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\partial_{\alpha_\mu}X^{\varepsilon}(\partial_{\beta_\mu}Y^{\alpha})}
\end{aligned}$$

$$-\underbrace{\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\partial_{\alpha_\mu}Y^{\varepsilon}\underbrace{\partial_{\overline{\alpha}}\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}}_{\delta^{\alpha}_{\gamma}}\partial_{\beta_\mu}X^{\gamma}}_{\Sigma^{q}_{\mu=1} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\partial_{\alpha_\mu}Y^{\varepsilon}(\partial_{\beta_\mu}X^{\alpha})}$$

$$= \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\left(\partial_{\varepsilon}X^{\sigma}\right)\left(\partial_{\sigma}Y^{\beta_\lambda}\right)+\sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}X^{\alpha}\partial_{\alpha}\partial_{\varepsilon}Y^{\beta_\lambda}-\sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\left(\partial_{\varepsilon}Y^{\sigma}\right)\left(\partial_{\sigma}X^{\beta_\lambda}\right)$$

$$-\sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}Y^{\alpha}\partial_{\alpha}\partial_{\varepsilon}X^{\beta_\lambda}+\underbrace{\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\left(-X^{\alpha}\partial_{\alpha_\mu}\partial_{\beta_\mu}Y^{\varepsilon}+\partial_{\beta_\mu}Y^{\alpha}\partial_{\alpha_\mu}X^{\varepsilon}+Y^{\alpha}\partial_{\alpha_\mu}\partial_{\beta_\mu}X^{\varepsilon}-\partial_{\beta_\mu}X^{\alpha}\partial_{\alpha_\mu}Y^{\varepsilon}\right)}_{-\Sigma^{q}_{\mu=1} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}(\partial_{\beta_\mu}\underbrace{(X^{\alpha}\partial_{\alpha_\mu}Y^{\varepsilon}-Y^{\alpha}\partial_{\alpha_\mu}X^{\varepsilon})}_{[X,Y]^{\varepsilon}})}$$

$$= \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\varepsilon}[X,Y]^{\beta_\lambda}-\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}(\partial_{\beta_\mu}[X,Y]^{\varepsilon})$$

by virtue of (4). On the other hand, we know that ${}^{cc}\widetilde{[X,Y]}$ have components

$${}^{cc}\widetilde{[X,Y]}=\begin{pmatrix}\widetilde{[X,Y]}^{b}\\ [X,Y]^{\beta}\\ \Sigma^{p}_{\lambda=1} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_{\varepsilon}[X,Y]^{\beta_\lambda}-\Sigma^{q}_{\mu=1} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\partial_{\beta_\mu}[X,Y]^{\varepsilon}\end{pmatrix}$$

with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t^{p}_{q}(B_m)$. Thus Theorem 4 is proved.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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